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A NEW DERIVATION OF THE KERR METRIC

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By applying the Arnowitt-Deser-Misner modified variational principle, a short and simple derivation is given of the Kerr metric for a stationary, axially symmetric gravitational field. Because the ADM technique involves only the metric components of a 3-dimensional space-like hyper-surface, the computations become much easier and lead very simply to the basic equations determining the metric functions.

1. INTRODUCTION

The Kerr metric for a stationary, axisymmetric gravitational field outside a rotating mass was formulated by Kerr (1963). The generalisation of this—the Kerr-Newman metric—for a charged, rotating mass was subsequently derived by Newman *et al.* (1965). In a recent paper Chandrasekhar (1978) noted that "there was no extent derivation of Kerr's solution that is direct and simple." Newman and Janis (1965) showed that the Kerr metric may be formally "derived" from the Schwarzschild metric by a special type of complex coordinate transformation in which the radial-and time-coordinates are allowed to take complex values. The "derivation" of the Kerr-Newman metric by Newman *et al.* mentioned above is achieved by applying the same algebraic "trick" (complex coordinate transformation) to the Reissner-Nordström metric. In his discussion of stationary, axi-symmetric fields Ernst (1968a) derived the field equations from an appropriate variational principle and showed that the equations can be reduced to the solution of a single equation for one complex function. By assuming a special form for this function, he showed that the problem reduces to the solution of Laplace's equation in spheroidal coordinates and noted that the simplest solution of this leads to the Kerr solution. In a subsequent paper (Ernst 1968b) he gave a similar derivation of the Kerr-Newman metric. In his paper referred to above Chandrasekhar gave a direct *ab initio* derivation of the Kerr metric starting from the general form of the metric for a stationary, axi-symmetric field and applying the (4-dimensional) Einstein field equations to determine the metric functions. The field equations in Boyer-Lindquist coordinates had been derived in a previous paper by Chandrasekhar and Friedman (1972). By a series of transformations of these equations, Chandrasekhar was able to derive Ernst's equation, and using the latter's simplest solution, derived the Kerr solution for the metric.

As mentioned above, Chandrasekhar's derivation of the Kerr metric is based on the field equations in their general 4-dimensional form. The object of the present paper is to show that a simpler and shorter derivation can be given by means of the Arnowitt-Deser-Misner $(3+1)$-dimensional formulation of the Hilbert-Palatini variational principle. (Misner *et al.* 1973). The ADM approach has been applied by Berger *et al.* (1972) to derive the metric for a spherically symmetric gravitational field, viz. the Schwarzschild and the Reissner-Nordström metrics. Because the ADM-modified variational principle involves only the metric components of a space-like 3-dimensional hyper-surface the computations become much simpler and the resulting field equations easier to handle. This procedure leads very simply to the basic equations for the determination of the metric functions, which are then shown to be identical with the corresponding equations derived by Chandrasekhar. Finally we give a direct derivation of Ernst's special solution from which the Kerr solution is known to follow. The analysis below might also be deemed to be of some interest as yet another non-trivial application of the ADM-technique.

2. The ADM-Variational Principle

The ADM formulation of the variational principle is as follows: Let the 3-metric of a space-like hyper-surface $x^0 \equiv t = \text{const.}$ be

$$dl^2 = {}^{(3)}g_{ij}\, dx^i\, dx^j \ (i,j = 1,2,3) \qquad \ldots(1)$$

the superscript $^{(3)}$ indicating that the quantity in front of it refers to the 3-geometry. Then the action principle is expressed as (Misner *et al.* 1973, pp. 520–25):

$$\left.\begin{array}{l} I = (1/16\pi) \int \mathscr{L} d^{(4)}x = \text{extremum} \\ \text{where } \mathscr{L} = \pi^{ij}\partial^{(3)}g_{ij}/\partial t - N\mathscr{H}(\pi^{ij}, {}^{(3)}g_{ij}) - N_i\mathscr{H}^{(i)}(\pi^{ij}, {}^{(3)}g_{ij}) \end{array}\right\}. \qquad \ldots(2)$$

