*Indian J. pure appl. Math.*, **13**(11) : 1354–1359, November 1982

# ON THE NÖRLUND SUMMABILITY OF CONJUGATE FOURIER SERIES

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(*Received* 5 *September* 1981)

In this note we improve the results on $\left(N, p_n^\alpha\right)$ summability of conjugate Fourier series by Prasad (1978).

## 1. DEFINITIONS AND NOTATIONS

Let $\Sigma a_n$ be a given infinite series with $\{S_n\}$ the sequence of partial sums, $\{p_n\}$ be a sequence of constants with $p_0 > 0$, and $p_n \geqq 0$ for $n > 0$, we define

$$p_n^\alpha = \sum_{k=0}^{n} \epsilon_{n-k}^{\alpha-1}\, p_k,\quad P_n^\alpha = \sum_{k=0}^{n} p_k^\alpha$$

where for $\alpha$ real $\epsilon_0^\alpha = 1$ and $\epsilon_n^\alpha = \dfrac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}$ $(n = 1, 2, \ldots)$.

The sequence to sequence transformation

$$t_n^\alpha = \frac{1}{P_n^\alpha} \sum_{k=0}^{n} p_{n-k}^\alpha\, S_k$$

defines the $\left(N, p_n^\alpha\right)$ mean of the sequence $\{S_n\}$. Further if $\lim_{n\to\infty} t_n = S$, the series $\Sigma a_n$ is said to be summable by $\left(N, p_n^\alpha\right)$ means. This method of summation is given by Cass (1969).

For $\alpha = 1$, this method reduces to $(N, p_n)$ method of summation, in addition to this, if $p_n = \dfrac{1}{n+1}$, this method reduces to harmonic summability. Also for $\alpha = 1$ and $p_n = \binom{n+\delta-1}{\delta-1}$, $\delta > 0$, this method reduces to $(C, \delta)$ summability.

Let $f(t)$ be a period function with period $2\pi$ and integrable $(L)$ over $(-\pi, \pi)$. The Fourier series associated with $f(t)$ is given by

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum A_n (x)$$

the derived Fourier series is

$$\sum_{n=1}^{\infty} n (b_n \cos nx - a_n \sin nx) = \sum nB_n (x)$$

and conjugate Fourier series is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum B_n (x)$$

where $a_n$ and $b_n$ are Fourier coefficients.

We will write $f(x+t) - f(x-t) - 2f'(x) = g(t)$,

$f(x+t) - f(x-t) = \psi(t)$,

$$N_n(t) = \frac{1}{2\pi P_n^\alpha} \sum_{k=0}^{n} p_k^\alpha \frac{\sin (n-k+\frac{1}{2}) t}{\sin t/2}$$

and

$$\overline{N}_n(t) = \frac{1}{2\pi P_n^\alpha} \sum_{k=0}^{n} p_k^\alpha \frac{\cos (n-k+\frac{1}{2}) t}{\sin t/2} .$$

## 2. Main Result

The basic results on summability of Fourier series are due to Iyengar (1943) and Siddiqi (1948) for harmonic summability, Pati (1961) and Singh (1964) on Nörlund summability. Similar results for conjugate Fourier series have been proved by these authors. Prasad (1978) proved the following two theorems on $(N, p_n^\alpha)$ summability of derived Fourier series and conjugate Fourier series.

*Theorem A*—The derived Fourier series $\Sigma nB_n(x)$ is summable by $(N, p_n^\alpha)$ means to the sum $f'(x)$ at the point $t = x$, at which

$$G(t) \equiv \int_0^t |d\, g(u)| = O\left(\frac{p_\tau}{P_\tau}\right) t \to 0\,, \quad \tau = [1/t]\,,$$

where $\{p_n\}$ is real is non-negative non-increasing sequence of constants such that $P_n \to \infty$, as $n \to \infty$.

*Theorem B*—The conjugate Fourier series $\Sigma B_n(x)$ is summable $(N, p_n^\alpha)$ to the sum $\frac{1}{2\pi} \int_0^\pi \psi(t) \cot t/2\, dt$ at $t = x$ at which this integral exists and

$$\Psi(t) \equiv \int_0^t |\psi(u)|\, du = o\left(\frac{p_\tau}{P_\tau}\right), \; t \to 0, \; \tau = [1/t]$$

where $\{p_n\}$ is the same sequence as in Theorem A.

Theorem A is generalisation of the result of Tripathi (1963), while Theorem B generalises the result of Singh (1964).

By proving that

$$\int_t^\delta \frac{|\phi(u)|}{u} P^\alpha_{[1/u]}\, du = o\left[P^\alpha_{(1/t)}\right], \; t \to 0$$

is a weaker condition than

$$\int_0^t |\phi(u)|\, du = o\left[\frac{p^\alpha_{(1/t)}}{P^\alpha_{(1/t)}}\right], \; t \to 0.$$

Pandey (1977) proved $(N, P^\alpha_n)$ summability of Fourier series.
Using this fact we prove the above two theorems under weaker conditions, in fact we prove the following

*Theorem I*—The derived Fourier series $\Sigma n B_n(x)$ is summable by $(N, p^\alpha_n)$ means to the sum $f'(x)$ at the point $t = x$, at which

$$\int_t^\delta \frac{|dg(u)|}{u} P^\alpha_{(1/u)} = o\left[P^\alpha_{(1/t)}\right] 0, \leqslant \delta \leqslant \pi, \; t \to 0$$

where $\{p_n\}$ is real nonnegative non-increasing sequence of coefficients such that $P_n \to \infty$, as $n \to \infty$.

