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# ON SOME NEW GENERATING FUNCTIONS

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In this paper some new generating functions for the Gottlieb, Meixner, Cesàro and generalized Sylvester polynomials are obtained.

## 1. INTRODUCTION

Let the sequence of functions $\{S_n(x) \mid n = 0, 1, 2, \ldots\}$ be generated by (Singhal and Srivastava 1972, p. 755, eqn. (1)] :

$$\sum_{n=0}^{\infty} A_{m,n}\, S_{m+n}(x)\, t^n = \frac{f(x, t)}{[g(x, t)]^m}\, S_m(h(x, t)) \qquad \ldots(1.1)$$

where $m$ is a non-negative integer, the $A_{m,n}$ are arbitrary constants, and $f$, $g$, $h$ are suitable functions of $x$ and $t$. The importance of a generating function of the form (1.1) in obtaining the bilateral and trilateral generating relations for the functions $S_n(x)$ was realised by several authors. For instance, using the generating functions of the type (1.1) for Hermite, Laguerre and Gegenbauer polynomials, Rainville [1960, p. 197, eqn. (1); p. 211, eqn. (9); p. 280, eqn. (23)] derived some bilateral and bilinear generating functions due to Mehler [Rainville 1960, p. 198, eqn. (2)], Brafman (Rainville 1960, pp. 198, 213), (see Brafman 1957 also), Hardy-Hille (Rainville 1960, p. 212, Theorem 69) and Meixner [Rainville 1960, p. 281 eqn. (24)]. Making use of their definition (1.1), Singhal and Srivastava (1972) proved the following general result on bilateral generating functions :

$$f(x, t)\, F\left[h(x, t), \frac{yt}{g(x, t)}\right] = \sum_{n=0}^{\infty} S_n(x)\, \sigma_n(y)\, t^n \qquad \ldots(1.2)$$

where

$$F[x, t] = \sum_{n=0}^{\infty} a_n\, S_n(x)\, t^n \qquad \ldots(1.3)$$

the $a_n \neq 0$ are arbitrary constants and $\sigma_n(y)$ is a polynomial of degree $n$ in $y$ defined by

$$\sigma_n(y) = \sum_{k=0}^{n} a_k \, A_{k,n-k} \, y^k. \qquad \ldots(1.4)$$

Later, by using the same formula (1.1), Srivastava and Lavoie (1975) generalized the formula (1.2) to the following form :

$$\sum_{n=0}^{\infty} S_n(x) \, \sigma_n^q(y) \, t^n = f(x, t) \, F_q\left[h(x, t), y \left\{\frac{t}{g(x, t)}\right\}^q\right] \qquad \ldots(1.5)$$

where

$$F_q[x, t] = \sum_{n=0}^{\infty} a_n \, S_{qn}(x) \, t^n$$

$q$ is an arbitrary positive integer, the $a_n \neq 0$ are arbitrary constants, and $\sigma_n^q(y)$ is a polynomial of degree $[n/q]$ in $y$ defined by

$$\sigma_n^q(y) = \sum_{k=0}^{[n/q]} a_k \, A_{qk,\, n-qk} \, y^k. \qquad \ldots(1.6)$$

For a further generalization of the bilateral generating function (1.5), see Srivastava and Lavoie [1975, p. 319, eqn. (107)].

Chatterjea (1977) used the Singhal-Srivastava definition (1.1) in a different direction. He gave the following extension of (1.2) :

$$\sum_{n=0}^{\infty} S_n(x) \, \sigma_n(y, z) \, t^n = f(x, t) \, F\left[h(x, t), y, \frac{zt}{g(x, t)}\right] \qquad \ldots(1.7)$$

where

$$F[x, y, t] = \sum_{n=0}^{\infty} a_n \, S_n(x) \, g_n(y) \, t^n \qquad \ldots(1.8)$$

is a bilateral generating relation, $a_n \neq 0$ are arbitrary constants and $g_n(y)$ are arbitrary classical polynomials or functions and

$$\sigma_n(y, z) = \sum_{k=0}^{\infty} a_k \, A_{k,\, n-k} \, g_k(y) \, z^k. \qquad \ldots(1.9)$$

Recently, making use of the Singhal-Srivastava definition (1.1), we have given the following generalization of the Chatterjea's formula (1.7) [Agarwal and Manocha 1980, p. 190, Theorem 2]

$$\sum_{n=0}^{\infty} S_n(x)\, \sigma_n^{\mu, r}(y, z)\, t^n = f(x, t)\, F_{\mu, r}\left[h(x, t), y, z\left(\frac{t}{g(x, t)}\right)^r\right] \quad \ldots(1.10)$$

where

$$F_{\mu,r}[x, y, t] = \sum_{n=0}^{\infty} a_{n, \mu}\, S_{rn}(x)\, g_{n+\mu}(y)\, t^n \quad \ldots(1.11)$$

$$\sigma_n^{\mu, r}(y, z) = \sum_{k=0}^{[n/r]} a_{k, \mu}\, A_{k, n-kr}\, g_{k+\mu}(y)\, z^k. \quad \ldots(1.12)$$

Here, as well as in what follows, $r$ is an arbitrary positive integer, $\mu$ is an arbitrary complex number, $a_{n, \mu} \neq 0$ are arbitrary constants and $g_\mu(y)$ are arbitrary functions of order $\mu$.