The quantities N, N^i are defined in terms of $^{(4)}g_{\mu\nu}$ of the 4-geometry by the relation:

$$\begin{pmatrix} {}^{(4)}g_{00}, & {}^{(4)}g_{0k} \\ {}^{(4)}g_{i0}, & {}^{(4)}g_{ik} \end{pmatrix} \equiv \begin{pmatrix} N_jN^j - (N)^2, & N_k \\ N_i \quad , & {}^{(3)}g_{ik} \end{pmatrix}; \ N_i = {}^{(3)}g_{im}N^m \qquad \ldots(3)$$

and

$$\left.\begin{array}{l} \mathscr{H}, \mathscr{H}^{(i)} \text{ by } \mathscr{H} = ({}^{(3)}g)^{-1/2}.\ [\pi^{ij}\pi_{ij} - \tfrac{1}{2}(\pi^i{}_i)^2] - ({}^{(3)}g)^{1/2\,(3)}\mathscr{R} \\ \qquad \mathscr{H}^{(i)} = -\ 2\pi^{ij}{}_{|j}. \end{array}\right\} \qquad \ldots(4,\text{a})$$

The subscript stroke indicates covariant differentiation w.r.t. the 3-metric

$$\left.\begin{array}{l} {}^{(3)}g = \det({}^{(3)}g_{ij}) \\ {}^{(3)}\mathscr{R} = \text{Riemann curvature invariant of the 3-metric (1).} \end{array}\right\} \qquad \ldots(4,\text{b})$$

The quantities to be varied independently to extremise the action are:

$^{(3)}g_{ij}$, π^{ij}, N and N^i.

3. Computation of $\mathscr{L}$ for a Stationary, Axi-symmetric metric

A general stationary, axi-symmetric metric can be written*

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi-\omega dt)^2 + e^{2\mu_2}(dx^2) + e^{2\mu_3}(dx^3)^2 \qquad \ldots(5)$$

where ϕ denotes the azimuthal angle (in the equatorial plane) and $x^2(=r)$, $x^3(=\theta)$ are the two remaining spatial coordinates.

By the assumption of stationarity and axi-symmetry, ν, ψ, ω, μ_2, μ_3 are functions of x^2, x^3 only. The 3-metric of the space-like hyper-surface t = const. of (5) is

$$dl^2 = e^{2\psi}(d\phi)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 \qquad \ldots(6)$$

so that the metric coefficients ${}^{(3)}g_{ij}$ are:

$$\left.\begin{array}{l} {}^{(3)}g_{11} = e^{2\psi},\ {}^{(3)}g_{22} = e^{2\mu_2},\ {}^{(3)}g_{33} = e^{2\mu_3} \\ {}^{(3)}g_{ij} = 0 \ \ i \neq j. \end{array}\right\} \qquad \ldots(7)$$

The lapse and shift functions N, N^i, N_i are, from (3),

$$\left.\begin{array}{l} N = e^{\nu},\ N^1 = -\omega,\ N^2 = N^3 = 0;\ N_1 = -\omega e^{2\psi} \\ N_2 = N_3 = 0. \end{array}\right\} \qquad \ldots(8)$$

For a stationary metric $\partial/\partial t \equiv 0$ and the variational principle (2) becomes

$$\int \mathscr{L} d^{(4)}x = I = (1/16\pi) \int [N\mathscr{H} + N_i\mathscr{H}^{(i)}]d^{(4)}x = \text{extremum.} \qquad \ldots(9)$$

The calculation of the Riemann curvature invariant for the metric (6) is carried out most simply by the method of curvature 2-forms. We thus find

$$\begin{aligned} {}^{(3)}\mathscr{H} = & -2e^{2\mu_2}[\psi_{,22} + (\psi_{,2} + \mu_{3,2})(\mu_{3,2} - \mu_{2,2}) + \psi_{,2}^2 + \mu_{3,22}] \\ & -2e^{2\mu_3}[\psi_{,33} + (\psi_{,3} + \mu_{2,3})(\mu_{2,3} - \mu_{3,3}) + \psi_{,3}^2 + \mu_{2,33}] \end{aligned} \qquad \ldots(10)$$

where a subscript comma denotes differentiation, *e.g.* $f_{,i} = \partial f/\partial x^i$.

