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ON THE EFFECT OF GRAVITY-INDUCED INITIAL STRESSESS ON THE DISTURBANCES IN AN ELASTIC HALF-SPACE DUE TO A MOVING LOAD

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The steady state problem of propagation of disturbances in an elastic half space due to a concentrated moving load on the surface is investigated. The medium is considered to be elastic and incompressible and in a state of initial stress due to gravity. Two cases have been considered, (i) the first, in which the load speed is greater than the shear wave velocity of the medium, (ii) the second, in which the load speed is less than the shear wave velocity of the medium. A solution for the stresses and displacements in the medium is obtained in both the cases and some numerical results showing the effect of gravity have been computed.

1. INTRODUCTION

The moving line load problem is a standard topic of importance in elasticity, seismology, and in engineering applications. Also the study of the effect of initial stresses in the medium has practical utility. The effect of initial stresses in the earth owing to gravity on the propagation of waves have been studied by Biot (1940). He has shown that the velocity of Rayleigh waves is greater due to the presence of initial stresses. Brunelle (1973) considered surface wave propagation under initial tension and compression, and obtained phase speeds of Rayleigh and Love waves which showed dramatic changes. The displacements and stresses produced by a uniformly moving line load on the boundary of a semi-infinite elastic medium have been obtained by Sneddon (1952), Cole and Huth (1958), Chakraborty (1958) and others. The extension of this problem to the case of a uniformly moving line load on the boundary of a semi-infinite elastic medium under initial stresses due to gravity has been made in this paper. Two cases have been considered according as the velocity of the moving load is less than or greater than the shear wave velocity in the unstressed medium. The formulation of Biot (1965) has been used, and in the evaluation of Fourier inverse transform integrals, generalised function approach has been used following Jones (1966).

2. FORMULATION

Let the space coordinates be x, y, z the half-space being $z \geqslant 0$, and a normal line-load of strength P be moving on the surface $z = 0$, in the negative x-direction with a speed v from $+\infty$ to $-\infty$. The initial stresses in the medium due to gravity g in the positive z-direction are assumed to be $S_{11} = S_{22} = S_{33} = S$ and $S_{12} = S_{23}$

$= S_{31} = 0$. From the stress equations of equilibrium and stress-free boundary conditions at $z = 0$, it follows that

$$S = -\rho g z. \qquad \text{...(2.1)}$$

Let the incremental stress tensor components due to the moving load be s_{ij} ; $i, j = 1, 2, 3$ and the displacement be (u, o, w). For initially stressed medium the governing equations of motion following Biot (1965) are

$$\frac{\partial s_{ij}}{\partial x_j} + \rho \Delta X_i - \rho \omega_{,k} X_k (x_e) + \rho e X_i (x_e) + S_{kj} \frac{\partial \omega_{ik}}{\partial x_j} + S_{ik} \frac{\partial \omega_{jk}}{\partial x_j} - e_{jk} \frac{\partial S_{ik}}{\partial x_j}$$

$$= \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \text{...(2.2)}$$

with the boundary conditions

$$[s_{ij} + S_{ij} + S_{kj}\,\omega_{ik} + S_{ij} e - S_{ik} e_{jk}]\, n_j = j_i \qquad \text{...(2.3)}$$

where X_i are the components of the body forces per unit mass, $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}),\ e = e_{11} + e_{22} + e_{33},\ \Delta X_i = \text{increment in } X_i \qquad \text{...(2.4)}$$

ρ is the density of the medium (= constant), n_i the unit normal to the boundary, f_i the final surface tractions.

In the present problem,

$$\left.\begin{aligned} &X_1 = X_2 = 0,\ X_3 = g, \\ \text{and}\quad &f_1 = f_2 = 0,\ f_3 = -P\delta(x + vt) \text{ on } z = 0 \end{aligned}\right\} \qquad \text{...(2.5)}$$

$\delta(x)$ being the Dirac delta function, and v the uniform velocity of the moving load.