*Theorem II*—The conjugate Fourier series $\Sigma\, B_n(x)$ is summable by $(N, p^\alpha_n)$ means to the sum $\frac{1}{2\pi} \int_0^\pi \psi(t) \cot t/2\, dt$ at $t = x$ at which this integral exists and

$$\int_t^\delta \frac{|\psi(u)|}{u} P^\alpha_{(1/u)}\, du = o\left[P^\alpha_{(1/t)}\right], \; 0 \leqslant \delta \leqslant \pi, \; t \to 0$$

where $\{p_n\}$ is the same sequence as defined in Theorem I.

## 3. Lemmas

For the proof of our theorems we require the following lemmas.

*Lemma* 1—If $\{p_n\}$ is a non-negative non-increasing sequence, then for $0 \leqslant a \leqslant b < \infty$, $0 < t < \pi$, we have

$$\left| \sum_{k=a}^{b} p_k^{\alpha} e^{i(n-k)t} \right| \leqslant k \left( P_{(1/t)}^{\alpha} \right).$$

Proof is on the lines of McFadden (1942).

*Lemma* 2—For the sequence $\{p_n\}$ satisfying the conditions of theorems,

$$N_n(t) = \frac{1}{2\pi P_n^{\alpha}} \sum_{k=0}^{n} p_k^{\alpha} \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2} = O\left[\frac{P_{(1/t)}^{\alpha}}{t P_n^{\alpha}}\right]$$

$$\overline{N}_n(t) = \frac{1}{2\pi P^{\alpha}} \sum_{k=0}^{n} p_k^{\alpha} \frac{\cos(n-k+\frac{1}{2})t}{\sin t/2} = O\left[\frac{P_{(1/t)}^{\alpha}}{t P_n^{\alpha}}\right]$$

and for $0 < T < \pi$ , $N(t) = O(n)$.

For proof see Pandey (1977) or Prasad (1978).

## 4. Proofs of the Theorems

*Proof of Theorem I*—If $S_n$ denotes the $n$th partial sum of derived Fourier series $\Sigma n B_n(x)$, then we have (Zygmund 1935)

$$S_n(x) - f'(x) = \frac{1}{2\pi} \int_0^{\pi} \frac{\sin(n+\frac{1}{2})t}{\sin t/2} \, dg(t).$$

Hence

$$t_n^{\alpha} = \frac{1}{P_n^{\alpha}} \sum_{k=0}^{n} p_k^{\alpha} \left\{ S_{n-k}(x) - f'(x) \right\}$$

$$= \frac{1}{P_n^{\alpha}} \sum_{k=0}^{n} p_k^{\alpha} \frac{1}{2\pi} \int_0^{\pi} \frac{\sin(n-k+\frac{1}{2})t}{\sin t/2} \, dg(t)$$

$$= \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right\} N_n(t) \, dg(t)$$

$$= I_1 + I_2 + I_3 + I_4, \text{ say.}$$

Now

$$I_1 = O\left[\int_0^{1/n} |N_n(t)| \, |dg(t)|\right]$$

$$= O(n) \int_0^{1/n} |dg(t)|$$

$$= O(n)\, o\left( p_n^{\alpha} / P_n^{\alpha} \right)$$

$$= o(1).$$

Next

$$I_2 = O\left( \int_{1/n}^{\delta} | N_n(t) | \; | dg(t) | \right)$$

$$= O\left( \frac{1}{P_n^{\alpha}} \int_{1/n}^{\delta} d\,g(t) \, . \; \frac{P_{(1/t)}^{\alpha}}{t} \right)$$

$$= o(1).$$

Finally, by Reimann-Lebesgue theorem and regularity of the method summation we have $I_3 = o(1)$.

Thus Theorem I is proved.

*Proof of Theorem II*—As in Theorem I we have

$$\bar{t}_n^{\alpha}(x) = \left\{ \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi} \right\} \psi(t)\, \bar{N}_n(t)\, dt = J_1 + J_2 + J_3 \text{, say.}$$

where

$$J_1 = O(n) \int_0^{1/n} | \psi(t) | \, dt$$

$$= o(1).$$

$$J_2 = O\left[ \int_{1/n}^{\delta} | \psi(t) | \; P_{(1/t)}^{\alpha} \cdot \frac{1}{t} \left( P_n^{\alpha} \right)^{-1} dt \right.$$

$$= o(1),$$

and $J_3 = o(1)$, as in $I_3$ in Theorem I.

Thus Theorem II is proved.

## Acknowledgement

The authors are thankful to Dr B. K. Beohar for the help during the preparation of this paper.

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