We write the π^{ij} in the form:

$$\left.\begin{array}{l} \pi^{11} = \pi_1 e^{-2\psi};\ \pi^{12} = \pi^{21} = \tfrac{1}{2}Re^{-\psi-\mu_2};\ \pi^{13} = \pi^{31} = \tfrac{1}{2}Qe^{-\psi-\mu_3} \\ \pi^{22} = \pi_2 e^{-2\mu_2};\ \pi^{23} = \pi^{32} = \tfrac{1}{2}Pe^{-\mu_2-\mu_3};\ \pi^{33} = \pi_3 e^{-2\mu_3} \end{array}\right\} \qquad \ldots(11)$$

Then

$$\left.\begin{array}{l} Tr\,\pi = \pi_1 + \pi_2 + \pi_3 \\ Tr(\pi^2) = \pi_1^2 + \pi_2^2 + \pi_3^2 + \tfrac{1}{2}(P^2 + Q^2 + R^2) \end{array}\right\} \qquad \ldots(12)$$

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* We employ throughout the notation used in Chandrasekhar's paper.

$$\left.\begin{aligned}
\pi^{1j}{}_{|j} &= \frac{1}{2}\left\{R_{,2} + R(\psi-\mu_2)_{,2}\right\} e^{-\psi-\mu_2} \\
&\quad + \frac{1}{2}\left\{ Q_{,3} + Q(\psi - \mu_3)_{,3}\right\} e^{-\psi-\mu_3} \\
\pi^{2j}{}_{|j} &= e^{-2\mu_2}\left\{ \pi_{2,2} - \mu_{2,2}\pi_2 - \psi_{,2}\pi_1 - \mu_{3,2}\pi_3\right\} \\
&\quad + \frac{1}{2}\left\{P_{,3} + P(\mu_2-\mu_3)_{,3}\right\} e^{-\mu_2-\mu_3} \\
\pi^{3j}{}_{|j} &= e^{-2\mu_3}\left\{ \pi_{3,3} - \mu_{3,3}\pi_3 - \psi_{,3}\pi_1 - \mu_{2,3}\pi_2\right\} \\
&\quad + \frac{1}{2}\left\{P_{,2} + P(\mu_3 - \mu_2)_{,2}\right\} e^{-\mu_2-\mu_3}
\end{aligned}\right\} \quad \ldots(13)$$

Substituting these expressions in (4) we get

$$\left.\begin{aligned}
\mathscr{H} &= \frac{1}{2}e^{-(\psi+\mu_2+\mu_3)}\Big[\pi_1^2 + \pi_2^2 + \pi_3^2 - 2\pi_1\pi_2 - 2\pi_2\pi_3 - 2\pi_3\pi_1 \\
&\quad + P^2 + Q^2 + R^2\Big] + 2e^{(\psi-\mu_2+\mu_3)}\,[\psi_{,22} + \psi_{,2}^2 \\
&\quad + (\psi_{,2} + \mu_{3,2})(\mu_{3,2} - \mu_{2,2}) + \mu_{3,22}] + 2e^{(\psi+\mu_2-\mu_3)}[\psi_{,33} \\
&\quad + \psi_{,3}^2 + (\psi_{,3} + \mu_{2,3})(\mu_{2,3} - \mu_{3,3}) + \mu_{2,33}]. \\
\mathscr{H}^{(1)} &= -\{R_{,2}+R(\psi-\mu_2)_{,2}\}e^{-\psi-\mu_2} - \{Q_{,3}+Q(\psi-\mu_3)_{,3}\}e^{-\psi-\mu_3} \\
\mathscr{H}^{(2)} &= -2e^{-2\mu_2}\{\pi_{2,2} - \mu_{2,2}\pi_2 - \psi_{,2}\pi_1 - \mu_{3,2}\pi_3\} \\
&\quad - \{P_{,3} + P(\mu_2 - \mu_3)_{,3}\}e^{-\mu_2-\mu_3} \\
\mathscr{H}^{(3)} &= -2e^{-2\mu_3}\{\pi_{3,3} - \mu_{3,3}\pi_3 - \psi_{,3}\pi_1 - \mu_{2,3}\pi_2\} \\
&\quad - \{P_{,2} + P(\mu_3 - \mu_2)_{,2}\}e^{-\mu_2-\mu_3}.
\end{aligned}\right\} \quad \ldots(14)$$

4. The Field Equations

The action (9) has to be extremized by varying in turn the π^{ij}, ${}^{(3)}g_{ij}$ and the lapse and shift functions N, N^i.