For incompressible isotropic, homogeneous, elastic medium, we take the constitutive equations as

$$\left.\begin{aligned} &s_{ij} = 2\mu e_{ij} \text{ for } i \neq j,\ s_{11} - s = 2\mu e_{11},\ s_{22} - s = 2\mu e_{22} \\ &s_{33} - s = 2\mu e_{33},\ \text{and } e_{11} + e_{22} + e_{33} = 0 \end{aligned}\right\} \qquad \text{...(2.6)}$$

μ being the rigidity modulus, s the mean incremental stress. With $u = u(x, z, t)$, $v = 0$, $w = w(x, z, t)$ and using eqn. (2.1) and (2.4) to (2.6), eqns. (2.2) become

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial s'}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2},\ \mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial z^2}\right) + \frac{\partial s'}{\partial z} = \rho \frac{\partial^2 \omega}{\partial t^2} \qquad \text{...(2.7)}$$

where

$$s' = s + \rho g \omega. \qquad \text{...(2.8)}$$

The condition of incompressibility is

$$\frac{\partial u}{\partial x} + \frac{\partial \omega}{\partial z} = 0 \qquad \text{...(2.9)}$$

The boundary conditions (2.3) no $z = 0$ give

$$\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial x} = 0,\ 2\mu \frac{\partial \omega}{\partial z} + s' - \rho g \omega = -P\delta(x + vt). \qquad \text{...(2.10)}$$

To study the steady-state motion, we use the transformation $\xi = x + vt$ when equations (2.7), (2.9) and the boundary conditions (2.10) become

$$\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{\mu}\frac{\partial s'}{\partial \xi} = \frac{v^2}{\beta^2}\frac{\partial^2 u}{\partial \xi^2}, \left(\frac{\partial^2 \omega}{\partial \xi^2} + \frac{\partial^2 \omega}{\partial z^2}\right) + \frac{1}{\mu}\frac{\partial s'}{\partial z} = \frac{v^2}{\beta^2}\frac{\partial^2 \omega}{\partial \xi^2}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial \omega}{\partial z} = 0 \quad \text{where } \beta^2 = \frac{\mu}{\rho} \qquad \ldots(2.11)$$

and

$$\frac{\partial u}{\partial z} + \frac{\partial \omega}{\partial \xi} = 0,\ 2\mu\frac{\partial \omega}{\partial z} + s' - \rho g\omega = -P\delta(\xi) \text{ at } z = 0. \qquad \ldots(2.12)$$

3. Supersonic Case : $v > \beta$

Applying the Fourier transform

$$\bar{u}(p, z) = \int_{-\infty}^{\infty} u(\xi, z)\, e^{ip\xi}\, d\xi$$

with the condition that u, w, s' are zero at $\xi = -\infty, +\infty$ eqns. (2.11) and (2.12) become

$$\left.\begin{aligned} &(\beta^2 - v^2)(-p^2)\,\bar{u} + \beta^2\frac{d^2\bar{u}}{dz^2} - ip\frac{\bar{s}'}{\rho} = 0 \\ &(\beta^2 - v^2)(-p^2)\,\bar{\omega} + \beta^2\frac{d^2\bar{\omega}}{dz^2} + \frac{1}{\rho}\frac{d\bar{s}'}{dz} = 0,\ -ip\bar{u} + \frac{d\bar{\omega}}{dz} = 0 \end{aligned}\right\} \qquad \ldots(3.1)$$

and

$$\frac{d\bar{u}}{dz} - ip\bar{\omega} = 0,\ 2\mu\frac{d\bar{\omega}}{dz} + \bar{s}' - \rho g\bar{\omega} = -P \text{ at } z = 0. \qquad \ldots(3.2)$$

We can take the solution of (3.1) as

$$\left.\begin{aligned} &\bar{u} = A_1 e^{-|p|z} + A_2 e^{ipqz},\ \bar{\omega} = -\frac{ip}{|p|}A_1 e^{-|p|z} + \frac{A_2}{q}e^{ipqz} \\ &\bar{s}' = -i\rho v^2 p A_1 e^{-|p|z} \quad \text{with } q^2 = \frac{v^2}{\beta^2} - 1 > 0. \end{aligned}\right\} \qquad \ldots(3.3)$$

