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TORSION OF A COMPOSITE, VISCOELASTIC PRISMATIC BAR OF TRIANGULAR CROSS-SECTION

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The generalization of Ilyushin's approximation method is used to solve the problem of torsion of a composite, layered viscoelastic prismatic bar of triangular cross-section. Numerical computations are carried out in a special case.

INTRODUCTION

Consider a composite-material consisting of two isotropic components: an elastic component with Young's modulus E and Poisson's ratio ν and a viscoelastic component characterized by the modulus of bulk K which is assumed to be constant and the dimensionless parameter ω related to the corresponding Poisson's ratio by the formula

$$\omega = (1 - 2\nu)/(1 + \nu).$$

It is known that the problem for such an inhomogeneous, isotropic medium can be reduced to that for a homogeneous, but anisotropic medium by the method of effective moduli (Pobedrya 1976). The solution of this new problem may then be obtained by making use of the generalization of Ilyushin's approximation (Ilyushin's and Pobedrya 1970, Allam and Pobedrya 1978). Then idea of this generalization may be summarised in the following : Let a solution of the corresponding elastic problem for the anisotropic medium be known and suppose an expression of the form $f(.)\,Q$ appears in this solution, where Q is a known quantity and $f(.)$ denotes a function of the moduli of elasticity. Substitution for these moduli in terms of E, ν, K and ω gives a function $f = f(\omega)$ which is then approximated by a expression of the form:

$$f(\omega) = \Sigma\, c_\alpha\, \psi_\alpha\, (\omega\, ((t)) = \Sigma\, c_\alpha\, \psi_\alpha\, (t)\ ;$$

here, each function ψ may be a constant or one of the kernels $\omega(t)$, $\pi(t)$ or $g_\beta(t)$, where $\pi = 1/\omega$ and $g_\beta(t) = 1/(1 + \beta\omega)$. The constants c_α have to be determined, for example, by the method of least squares. The viscoelastic solution may now be determed by recording the expression $f(.)\,Q$ in the form

$$c_\alpha \int_0^t \psi_\alpha\, (t - \tau)\, dQ\, (\tau).$$

FORMULATION OF THE PROBLEM

A prismatic bar with cross-section in the form of an isoceles triangle of base $2d$ and height h is subjected to a torsional moment $M(t)$ acting on one of its bases, the other (Fig. 1) base being fixed. The bar is made of a viscoelastic material reinforced by elastic layers parallel to the base of the cross-section.

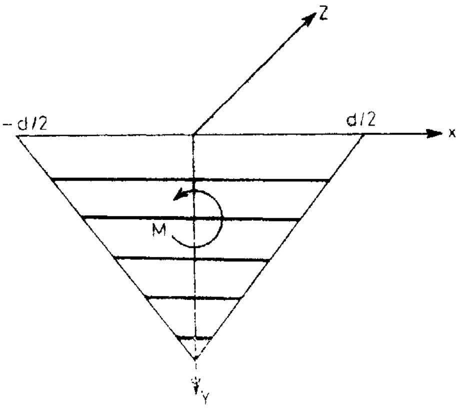

FIG. 1

Choose a system of orthogonal cartesian coordinates xyz with z-axis directed along the bar and xy-plane coinciding with the free base of the bar, as shown in the figure. The elastic solution for stresses of the corresponding problem for the orthotropic case is (Lekhnitskyi 1971)

$$\tau_{xz} = \frac{80\,M}{d^3}\left[\frac{y^2}{d^2}g^2 + \frac{1}{4}\left(1 - \frac{4x^2}{d^2}\right)g - \frac{2y}{\sqrt{3}d}\,g^{3/2}\right] \quad \text{...(1)}$$

and

$$\tau_{yz} = \frac{80\,M}{d^3}\,\frac{xy}{d^2}\,g$$

where

$$g = G_{13}/G_{23} \quad \text{...(2)}$$

and G_{13}, G_{23} are the shear moduli for the xz- and yz-planes respectively.

In order to transform to the viscoelastic case under consideration, one should use the effective moduli to replace G_{13} and G_{23} by proper viscoelastic operators which, in this case, reduce to the single operator of relaxation ω :

$$G_{13} = \frac{\gamma E}{2\,(1+\nu)} + \frac{3}{2}\,(1-\gamma)\,K\omega$$

and

$$G_{23} = \frac{3\,KE\omega}{2\,[\,(1-\gamma)\,E + 3\gamma\,(1+\nu)\,K\omega]} \quad \text{...(3)}$$

where $\gamma = V_e/V$ is the volume ratio of the elastic reinforcing material to the whole volume. Substituting from (3) into (2), the following formula is obtained for g :

$$g = A_1 + A_2\,\omega + (A_3/\omega) \quad \text{...(4)}$$

where

$$A_1 = 1 - 2m, \ A_2 = 3m (1 + \nu) M, \ A_3 = m/3 (1 + \nu) M$$

and

$$m = \gamma (1 - \gamma), \quad M = K/E.$$

To proceed further, it is desirable to approximate function $g^{3/2}$ appearing in the r.h.s. of eqn. (1) in a suitable way. According to the above-mentioned scheme, let

$$\omega g^{3/2} = C_1 + C_2 \omega + C_3 \omega^2 + C_4 (2 + \omega)^{-1} \qquad \text{...(5)}$$

and coefficients C_i, $i = 1, \ldots, 4$ are to be determined so as to realize an extremum for the integral

$$I = \int_0^1 [g^{3/2} - (C_1 + C_2 \omega + C_3 \omega^2 + C_4 (w + \omega)^{-1})]^2 \, d\omega .$$

The conditions for the extremum

$$\frac{\partial I}{\partial C_i} = 0, \quad i = 1, \ldots, 4 \qquad \text{...(6)}$$

reduce to a system of four linear, inhomogeneous algebraic equations in the unknowns C_i, the free terms being integrals involving $g^{3/2}$.