Varying N, N_1, N_2, N_3 gives the four initial-value equations:

$$\mathscr{H} = 0,\ \mathscr{H}^{(1)} = 0,\ \mathscr{H}^{(2)} = 0,\ \mathscr{H}^{(3)} = 0 \quad \ldots(15)$$

Varying the π^{ij} independently is equivalent to varying the π_i and P, Q, R. The π_i variation gives the three equations

$$Ne^{-(\psi+\mu_2+\mu_3)}(\pi_1 - \pi_2 - \pi_3) + 2N_2\psi_{,2}e^{-\mu_2} + 2N_3\psi_{,3}e^{-2\mu_3} = 0$$

$$Ne^{-(\psi+\mu_2+\mu_3)}(-\pi_1 + \pi_2 - \pi_3) + 2N_2\mu_{2,2}e^{-2\mu_2} + 2N_3\mu_{2,3}e^{-2\mu_3} + (2N_2e^{-2\mu_2})_{,2} = 0$$

$$Ne^{-(\psi+\mu_2+\mu_3)}(-\pi_1 - \pi_2 + \pi_3) + 2N_2\mu_{3,2}e^{-2\mu_2} + 2N_3\mu_{3,3}e^{-2\mu_3} + (2N_3e^{-2\mu_3})_{,3} = 0.$$

Using the fact that $N_2 = N_3 = 0$, [eqn. (18)], these give

$$\pi_1 = \pi_2 = \pi_3 = 0. \quad \ldots(16)$$

Next varying P, Q, R yields the three equations

$$\left.\begin{aligned} &Ne^{-(\psi+\mu_2+\mu_3)}P - N_2(\mu_2 - \mu_3)_{,3}e^{-\mu_2-\mu_3} + N_3(\mu_2 - \mu_3)_{,2}e^{-\mu_2-\mu_3} \\ &\qquad + [N_3e^{-\mu_2-\mu_3}]_{,2} + [N_2e^{-\mu_2-\mu_3}]_{,3} = 0 \\ &Ne^{-(\psi+\mu_2+\mu_3)}Q - N_1(\psi - \mu_3)_{,3}e^{-\psi-\mu_3} + (N_1e^{-\psi-\mu_3})_{,3} = 0 \\ &Ne^{-(\psi+\mu_2+\mu_3)}R - N_1(\psi-\mu_2)_{,2}e^{-\psi-\mu_2} + (N_1e^{-\psi-\mu_2})_{,2} = 0 \end{aligned}\right\} \quad \ldots(17)$$

The first of these equations combined with $N_2 = N_3 = 0$ shows that

$$P = 0 \quad \ldots(18)$$

With $P = 0$, $N_2 = 0$, $N_3 = 0$, we see from (14) that the equations $\mathscr{H}^{(2)} = 0$, $\mathscr{H}^{(3)} = 0$ of (15) are identically satisfied.

The equation $\mathscr{H}^{(1)} = 0$ gives

$$[Re^{(\psi-\mu_2)}]_{,2} + [Qe^{(\psi-\mu_3)}]_{,3} = 0 \quad \ldots(19)$$

showing that we can put

$$Qe^{(\psi-\mu_3)} = F_{,2}$$

$$Re^{(\psi-\mu_2)} = -F_{,3}$$

or

$$\left.\begin{aligned} Q &= F_{,2}e^{(-\psi+\mu_3)} \\ R &= -F_{,3}e^{(-\psi+\mu_2)} \end{aligned}\right\} \quad \ldots(20)$$

Substituting for Q and R in the second and third equations (17) we get,

$$\left.\begin{aligned} \omega_{,2} &= -(N_1e^{-2\psi})_{,2} = -Ne^{(-3\psi+\mu_2-\mu_3)}F_{,3} \\ \omega_{,3} &= -(N_1e^{-2\psi})_{,3} = Ne^{(-3\psi-\mu_2+\mu_3)}F_{,2} \end{aligned}\right\} \quad \ldots(21)$$

Finally we write down the equations corresponding to the variation of ψ, μ_2, μ_3 (taking into account the already determined values $N_2 = N_3 = P = 0$):

$$\begin{aligned} -\frac{N}{2}e^{-(\psi+\mu_2+\mu_3)}(Q^2+R^2) &+ 2Ne^{(\psi-\mu_2+\mu_3)}\{\psi_{,22}+\psi_{,2}^2+(\psi_{,2}+\mu_{3,2})(\mu_{3,2}-\mu_{2,2}) \\ &+\mu_{3,22}\} + 2Ne^{(\psi+\mu_2-\mu_3)}\{\psi_{,33}+\psi_{,3}^2+(\psi_{,3}+\mu_{2,3})(\mu_{2,3}-\mu_{3,3}) \\ &+\mu_{2,33}\} - \left[2Ne^{(\psi-\mu_2+\mu_3)}(2\psi_{,2}-\mu_{2,2}+\mu_{3,2}) - N_1Re^{-\psi-\mu_2}\right]_{,2} \\ &-\left[2Ne^{(\psi+\mu_2-\mu_3)}(2\psi_{,3}-\mu_{3,3}+\mu_{2,3}) - N_1Qe^{-\psi-\mu_3}\right]_{,3} \\ &+\left[2Ne^{(\psi-\mu_2+\mu_3)}\right]_{,22} + \left[2Ne^{(\psi+\mu_2-\mu_3)}\right]_{,33} = 0 \quad \ldots(22,a) \end{aligned}$$

