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ON OSCILLATIONS AND STABILITY OF THIRD ORDER DIFFERENTIAL EQUATIONS

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A general third order differential equation is considered. A necessary and sufficient conditions for its bounded solution to be oscillatory and the conditions under which the solution is non-oscillatory are given.

§1. This paper deals with autonomous nonlinear third order differential equation. This equation is a generalization of the well known Van der Pol and Lienard equations to third order in contrast to generalization of Levinson and Smith which remains within the framework of the second order.

The differential system considered in this paper has the form

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3]^T = [x_2 - f_1(x_1), x_3 - f_2(x_1), - f_3(x_1)]^T,$$

$$\text{i.e. } \dot{x} = F(x) \qquad (1)$$

which is equivalent to the single third order differential equation

$$\dddot{x} + \frac{d}{dt}(f'_1(x)\dot{x} + f_2(x)) + f_3(x) = 0. \qquad ..(2)$$

The functions f_1, f_2 and f_3 are real and depend only on x. The dots and dashes indicate differentiations with respect to t and x respectively.

It will be assumed that :

(i) f_1, f_2 and f_3 are in C^1 class for all $x \in (-\infty, \infty)$. and of such nature that the existence and uniqueness of solutions, as well as their continuous dependence on the the initial values is assured.

$$\text{(ii) } f_i(0) = 0, \quad i = 1, 2, 3. \qquad ..(3)$$

$$\text{(iii) } f'_1(0) < 0, \; f'_3(0) < 0, f'_1(0), \; f'_2(0) - f'_3(0) < 0. \qquad ..(4)$$

(iv) If the point $P(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, $x_1(t_0) \geqslant 0$ $(x_1(t_0) \leqslant 0)$ and the bounded solution $P(t)$ passes through $p(t_0)$, then it cannot satisfy $x_1(t) \geqslant 0$ $(x_1(t) \leqslant 0)$ for $t \in (t_0, \infty)$, where D is the exterior of the sphere $S:V(p) = a$ such that if $P(x_1, x_2, x_3) \in D$ then $x_1^2 + x_2^2 + x_3^2 \geqslant C^2$, where C is a positive constant.

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Abdel Karim and Gregus (1969), El-Owaidy *et al.* (1977), Harrow (1968), Mulholland (1971), Rauch (1950) and others have studied different properties of the solutions of system like (1).

Definition 1—A solution $P(t) = (x_1(t), x_2(t), x_3(t))$ is bounded if and only if:

$$|x_1(t)| < A_1,\ |x_2(t)| < A_2,\ |x_3(t)| < A_3$$

where A_i, $i = 1, 2, 3,$— are arbitrary positive constants.

The equilibrium state of the system (1) are points at which the following equations are simultaneously satisfied:

$$x_2 - f_1(x_1) = 0,\ x_3 - f_2(x_1) = 0,\ f_3(x_1) = 0.$$

Thus, system (1) has a unique equilibrium state (singular point), the coordinate origin.

In two-dimensional real autonomous equations where the solution $x(t)$ and its derivative (i.e. $\dot{x}(t)$) are bounded, the general theory of Bendixon—Poincare [Coddington and Levinson (1955, Chap. 16)] asserts that this solution must spiral asymptotically toward a periodic solution of the equation. The solution $x(t)$ is oscillatory since every periodic solution contains critical points and therefore $x(t)$ has an infinite number of zeros.

The associated variational system of (1) corresponding to the solution at the origin has the form

$$y' = F_x(0)y$$

$$\text{i.e} \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -f_1'(0) & 1 & 0 \\ -f_2'(0) & 0 & 1 \\ -f_3'(0) & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \quad \text{...(5)}$$

The characteristic equation corresponding to the matrix $F_x(0)$ is given by

$$r^3 + f_1'(0)\, r^2 + f_2'(0) r + f_3'(0) = 0. \quad \text{...(6)}$$

The conditions of Hurwitz criterion for the real parts of all roots of (6) to be positive are those given by (4) and thus the origin is negatively asymptotic stable.

It is clear that the nonlinear system (1) can be linearized and in virtue of (5) can be written as

$$\dot{x}_1(t) = x_2(t) - f_1'(0) + 0(|x_1|),$$

$$\dot{x}_2(t) = x_3(t) - f_2'(0) + 0(|x_1|),\ \dot{x}_3 = -f_3'(0)x_1(t) + 0(|x_1|). \quad \text{...(7)}$$

According to Lyapunov theorems (Cesari 1963), if there exists a function V which is positive quadratic form in x_1, x_2, x_3 and its derivative with respect to time, owing to system (7), i. e.