Now from (3.3) we also get

$$\left.\begin{aligned} &\overline{\frac{\partial u}{\partial \xi}} = -ip\bar{u},\ \overline{\frac{\partial \omega}{\partial \xi}} = -ip\bar{\omega},\ \overline{\frac{\partial u}{\partial z}} = -|p|A_1 e^{-|p|z} + ipqA_2 e^{ipqz} \\ &\overline{\frac{\partial \omega}{\partial z}} = -\overline{\frac{\partial u}{\partial \xi}}. \end{aligned}\right\} \qquad \ldots(3.4)$$

Substitution of (3.3) in (3.2) gives A_1, A_2. Putting these values of A_1, A_2 in (3.3), (3.4) and then on taking Fourier inverse transform we obtain the expression for u, w in terms of the integrals I_1, I_2, I_3 and also $\frac{\partial u}{\partial \xi}, \frac{\partial \omega}{\partial \xi}, \frac{\partial u}{\partial z}, \frac{\partial \omega}{\partial z}$, s' in terms of the integrals J_1, J_2, J_3 respectively, where

$$I_1 = \int_0^\infty \frac{e^{-pz}}{\alpha_1^2 + \beta_1^2} (\alpha_1 \sin p\xi - \beta_1 \cos p\xi)\, dp$$

$$I_2 = \int_0^\infty \frac{\alpha_1 \cos p(\xi - qz) + \beta_1 \sin p(\xi - qz)}{\alpha_1^2 + \beta_1^2}\, dp$$

$$I_3 = \int_0^\infty \frac{e^{-pz}}{\alpha_1^2 + \beta_1^2} (\alpha_1 \cos p\xi + \beta_1 \sin p\xi)\, dp \qquad \ldots(3.5)$$

$$J_1 = \int_0^\infty \frac{e^{-pz} p}{\alpha_1^2 + \beta_1^2} (\alpha_1 \cos p\xi + \beta_1 \sin p\xi)\, dp$$

$$J_2 = \int_{-\infty}^\infty \frac{ip(\alpha_1 + i\beta_1)}{\alpha_1^2 + \beta_1^2} e^{-ip(\xi - qz)}\, dp$$

$$J_3 = \int_0^\infty \frac{e^{-pz} p}{\alpha_1^2 + \beta_1^2} (\beta_1 \cos p\xi - \alpha_1 \sin p\xi)\, dp$$

where

$$\alpha_1 = a\,|p| + b, \quad \beta_1 = cp,\ a = \rho(2\beta^2 - v^2)(q^2 - 1),\ b = \rho g(q^2 + 1),\ c = 4\mu q. \qquad \ldots(3.6)$$

Assuming $b < |a|$, and $b < c$, which is valid for g small, we expand the integrands in powers of b, and retain only upto the first order terms and then we evaluate the resulting integrals in the sense of generalised functions, using the formulae : (Lavoine 1959)

$$\int_0^\infty \frac{e^{-\alpha y} \sin \alpha x}{\alpha^{n+1}}\, d\alpha = (-1)^n \frac{r^n}{n!}\left[\theta \cos n\theta + \left\{\log r - \psi(n+1)\right\} \sin n\theta\right]$$

$$\int_0^\infty \frac{e^{-\alpha y} \cos \alpha x}{\alpha^{n+1}}\, d\alpha = -(-1)^n \frac{r^n}{n!}\left[\cos n\theta \left\{\log r - \psi(n+1)\right\} - \theta \sin n\theta\right] \qquad \ldots(3.7)$$

where γ = Euler's constant, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \dfrac{x}{y}$,

for $|\arg \alpha| < \dfrac{\pi}{2}$, $\psi(n+1) = \sum_{k=1}^{n} \dfrac{1}{k} - \gamma$, $\quad \psi(1) = -\gamma, n > 0$.