$$\begin{aligned} -\frac{N}{2}e^{-(\psi+\mu_2+\mu_3)}(Q^2+R^2) &- 2Ne^{(\psi-\mu_2+\mu_3)}\Big\{\psi_{,22}+\psi^2_{,2}+(\psi_{,2}+\mu_{3,2})(\mu_{3,2}-\mu_{2,2}) \\ &+\mu_{3,22}\Big\} + 2Ne^{(\psi+\mu_2-\mu_3)}\{\psi_{,33}+\psi_{,3}^2+(\psi_{,3}+\mu_{2,3})(\mu_{2,3}-\mu_{2,33}) \end{aligned}$$

(*equation continued on p.* 1327)

$$+\mu_{2,33}\} + \left[2Ne^{(\psi-\mu_2+\mu_3)}(\psi+\mu_3)_{,2} - N_1 R e^{-\psi-\mu_2} \right]_{,2}$$

$$-\left[2Ne^{(\psi+\mu_2-\mu_3)}(\psi_{,3}+2\mu_{2,3}-\mu_{3,3}) \right]_{,3} + N_1\{R_{,2}+$$

$$+ R(\psi-\mu_2)_{,2}\}e^{-\psi-\mu_2} + \left[2N e^{(\psi+\mu_2-\mu_3)}\right]_{,33} = 0 \qquad \ldots(22,b)$$

$$-\frac{N}{2}e^{-(\psi+\mu_2+\mu_3)}(Q^2+R^2)+2Ne^{(\psi-\mu_2+\mu_3)}\Big\{\psi_{,22}+\psi_{,2}^2+(\psi_{,2}+\mu_{3,2})(\mu_{3,2}-\mu_{2,2})$$

$$+\mu_{3,22}\Big\} - 2Ne^{(\psi+\mu_2+\psi_3)}\{\psi_{,33}+\psi_{,3}^2+(\psi_{,3}+\mu_{2,3})\ (\mu_{2,3}-\mu_{3,5})$$

$$+\mu_{2,33}\} - \left[2Ne^{(\psi-\mu_2+\mu_3)}(\psi_{,2}+2\mu_{3,2}-\mu_{2,2})\right]_{,2}$$

$$+\left[2Ne^{(\psi+\mu_2-\mu_3)}(\psi_{,3}+\mu_{2,3})-N_1 Q e^{-\psi-\mu_3}\right]_{,3}$$

$$+N_1\{Q_{,3}+Q(\psi-\mu_3)_{,3}\}e^{-\psi-\mu_3} + \left[2Ne^{(\psi-\mu_2+\mu_3)}\right]_{,22} = 0.$$

$$\ldots(22,c)$$

5. Coordinate Conditions and Final Form of the Equations

To proceed further we now impose, following Chandrasekhar, the coordinate conditions:

$$\left.\begin{aligned} e^{2(\mu_3-\mu_2)} &= \Delta(r) = r^2 + a^2 - 2Mr \\ Ne^{\psi} &= e^{\nu+\psi} = \Delta^{1/2}\sin\theta. \end{aligned}\right\} \qquad \ldots(23)$$

As shown by Chandrasekhar in his paper this choice of gauge is consistent with the existence of a smooth event horizon i.e. a 2-dimensional null-surface spanned by the two Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ admitted by the assumed form of the space-time metric.