The bilateral generating function (1.10) is contained, as an obvious special case, in a substantially more general result due to Srivastava [1980, p. 224, eqn. (1.10)]. The object of this paper is to derive trilateral generating functions for the Gottlieb, Meixner, Cesàro and generalized Sylvester polynomials. These cases have not been considered so far (see Singhal and Srivastava 1972, Srivastava and Lavoie 1975, Chatterjea 1977, Agarwal and Manocha 1980) perhaps because of the non-availability of the generating functions of the type (1.1) for these polynomials. In the present paper we first obtain generating functions of the type (1.1) for these polynomials and then follow the method of the proof of the aforementioned general formulas to obtain the trilateral generating functions.

## 2. Gottlieb Polynomials

Gottlieb polynomials are defined by the generating relation (1938, p. 455)

$$(1-t)^x (1-te^{-\lambda})^{-x-1} = \sum_{n=0}^{\infty} \varphi_n(x; \lambda)\, t^n. \quad \ldots(2.1)$$

Replacing $t$ by $t+u$ in (2.1), we get

$$(1-t)^x (1-te^{-\lambda})^{-x-1} \left(1 - \frac{u}{1-t}\right)^x \left(1 - \frac{u\, e^{-\alpha}}{1-t}\right)^{-x-1}$$

where, for convenience, $\alpha = \log_e\left(\frac{e^\lambda - t}{1-t}\right)$

$$= \sum_{n=0}^{\infty} \varphi_n(x; \lambda)\, (t+u)^n$$

(*equation continued on p.* 1372)

$$= \sum_{n,k=0}^{\infty} \binom{n+k}{k} \varphi_{n+k}(x; \lambda)\, t^n u^k$$

or

$$(1-t)^x (1-te^{-\lambda})^{-x-1} \sum_{k=0}^{\infty} \varphi_k(x; \alpha) \left(\frac{u}{1-t}\right)^k$$

$$= \sum_{n,k=0}^{\infty} \binom{n+k}{k} \varphi_{n+k}(x; \lambda)\, t^n u^k.$$

Equating the coefficient of $u^k$, we obtain the following formula of the type (1.1) for the Gottlieb polynomials :

$$\sum_{n=0}^{\infty} \binom{n+k}{k} \varphi_{n+k}(x; \lambda)\, t^n$$

$$= (1-t)^{x-k} (1-te^{-\lambda})^{-x-1} \varphi_k \left( x; \log_e \left(\frac{e^{\lambda}-t}{1-t}\right)\right). \quad \text{.. (2.2)}$$

Following Agarwal and Manocha (1980) and Srivastava (1980) we obtain the desired trilateral generating function for the Gottlieb polynomials, given by ;

*Theorem* 2.1 — Let

$$G_{r,\mu}[x, \lambda, y, t] = \sum_{n=0}^{\infty} a_{n,\mu}\, \varphi_{nr}(x; \lambda)\, g_{n+\mu}(t)\, t^n \quad \text{...(2.3)}$$

be a bilateral generating function. Then the following trilateral generating relation holds :

$$\sum_{n=0}^{\infty} \varphi_n(x; \lambda)\, \Omega_n^{\mu,r}(y, z)\, t^n = (1-t)^{-x} (1-te^{-\lambda})^{-x-1}$$

$$G_{r,\mu}\left[x, \log_e\left(\frac{e^{\lambda}-t}{1-t}\right), y, z\left(\frac{t}{1-t}\right)^r\right], \quad \text{...(2.4)}$$

where, as well as throughout this paper,

$$\Omega_n^{\mu,r}(y, z) = \sum_{k=0}^{[n/r]} \binom{n}{rk} a_{k,\mu}\, g_{k+\mu}(y)\, z^k. \quad \text{...(2.5)}$$

## 3. Meixner Polynomials

The Meixner polynomials $m_n(x; \beta, c)$ are defined by the generating relation [Erdélyi *et al.* 1953, p. 225, eqn. (13)]

$$\sum_{n=0}^{\infty} m_n(x;\beta,c)\frac{t^n}{n!}=\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\beta},\ |t|<\min(1,|c|). \quad ...(3.1)$$