Finally, we get the displacements and stresses as :

$$u = \frac{P(q^2-1)}{\pi}\Big[A\tan^{-1}\frac{\xi}{z} + C\,(\log\sqrt{\xi^2+z^2}+\gamma) + B\Big\{z\tan^{-1}\frac{\xi}{z}$$
$$+\xi\,(\log\sqrt{\xi^2+z^2}+\gamma)\Big\} + D\Big\{z\,(\log\sqrt{\xi^2+z^2}-1+\gamma) - \xi\tan^{-1}\frac{\xi}{z}\Big\}\Big]$$
$$+\frac{Pq}{\pi}\Big[-2A\,(\log|\xi-qz|+\gamma) + C\pi\Big\{2H(\xi-qz)-1\Big\}$$
$$+B\pi|\xi-qz| + 2D(\xi-qz)(\log|\xi-qz|-1+\gamma)\Big] \qquad \ldots(3.8)$$

$$\omega = \frac{P(q^2-1)}{\pi}\Big[-A\,(\log\sqrt{\xi^2+z^2}+\gamma) + C\tan^{-1}\frac{\xi}{z}$$
$$-B\Big\{(\log\sqrt{\xi^2+z^2}-1+\gamma) - \xi\tan^{-1}\frac{\xi}{z}\Big\} + D\Big\{z\tan^{-1}\frac{\xi}{z}$$
$$+\xi\,(\log\sqrt{\xi^2+z^2}-1+\gamma)\Big\}\Big] + \frac{P}{\pi}\Big[-2A\,(\log|\xi-qz|+\gamma)$$
$$+C\pi\Big\{2H(\xi-qz)-1\Big\} + B\pi|\xi-qz|$$
$$+2D(\xi-qz)(\log|\xi-qz|-1+\gamma)\Big],$$

$$s_{11} = \frac{P(q^2-1)(2\mu+\rho v^2)}{\pi}\Big[\frac{Az}{\xi^2+z^2} + \frac{C\xi}{\xi^2+z^2} + B\,(\log\sqrt{\xi^2+z^2}+\gamma)$$
$$-D\tan^{-1}\frac{\xi}{z}\Big] - \rho g\omega - \frac{2Pq\mu}{\pi}\Big[\frac{2A}{\xi-qz} - 2\pi C\delta(\xi-qz)$$
$$-B\pi\Big\{2H(\xi-qz)-1\Big\} - 2D\,(\log|\xi-qz|+k)\Big],$$

$$s_{13} = \frac{2\mu P(q^2-1)}{\pi}\Big[A\Big(\frac{1}{\xi-qz} - \frac{\xi}{\xi^2+z^2}\Big) + C\Big\{\frac{z}{\xi^2+z^2} - \pi\delta(\xi-qz)\Big\}$$
$$+B\Big\{\tan^{-1}\frac{\xi}{z} - \pi H(\xi-qz) + \frac{\pi}{2}\Big\}$$
$$+D\,(\log\sqrt{\xi^2+z^2} - \log|\xi-qz| + k)\Big],$$

$$s_{33} = \frac{P(q^2-1)(\rho v^2-2\mu)}{\pi}\Big[\frac{Az}{\xi^2+z^2} + \frac{C\xi}{\xi^2+z^2} + B\,(\log\sqrt{\xi^2+z^2}+\gamma)$$
$$-D\tan^{-1}\frac{\xi}{z}\Big] - \rho g\omega + \frac{2\mu qP}{\pi}\Big[\frac{2A}{\xi-qz} - 2\pi C\delta(\xi-qz)$$
$$-B\pi\Big\{2H(\xi-qz)-1\Big\} - 2D\,(\log|\xi-qz|+k)\Big],$$

(equation continued on p. 1337)

$$s_{22}=\frac{P\,(q^2-1)\,\rho v^2}{\pi}\left[\frac{Az+C\xi}{\xi^2+z^2}+B\,(\log\sqrt{\xi^2+z^2}+\gamma)-D\tan^{-1}\frac{\xi}{z}\right]-\rho g\omega,$$

$$s_{12}=s_{23}=0,$$

$$\text{where } A=\frac{a}{a^2+c^2},\ B=\frac{b\,(a^2-c^2)}{(a^2+c^2)^2},\ C=\frac{c}{a^2+c^2},\ D=\frac{2abc}{(a^2+c^2)^2}. \qquad \ldots(3.9)$$