Using the conditions (23), eqns. (20) and (21) become

$$Q = [e^{(3\psi+\mu_3)}/\Delta\sin\theta]\omega_{,3};\ R = [e^{(3\psi+\mu_2)}/\sin\theta]\omega_{,2} \qquad \ldots(24)$$

$$\left.\begin{aligned} \ldots\omega_{,2} &= (N_1 e^{-2\psi})_{,2} = e^{-4\psi}\sin\theta\, F_{,3} \\ -\omega_{,3} &= (N_1 e^{-2\psi})_{,2} = -e^{-4\psi}\Delta\sin\theta\, F_{,2} \end{aligned}\right\}. \qquad \ldots(25)$$

Substituting in (19) for Q and R from (24) we have

$$\left(\frac{e^{4\psi}}{\sin\theta}\omega_{,2} \right)_{,2} + \left(\frac{e^{4\psi}}{\Delta\sin\theta}\omega_{,3} \right)_{,3} = 0$$

$$\text{or}\quad \Delta\,(e^{4\psi}\omega_{,2})_{,2} + \sin\theta\left(\frac{e^{4\psi}}{\sin\theta}\,\omega_{,3} \right)_{,3} = 0. \qquad \ldots(26)$$

Similarly substituting from (24), (25) eqns. (22a, b, c) become

$$2\sin\theta\ \Big[\Big(\Delta\psi_{,22}+\psi_{33}-\Delta\psi_{,2}^2-\psi_{,3}^2+\frac{3}{2}\Delta'\psi_{,2}+2\cot\theta\,\psi_{,3}\Big) - (\Delta\mu_{3,22}+\tfrac{1}{2}\Delta'\mu_{3,2}$$

(*equation continued on p.* 1328)

$$+\mu_{3,33}\Big)\Big] = -\frac{e^{4\psi}}{2\Delta\sin\theta}\left(\Delta\,\omega_{,2}^{2}+\omega_{,3}^{2}\right)-\left(\frac{e^{4\psi}}{\sin\theta}\,\omega\omega_{,2}\right)_{,2}$$

$$-\left(\frac{e^{4\psi}}{\Delta\sin\theta}\,\omega\omega_{,3}\right)_{,3} \qquad ...(27)$$

$$2\sin\theta\left[\Delta\psi_{,2}^{2}-\psi_{,3}^{2}-\frac{1}{2}\Delta'\psi_{,2}+\cot\theta\psi_{,3}-\frac{1}{2}\Delta'\mu_{3,2}+\cot\theta\mu_{3,3}+1\right]$$

$$=-\frac{e^{4\psi}}{2\Delta\sin\theta}\left(\Delta\omega_{,2}^{2}+\omega_{,3}^{2}\right)+\left(\frac{e^{4\psi}}{\sin\theta}\,\omega_{,2}^{2}\right) \qquad ...(28)$$

$$-2\sin\theta\left[\Delta\psi_{,2}^{2}-\psi_{,3}^{2}-\frac{1}{2}\Delta'\psi_{,2}+\cot\theta\psi_{,3}-\frac{1}{2}\Delta'\mu_{3,2}+\cot\theta\mu_{3,3}+1\right]$$

$$=-\frac{e^{4\psi}}{2\Delta\sin\theta}\left(\Delta\omega_{,2}^{2}+\omega_{,3}^{2}\right)+\left(\frac{e^{4\psi}}{\Delta\sin\theta}\,\omega_{,3}^{2}\right) \qquad ...(29)$$

$[\Delta' = d\Delta/dr]$.

Also the initial value equation $\mathcal{H}=0$ [eqn. (14)] becomes

$$2\sin\theta\left[\left(\Delta\psi_{,22}+\psi_{,33}+\Delta\psi_{,2}^{2}+\frac{1}{2}\Delta'\psi_{,2}+\psi_{,3}^{2}\right)+\left(\Delta\mu_{3,22}+\frac{1}{2}\Delta'\mu_{3,2}+\mu_{3,33}\right)\right]$$

$$=-\frac{e^{4\psi}}{2\Delta\sin\theta}\left(\Delta\omega_{,2}^{2}+\omega_{,3}^{2}\right) \qquad ...(30)$$

Adding together eqns. (27) and (30) we get

$$\Delta\psi_{,22}+\psi_{,33}+\Delta'\psi_{,2}+\cot\theta\psi_{,3}=-\frac{e^{4\psi}}{2\Delta\sin^{2}\theta}\left(\Delta\omega_{,2}^{2}+\omega_{,3}^{2}\right). \qquad ..(31)$$

Equations (28) and (29) are seen to be identical and we have

$$\frac{1}{2}\Delta'\mu_{3,2}-\cot\theta\mu_{3,3}=\left(\Delta\psi_{,2}^{2}-\frac{1}{2}\Delta'\psi_{,2}-\psi_{,3}^{2}+\cot\theta\psi_{,3}+1\right)$$

$$-\frac{e^{4\psi}}{4\Delta\sin^{2}\theta}\left(\Delta\omega_{,2}^{2}-\omega_{,3}^{2}\right). \qquad ...(32)$$