Applying the method of derivation of (2.2), we obtain the following formula of the type (1.1) for the Meixner polynomials :

$$\sum_{x=0}^{\infty}\frac{m_{n+k}(x;\beta,t)\,t^n}{n!}=\left(1-\frac{t}{c}\right)^x(1-t)^{-x-\beta-k}\,m_k(x;\beta,(c-t)(1-t)^{-1})) \quad ...(3.2)$$

which leads us, as before, to

*Theorem* 3.1 — Let

$$H_{r,\mu}[x,\beta,c,y,t]=\sum_{n=0}^{\infty}a_{n,\mu}\,m_{rn}(x;\beta,c).\,g_{n+\mu}(y)\frac{t^n}{(rn)!} \quad ...(3.3)$$

be a bilateral generating function. Then the following trilateral generating relation holds :

$$\sum_{n=0}^{\infty}m_n(x;\beta,c)\,\Omega^{r,\mu}(y,z)\frac{t^n}{n!}$$

$$=\left(1-\frac{t}{c}\right)^x(1-t)^{x-\beta}H_{r,\mu}\left[x,\beta,(c-t)(1-t)^{-1},y,z\left(\frac{t}{1-t}\right)^r\right]. \quad ...(3.4)$$

## 4. Cesàro Polynomials

Cesàro polynomial $g_n^{(m)}(x)$ (which, in fact, is the $m$th mean of the first $n$ partial sums of the series $1+x+x^2+...$) is defined by [Basu 1976, p. 1107, eqn. (1.3)]

$$\sum_{n=0}^{\infty}g_n^{(m)}(x)\,t^n=(1-t)^{-m-1}(1-xt)^{-1}. \quad ...(4.1)$$

From (4.1) it is easy to obtain the following formula of the type (1.1) for the Cesàro polynomials :

$$\sum_{n=0}^{\infty}\binom{n+k}{k}g_{n+k}^{(m)}(x)\,t^n=(1-t)^{-m-1-k}(1-xt)^{-1}g_k^{(m)}\left(\frac{x(1-t)}{1-xt}\right) \quad ...(4.2)$$

we have thus got the formula required to derive the following trilateral generating function for Cesàro polynomials :

*Theorem* 4.1 — Let

$$Y_{r,\mu}[x,y,t]=\sum_{n=0}^{\infty}a_{n,\mu}\,g_{rn}^{(m)}(x)\,g_{n+\mu}(y)\,t^n \quad ...(4.3)$$

be a bilateral generating function. Then the following trilateral generating relation holds :

$$\sum_{n=0}^{\infty} g_n^{(m)}(x)\, \Omega_n^{\mu,r}(y, z)\, t^n$$

$$= (1-t)^{-m-1}(1-x\,t)^{-1} Y_{r,\mu}\left[\frac{x(1-t)}{1-x\,t}, y, z\left(\frac{t}{1-t}\right)^r\right]. \qquad ...(4.4)$$

## 5. Generalized Sylvester Polynomials

Define the polynomial $f_n(x; a)$, where a $\neq 0$ is an arbitrary constant, by means of the generating relation

$$\sum_{n=0}^{\infty} f_n(x; a)\, t^n = (1-t)^{-x}\, e^{axt}. \qquad ...(5.1)$$

We call the polynomials $f_n(x; a)$ generalized Sylvester polynomials in view of the relation

$$f_n(x; 1) = \varphi_n(x) \qquad ...(5.2)$$

where $\varphi_n(x)$ is the Sylvester polynomial (see Rainville 1960, p. 302). It can be easily verified that

$$f_n(x; a) = \frac{(ax)^n}{n!}\, {}_2F_0\left[-n, x; ...; -\frac{1}{ax}\right] \qquad ...(5.3)$$

and

$$(1-axt)^{-c}\, {}_2F_0\left[c, x; ...; \frac{t}{1-axt}\right] \cong \sum_{n=0}^{\infty} (c)_n f_n(x; a)\, t^n. \qquad ...(5.4)$$

Starting, as usual, from (5.1) we get the following formula of the type (1.1) for the polynomials $f_n(x; a)$ :

$$\sum_{n=0}^{\infty} \binom{n+k}{k} f_{n+k}(x; a)\, t^n = (1-t)^{-x-k}\, e^{axt} f_k(x; a(1-t)) \qquad ...(5.5)$$

which provides us with the basic tool to deduce the following theorem on trilateral generating functions for the polynomials $f_n(x; a)$ :

*Theorem* 5.1 — Let

$$Z_{r,\mu}[x, a, y, t] = \sum_{n=0}^{\infty} a_{n,\mu}\, f_{rm}(x; a) g_{n+\mu}(y)\, t^n \qquad ...(5.6)$$

be a bilateral generating function. Then the following trilateral generating relation holds :

$$\sum_{n=0}^{\infty} f_n(x; a)\, \Omega_n^{r,\mu}(y, z)\, t^n = (1-t)^{-x}\, e^{axt}\, Z_{r,\mu}\left[x; a(1-t), y, z\left(\frac{t}{1-t}\right)^r\right]. \qquad \ldots(5.7)$$


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