4. Subsonic Case : $v < \beta$

The equations of motion (2.11) with the boundary conditions (2.12) have the solution in the form

$$u=\int_0^\infty (Le^{-pz}+Mq_1e^{-pq_1z})\sin p\xi dp,\ \omega=\int_0^\infty (Le^{-pz}+Me^{pq_1z})\cos p\xi dp$$

$$s'=\rho v^2\int_0^\infty L\,pe^{-pz}\cos p\xi dp \qquad \ldots(4.1)$$

$$\text{where } L=\frac{P\,(q_1^2+1)}{\pi\,(a_1p-b_1)},\ M=\frac{-2P}{\pi\,(a_1p-b_1)},\ q_1^2=1-v^2/\beta^2$$

$$a_1=\mu\,(1+q_1^2)^2-4\mu q,\ b_1=\rho g\,(1-q_1^2). \qquad \ldots(4.2)$$

Expanding L, M in powers of b_1 and retaining upto the first power of b_1 and then evaluating the integrals with the help of (3.7), we obtain finally

$$u=L_1\tan^{-1}\frac{\xi}{z}+L_2q_1\tan^{-1}\frac{\xi}{q_1z}-M_1\left[z\tan^{-1}\frac{\xi}{z}\right.$$

$$\left.+\xi\,(\log\sqrt{\xi^2+z^2}+\gamma-1)\right]-M_2q_1\left[q_1z\tan^{-1}\frac{\xi}{q_1z}\right.$$

$$\left.+\xi\,(\log\sqrt{\xi^2+q_1^2z^2}+\gamma-1)\right]$$

$$\omega=-L_1(\log\sqrt{\xi^2+z^2}+\gamma)-L_2(\log\sqrt{\xi^2+q_1^2z^2}+\gamma)$$

$$+M_1\left[z\,(\log\sqrt{\xi^2+z^2}+\gamma-1)-\xi\tan^{-1}\frac{\xi}{z}\right]$$

$$+M_2\left[q_1z\,(\log\sqrt{\xi^2+q_1^2z^2}-1+\gamma)-\xi\tan^{-1}\frac{\xi}{zq_1}\right] \qquad \ldots(4.3)$$

$$s_{11}=(\rho v^2+2\mu)\left[L_1\frac{z}{\xi^2+z^2}-M_1\,(\log\sqrt{\xi^2+z^2}+\gamma)\right]+2L_2\mu q_1^2\frac{z}{\xi^2+q_1^2z^2}$$

$$-2M_1\mu q_1\,(\log\sqrt{\xi^2+q_1^2z^2}+\gamma)-\rho g\omega$$

$$s_{22}=\rho v^2L_1\frac{z}{\xi^2+z^2}-\rho v^2M_1\,(\log\sqrt{\xi^2+z^2}+\gamma)-\rho g\omega$$

$$s_{33}=(\rho v^2-2\mu)\left[L_1\frac{z}{\xi^2+z^2}-M_1\,(\log\sqrt{\xi^2+z^2}+\gamma)\right]-2L_2\mu q_1^2\frac{z}{\xi^2+q_1^2z^2}$$

$$+2M_2\mu q_1\,(\log\sqrt{\xi^2+q_1^2z^2}+\gamma)-\rho g\omega$$

(equation continued on p. 1338)

$$s_{13} = -2\mu\left[\frac{L_1\xi}{\xi^2+z^2} + M_1 \tan^{-1}\frac{\xi}{z}\right] - \mu(1+q_1^2)\left[\frac{L_2\xi}{\xi^2+q_1^2z^2} + M_2 \tan^{-1}\frac{\xi}{q_1 z}\right]$$

$$s_{12} = s_{23} = 0$$

where $L_1 = \dfrac{P(1+q_1^2)}{\pi a_1}$, $M_1 = \dfrac{P(1+q_1^2)\,b_1}{\pi a_1^2}$, $L_2 = \dfrac{-2P}{\pi a_1}$, $M_2 = \dfrac{-2Pb_1}{\pi a_1^2}$ ...(4.4)