Equations (26), (31) and (32) constitute three equations for ψ, μ_3 and ω, These equations are the same as the equations derived by Chandrasekhar. He writes them in a slightly different form by introducing the function:

$$\chi = e^{-2\psi}\,\Delta^{1/2}\sin\theta. \qquad ...(33)$$

Then it is readily verified that

$$\left(\frac{\Delta\chi_{,2}}{\chi}\right)_{,2}+\cot\theta\frac{\chi_{,3}}{\chi}+\left(\frac{\chi_{,3}}{\chi}\right)_{,3}=-2\{(\Delta\psi_{,2})_{,2}+\psi_{,33}+\cot\theta\psi_{,3}\}$$

so that (31) becomes

$$\chi^{2}\left[\left(\frac{\Delta\chi_{,2}}{\chi}\right)_{,2}+\left(\frac{\delta\chi_{,3}}{\chi}\right)_{,3}\right]=\omega_{,2}^{2}+\delta\omega_{,3}^{2} \qquad ...(34)$$

wherein we have used $\mu = \cos\theta$ as the variable x^3 instead of θ and also put

$$\delta = 1-\mu^{2} = \sin^{2}\theta. \qquad ...(35)$$

Also eqn. (26) now becomes

$$\left(\frac{\Delta}{\chi^2} \omega_{,2} \right)_{,2} + \left(\frac{\delta}{\chi^2} \omega_{,3} \right)_{,3} = 0. \qquad \ldots(36)$$

Further it is readily verified that

$$\Delta\, \psi_{,2}^2 - \frac{1}{2} \Delta' \psi_{,2} - \delta\, \psi_{,3}^2 - \cos\theta\, \psi_{,3} + 1$$

$$= 1 - \left(\Delta' \right)^2 \Big/ 16\Delta + \frac{1}{4} \cot^2\theta - \frac{1}{4\chi^2} \left(\Delta\chi_{,2}^2 - \delta\chi_{,3}^2 \right). \qquad \ldots(37)$$

Observing that $\mu_{3,2} - \mu_{2,2} = (\Delta'/2\Delta)$ and using (32) we find

$$\frac{1}{2} \Delta' \left(\mu_2 + \mu_3 \right)_{,2} + \cos\theta \left(\mu_2 + \mu_3 \right)_{,3}$$

$$= \frac{1}{2\chi^2} \left(\Delta\chi_{,2}^2 - \delta\chi_{,3}^2 \right) - \frac{1}{2\chi^2} \left(\Delta\, \omega_{,2}^2 - \delta\omega_{,3}^2 \right) - \frac{3}{2} \frac{(M^2 - a^2)}{\Delta} + \frac{1}{2\delta}. \qquad \ldots(38)$$

Equations (34), (36) and (38) are identical with the equations derived by Chadrasekhar. These equations along with (23) determine ν, ω, ψ, μ_2 and μ_3. To effect this solution Chandrasekhar showed that (34) and (36) can be combined into a single equation (Ernst's equation)

$$(1 - \mathscr{E}^*) [(\Delta\, \mathscr{E}_{,2})_{,2} + (\delta\, \mathscr{E}_{,3})_{,3}] = -2\, \mathscr{E}^* [\Delta\, (\mathscr{E}_{,2})^2 + \delta\, (\mathscr{E}_{,3})^2] \qquad \ldots(39)$$

where $\mathscr{E} = (Z-1)/(Z+1)$, $Z = f + ig$, $\mathscr{E}^*$ = complex conjugate of $\mathscr{E}$, and f, g are defined by

$$\left. \begin{aligned} f &= (\Delta\delta)^{1/2} \cdot \frac{(\chi^2 - \omega^2)}{\chi}\ ; \\ g_{,2} &= (f^2/\Delta)\, [\omega/(\chi^2 - \omega^2)]_{,3}\ ;\ g_{,3} = -(f^2/\delta)\, [\omega/(\chi^2 - \omega^2)]_{,2}\ . \end{aligned} \right\} \qquad \ldots(40)$$

Ernst wrote eqn. (39) in the form

$$\left. \begin{aligned} &(1 - \mathscr{E}\mathscr{E}^*) \{[(\eta^2 - 1)\, \mathscr{E}_{,\eta}]_{,\eta} + [(1 - \mu^2)\, \mathscr{E}_{,\mu}]_{,\mu}\} \\ &\quad = -2\, \mathscr{E}^* [(\eta^2 - 1)\, (\mathscr{E}_{,\eta})^2 + (1 - \mu^2)\, (\mathscr{E}_{,\mu})^2] \end{aligned} \right\} \qquad \ldots(41)$$

where $\eta = (r - M) / (M^2 - a^2)^{1/2}$,

and showed that the Kerr metric follows from the simplest solution of (41), viz.