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \frac{\partial V}{\partial x_3}\dot{x}_3 \qquad \ldots(8)$$

is always negative (positive), then the system (7) or (2) is asymptotically (negative asymptotically) stable, substituting the corresponding values for $\dot{x}_1$, $\dot{x}_2$ and $\dot{x}_3$ from (7) and after simple calculations we obtain

$$\frac{dV}{dt} \geqslant b\,(x_1^2 + x_2^2 + x_3^2) - 0\,(x_1^2 + x_2^2 + x_3^2) \geqslant 0$$

where b is an arbitrary positive constant and $x_1^2 + x_2^2 + x_3^2 \leqslant a^2$, and a is sufficiently small constant.

Theorem 1—Under the assumptions stated before the first component $x_1(t)$ of the solution $P(t)$ of (1) is oscillatory.

PROOF : We divide the domain D into two subdomains by the plane $x = 0$, namely:

(i) D_1: consists of the points $(0, x_2, x_3)$ such that

$$(0, x_2, x_3) \in D_1,\ x_2 > 0, \qquad \ldots(9)$$

(ii) D_2: consists of the points $(0_1, x_2, x_3)$ such that

$$(0_1, x_2, x_3) \in D_2,\ x_2 < 0. \qquad (10)$$

If the point $P(t_0) \in \bar{D}_1$ (the closure of D_1) and $x_2(t_0) \neq 0$ then from (1) $\dot{x}_1(t) = x_2(t_0) > 0$. Consequently, $x_1(t) > 0$ (<0) for t immediately to the right (left) of t_0. Similarly if $P(t_0) \in \bar{D}_2$ and $x_2(t_0) \neq 0$ then $x_1(t) < 0$ (>0) for t immediately to the right (left) of t_0. Also if $p(t_0)$ - $(0, 0, x_3(t_0))$, $x_3(t_0) > 0$, then from (2) we have

$$\dot{x}_1(t_0) = 0 \text{ and } \dot{x}_3(t_0) > 0.$$

Thus $x(t)$ has a minimum at $t = t_0$, so that $x(t) > 0$ for t in a neighbourhood of t_0. In the same way if $x_3(t_0) < 0$ we can show that $x(t)$ has a maximum point at $t = t_0$ so that $x(t) < 0$ for t in a neighbourhood of t_0.

Since the solution $p(t)$ is strongly bounded then, from condition (iv), it follows that there exists a sequence of points $\{\bar{t}_n\} \to \infty$, such that $x_1(\bar{t}_n) = 0$ and $\dot{x}_1(\bar{t}_n) \neq 0$; i.e. $p(\bar{t}_n) \in \bar{D}_1 \, U \, \bar{D}_2$, $x_2(\bar{t}_n) \neq 0$.

Thus $x_1(t)$ is oscillatory and this completes the proof.

§2. In this section we shall study the case in which the solution is non-oscillatory and we need the following definition.

Definition 2 –The solution $x(t)$ of (2) is called non-oscillatory on (d, ∞), if it has at most one double zero point or at most two simple zeros.

Theorem 2—If the functions f_1, f_2 and f_3 are continuous for $x \in (-\infty, \infty)$,

$$\int_a^t [f_2(x) - (t-x)\, f_3(x)]dx < 0,\ f_1(x) < 0 \qquad \ldots(11)$$

$$x(a) = \dot{x}(a) = 0, \ \ddot{x}(a) = 1 \qquad \ldots(12)$$

then the solution $x(t)$ of (2) is non-oscillatory.

PROOF : According to Theorem 1, (cf. Karim and Gregus 1969), it is enough to prove that $x_1(t)$ which satisfies the condition (12) has no zero on the right of a. Then from (3) we have

$$\ddot{x} + (f_1\dot{x}) + f_2 + \int_a^t f_3(x)\, dx = 1 + f_2(a). \qquad \ldots(13)$$

Integrating once more and changing the order of integration

$$x + f_1 + \int_a^t [f_2(u) + (t-u) f_3(u)]\, du$$

$$- (1 + f_2(0))(t-a) - f_1(0) = 0. \qquad \ldots(14)$$

Suppose the converse is true, i.e. $t_1 > a$ is the first zero point of x on the right of a. Then

$$x(t_1) = 0, \ \dot{x}(t_1) < 0. \qquad \ldots(15)$$

Then from (14) and (15) we obtain a contradiction. This completes the proof.

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