5. Discussion and Numerical Results

In (3.8), the terms containing B and D give the effect of gravity on the incremental displacements and stresses in the supersonic case, while the terms containing M_1 and M_2 in equations (4.3) give the effect on the incremental displacements and stresses in the subsonic case. The classical results obtained by Cole and Huth (1958) can be deduced by putting $g = 0$. The stresses in the classical case are product of z and functions of ξ/z. But when the effect of gravity is taken into account, this is seen not to be valid. Figures 1–4 give the displacement component u as a function of ξ in the subsonic cases at fixed z, with $v^2/\beta^2 = 0.8$, $z = 1$; $v^2/\beta^2 = 0.8$, $z = 2$; $v^2/\beta^2 = 0.2, z = 1$; $v^2/\beta^2 = 0.2$, $z = 2$; respectively. Figures 5 and 6 give the displacement component u in the supersonic cases with $v^2/\beta^2 = 1.05$ for $z = 1, 2$ respectively. The dotted lines in each corresponds to the case when the effect of gravity is neglected. In the subsonic case it is seen that the general effect of gravity is to reduce the magnitude of displacement, the amount of reduction increasing with the increase of g/β^2. Also the effect of gravity is more significant if v^2/β^2 is near to 1, rather than when v^2/β^2 is very small. In the supersonic case, however, it is seen that the effect of gravity is to increase the displacement for some values of ξ but the amount of increment decreases with g/β^2 increasing.

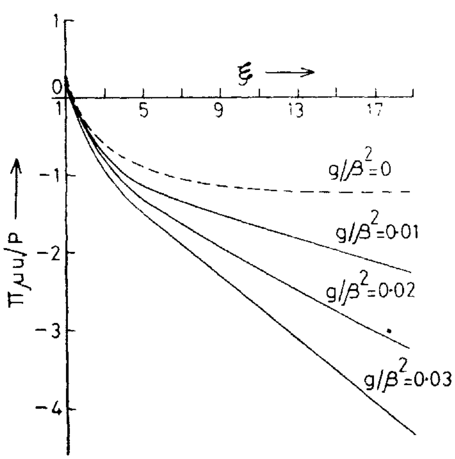

Fig. 1

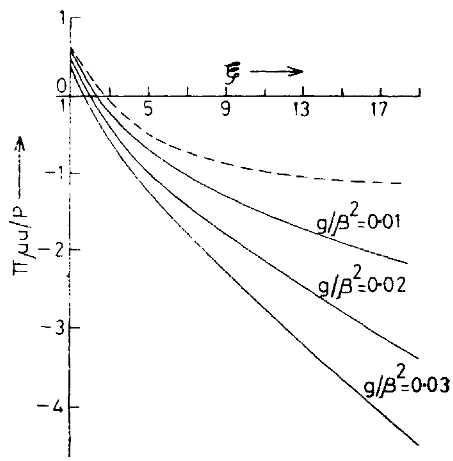

Fig. 2

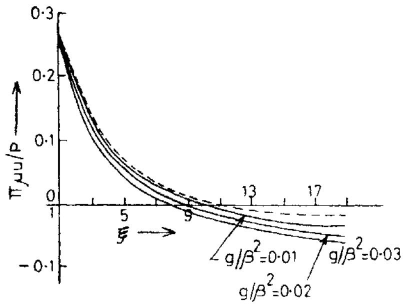

FIG. 3

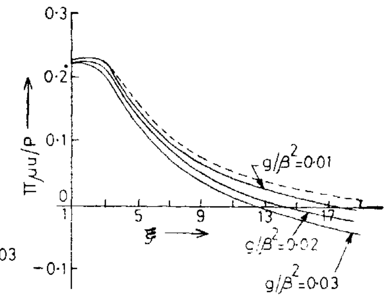

FIG. 4

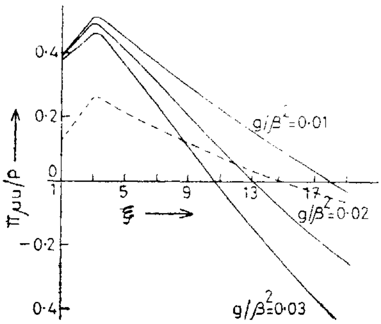

FIG. 5
[$\nu^2/\beta^2 = 1.05$, $z = 1$]

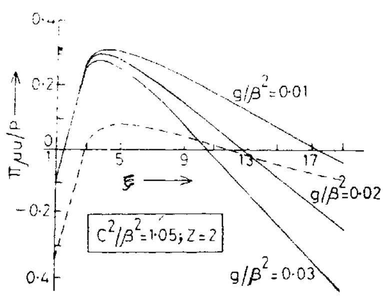

FIG. 6
[$\nu^2/\beta^2 = 1.05$, $z = 2$]

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