Substituting from (4) and (5) into (1) and collecting similar terms, one fiinally arrives at the following formulae :

$$\tau_{xz} = \frac{80}{d^3} \left(\frac{B_1}{\omega^2} + \frac{B_2}{\omega} + \frac{B_3}{2 + \omega} + B_4 + B_5 \omega + B_6 \omega^2 \right) M \qquad \text{...(7)}$$

and

$$\tau_{yz} = \frac{80}{d^3} \eta\zeta \left(A_1 + A_2 \omega + \frac{A_3}{\omega} \right) M$$

where the following dimensionless parameters were introduced

$$\eta = 2x/d, \quad \zeta = y/h, \quad r = h/d$$

and

$$B_1 = A_3^2, \ B_2 = 2A_1 A_3 r^2 \zeta^2 + \tfrac{1}{4} A_3 (1 - \eta^2) - \frac{1}{\sqrt{3}} (2C_1 + C_4) r\zeta$$

$$B_3 = \frac{C_4}{\sqrt{3}} r\zeta, \ B_4 = \left(A_1^2 + 2A_2 A_3 \right) r^2 \zeta^2 + \tfrac{1}{4} A_1 (1 - \eta^2) - \frac{2C_2}{\sqrt{3}} r\zeta$$

$$B_5 = 2A_1 A_2 + \tfrac{1}{4} A_2 (1 - \eta^2) - \frac{2C_3}{\sqrt{3}} r\zeta \ B_6 = A_2^2 .$$

In what follows, the torsional moment is taken in the form

$$\omega M(t) = M_0 H(t) \qquad \text{...(8)}$$

where $H(t)$, is the Heaviside step function; also let

$$\omega(t) = a + b\, e^{-\alpha t} = a + b\, e^{-\tau} \qquad \text{...(9)}$$

where τ is the dimensionless time. Constants a and b have to be determined from experiment.

At this stage, the quantities between brackets in the r.h.s. of eqns. (7) should be considered as operators of the form $f(\omega)$ acting on the known function $M(t)$. As is well known, the operator $\tau = 1/\omega$ has to be calculated by making use of the formula

$$\tau^* \omega^* = 1$$

where (*) denotes the Laplace-Karson transform. All other operators may be calculated by the same way. The details of calculations will be omitted, since they can be found in Allam and Pobedrya (1978).

After the recoding has been carried out as explained in the introduction, the following formulae are finally obtained for the viscoelastic stresses :

$$\tau_{xz}(\eta, \zeta, \tau) = \frac{80 M_0}{d^3} (B_1 F_1 + B_2 F_2 + B_3 F_3 + B_4 F_4 + B_5 F_5 + B_6 F_6)$$

and

$$\tau_{yz}(\eta, \zeta, \tau) = \frac{80 M_0}{d^3} r\eta\zeta (A_1 + A_2 F_5 + A_3 F_2)$$

where

$$F_1 = \frac{b}{a(a+b)} \left[\frac{1}{a} - \left(\frac{1}{a} + \frac{b\tau}{(a+b)^2} \right) e^{-a\tau/(a+b)} \right]$$

$$F_2 = \frac{1}{a} - \frac{b}{a(a+b)} e^{-a\tau/(a+b)}$$

$$F_3 = \frac{1}{2+a} - \frac{2b}{(2+a)(2+a+b)} e^{8(2+a)\tau/(2+a+b)}$$

$$f_4 = 1,\ f_5 = a + be^{-\tau},\ F_6 = -b\,[a - (a - b\tau)\, e^{-\tau}.$$

EXAMPLE

Numerical calculations were carried out for the following values of the parameters :

$$\gamma = 0.1, \quad \nu = 0.3, \quad M = 0.1, \quad r = 0.5.$$

The values of coefficients a and b were taken following Allam and Pobedrya (1978).

$$a = 0.01 \text{ and } b = 0.99.$$

For this concrete case, it was found that

$$C_1 = -0.8686558 \times 10^1, \qquad C_2 = 0.3584471 \times 10^1,$$

$$C_3 = -0.1198764 \times 10^{-2}, \qquad C_4 = 0.1896850 \times 10^2,$$

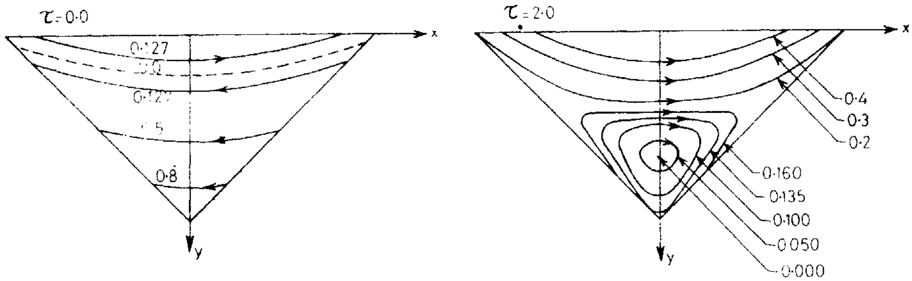

FIG. 2. Shear lines.

Figure 2 shows the lines of equal values of the total dimensionless shear.

$$\tau_s = \frac{d^3}{80\, M_0} \left(\tau_{xz}^2 + \tau_{yz}^2 \right)^{1/2}$$

For $\tau = 0.0$, the maximal value of τ_s is attained at the vertex of the cross-section, while at $\tau = 2.0$, it is attained at the origin. Lines zero values of τ_s are noticed on the figure.

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