$$\mathscr{E} = -p\eta - iq\mu\ , \ (p, q \text{ constants}, p^2 + q^2 = 1) \qquad \ldots(42)$$

which he obtained as a particular case of a more general solution of (41) in spheroidal coordinates. Chandrasekhar adopted this "readily verifiable solution" of (41) and from it derived solutions of the basic equations above for the metric functions, which are then shown to lead to the well-known form of the Kerr metric.

6. Direct Derivation of (42)

Here we shall give an alternative direct derivation of the solution (42) of (41). Put $x = p_1\eta$, $y = q_1\mu$ with p_1, q_1 real constants.

Then (41) becomes

$$\left(1-\mathcal{E}\mathcal{E}^*\right)\left\{\left[\left(x^2-p_1^2\right)\mathcal{E}_{,x}\right]_{,x}+\left[\left(q_1^2-y^2\right)\mathcal{E}_{,y}\right]_{,y}\right\}$$
$$=-2\,\mathcal{E}^*\left\{\left(x^2-p_1^2\right)\left(\mathcal{E}_{,x}\right)^2+\left(q_1^2-y^2\right)\left(\mathcal{E}_{,y}\right)^2\right\} \qquad \ldots(43)$$

Writing $\zeta = x + iy$, $\zeta^* = x - iy$ we have

$$\mathcal{E}_{,x} = \mathcal{E}_{,\zeta} + i\,\mathcal{E}_{\zeta^*}\ ;\ i\,\mathcal{E}_{,y} = \mathcal{E}_{,\zeta^*} - \mathcal{E}_{,\zeta}$$

Assuming that $\mathcal{E}$ should be an analytic function of ζ the condition for this is

$$\mathcal{E}_{,\zeta^*} = 0 \text{ (Ahlfors 1966, p. 27).} \qquad \ldots(44)$$

Changing the variables to ζ, ζ^* and using the condition (44) we find that (43) reduces to

$$\left(1-\mathcal{E}\mathcal{E}^*\right)\left\{\left(\zeta\zeta^*-p_1^2-q_1^2\right)\mathcal{E}_{,\zeta\zeta}+2\zeta^*\mathcal{E}_{,\zeta}\right\}$$
$$=-2\,\mathcal{E}^*\left(\zeta\zeta^*-p_1^2-q_1^2\right)\left(\mathcal{E}_{,\zeta}\right)^2 \qquad \ldots(45)$$

Since $\mathcal{E}$ is assumed to be analytic near $\zeta = 0$ we can expand it in a power series :

$$\mathcal{E} = a_0 + a_1\zeta + a_2\zeta^2 + \ldots a_i\zeta^i + \ldots$$

Substituting in (45) and equating coefficients of powers of ζ on both sides we get a set of equations for determining a_i, $(i \geqslant 1)$, in terms of a_0.
In particular for $a_0 = 0$ we find that

$$a_i = 0\ (i \geqslant 2)\ ,\ a_1 \neq 0$$

so that the corresponding solution is

$$\mathcal{E} = a_1\zeta \qquad \ldots(46)$$

where a_1 satisfies the condition :

$$|a_1|^2\left(p_1^2+q_1^2\right) = 1. \qquad \ldots(47)$$

If we put

$$p = |a_1|\,p_1\ ,\ q = |a_1|\,q_1\ .\ a_1 = |a_1|\,e^{i\alpha}$$

then the solution is

$$\mathcal{E} = a_1\zeta = e^{i\alpha}\,(p\eta + iq\mu),\ p^2 + q^2 = 1. \qquad \ldots(48)$$

Taking $\alpha = 0$ we get the solution adopted by Ernst and Chandrasekhar. As shown in Chandrasekhar's paper, from this solution for $\mathcal{E}$ we get,

$$f = (\Delta - \alpha^2\delta)/\rho^2\ ,\ g = 2aM\mu/\rho^2\ ,\ \rho^2 = r^2 + a^2\mu^2 \qquad \ldots(49)$$

and these lead to the usual expressions for ν, ω, ψ; μ_2, μ_3 characterising the Kerr metric.

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