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OPERATOR EQUATIONS $AB + BA^* = A^*B + BA = I$

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Let A and B be bounded linear operators on a Hilbert space H satisfying the equations of the title. The problem considered here is that of finding (i) sufficient conditions for the solution A, assuming that such a solution exists, to be normal; (ii) necessary and sufficient conditions for the existence of a solution; and (iii) conditions for the solution to be unique.

1. INTRODUCTION

Let A and B be bounded linear operators on a Hilbert space H satisfying the operator equations

$$AB + BA^* = I = A^*B + BA. \qquad \text{.. (1)}$$

The problem of finding sufficient conditions for the solution A to (1), assuming such a solution exists, to be normal has been considered by a number of authors, amongst them Kamei and Kato (1978), and Duggal and Khalagai (1981). Duggal and Khalagai have shown that if either $0 \notin W(B)$ (= the numerical range of B) or $\sigma(B) \cap \sigma(-B) = \phi$ ($\sigma(B)$ = the spectrum of B) then the solution A to (1), whenever it exists, is normal. They have also shown that if either $0 \notin \sigma(A) + \sigma(A^*)$ or $\sigma(A) \cap \sigma(-A^*) = \phi$, then there always exists a solution B to (1). We continue this study here, and prove a number of further results on (i) sufficient conditions for the normality of the solutions A (assuming that they exist); (ii) necessary/sufficient conditions for the existence of the solutions to equation (1); and (iii) the uniqueness of the solutions.

2. NOTATION AND COMPLEMENTARY RESULTS

In the following we shall denote the real (number) line by $\mathbf{R}$, and the complex plane by $\mathbb{C}$. The sets π_+ and π_0 will be defined by $\pi_+ = \{z \in \mathbb{C} : \text{Re } z > 0\}$ and $\pi_0 = \{z \in \mathbb{C} : \text{Re } z = 0\}$, where Re z denotes the real part of z. $W(A)$, $\sigma(A)$ and $\rho(A)$ will denote, respectively, the numerical range, the spectrum and the resolvent set of the operator A. $\sigma_\delta(A)$ will denote the approximate defect spectrum, i. e. $\sigma_\delta(A) = \{z \in \mathbb{C} : (A - zI) \text{ is not surjective}\}$, of the operator A. Ran (A) and Ker (A) will denote, respectively, the range and the kernel of the operator A. $B(H)$ will denote

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the algebra of bounded linear operators on the Hilbert space H, and A, B, D, X etc. will denote elements of $B(H)$. We say that $A \in B(H)$ is positive definite, written $A \gg 0$, if there exists $a \in \mathbb{R}$, $a > 0$, such that $((A-aI)\ x, x) \geqslant 0$ for all $x \in H$. Here $(.\,,\,.)$ denotes the inner product on H. We define $[A, B] = AB-BA$.

Let A, B satisfy (1). Then $[B^2, A^*+A] = 0$ and $[B, A^*-A] = 0$. Hence $[B^2, A] = 0$ and $[B^2, A^*] = 0$. Again, we have from (1) that

$$BA^*A = A-ABA = A\,(I-BA) = AA^*B \qquad ..\,(2)$$

and similarly that

$$B^*A^*A = AA^*B^*. \qquad ...(3)$$

Together, (2) and (3) imply that

$$(B + B^*)\,A^*A = AA^*\,(B + B^*) \qquad ...(4)$$

$$(B - B^*)\,A^*A = AA^*\,(B - B^*). \qquad ...(5)$$

The following theorem of Nakamoto (1977) will be required.

Theorem N — Let $H_n = A^*B^nA$, where B is normal. If (i) $0 \notin W(A)$ and (ii) $[B, H_n] = 0$ for $n = 0, 1$, then $[B, A] = 0$.

3. Normal Solutions—Sufficient Conditions

We assume in the following that there exists a solution A to (1). It is immediate from (4) and (5) that if Ran (Re B) (Ran (Im B)) is dense in H, and if either $[\text{Re}\,B, A^*A] = 0$ or $[\text{Re}\,B, AA^*] = 0$ (respectively, either $[\text{Im}\,B, A^*A] = 0$ or $[\text{Im}\,B, AA^*] = 0$), then the solution A to (1) is normal. Kamei and Kato (1978) have shown that if $\sigma(\text{Re}\,A) \cap \sigma(-\text{Re}\,A) = \phi$, then A is normal: we show that the same conclusion is obtained upon replacing Re A by Re B.

Theorem 1 — If $\sigma(\text{Re}\,B) \cap \sigma(-\text{Re}\,B) = \phi$, then the solution A to (1) is normal.

Proof : Set $2\,\text{Re}\,B = C$. Then we have from (4) that $CA^*A = AA^*C$, and hence also that $A^*AC = CAA^*$. Setting $A^*A - AA^* = X$, this implies that $CX + XC = 0$. Clearly, $X = 0$ is a solution of this equation. Since $\sigma(C) \cap \sigma(-C) = \phi$, we have by a result of Rosenblum (1956) that $X = 0$ is the only solution. Hence $A^*A = AA^*$, i.e. A is normal.

The condition $0 \notin W(\text{Re}\,B)$ is not comparable with the condition that $0 \notin W(B)$. Duggal and Khalagai (1981) have shown that if $0 \notin W(B)$, then the solution A to (1) is normal : we show that the result holds true for the case in which $0 \notin W(\text{Re}\,B)$.

Theorem 2 — The solution A to (1) is normal if any one of the following conditions is satisfied :

(i) $0 \notin W(\text{Re}\,B)$; (ii) $0 \notin W(\text{Im}\,B)$.

Proof : The proof of (i) and (ii) follows from the same argument : we prove (i).

Set $T = \begin{pmatrix} \text{Re}\,B & 0 \\ 0 & \text{Re}\,B \end{pmatrix}$ and $K = \begin{pmatrix} 0 & A^*A \\ AA^* & 0 \end{pmatrix}$. Then $0 \notin W(T)$. It is easily verified, using (4), that $TK = K^*T$ and $TK^* = KT$. The operator $(K-K^*)$ is normal, and satisfies the relations $[T^* T, K-K^*]=0$ and $[T^* (K-K^*) T, K\ \ K^*] = 0$. It follows from Theorem N that $[T, K-K^*] = 0$ i.e. $TK - TK^* = KT-K^*T$, and hence that $TK = TK^*$. Since $0 \notin W(T)$, this implies that $K = K^*$, and hence that A is normal.

Corollary 1 — The solution A to (1) is normal if any one of the following conditions is satisfied :
(i) $\sigma(\text{Re } B) \subseteq \pi_+$; (ii) B is normal and $\sigma(B)$ lies strictly on one side of the origin; (iii) B is hyponormal (i. e., $[B^*, B] \geqslant 0$) and $0 \notin \text{Re } \sigma(B)$; (iv) $[B^* B \text{ Re } B] = 0$ and $\text{Re } \sigma(B) \subseteq \pi_+$; (v) $\| (B - aI)^{-1} \| = 1/d(a, \sigma(B))$, $a \notin \sigma(B)$ (i. e. B satisfies the growth condition G_1), $\text{Re } \sigma(B) \subseteq \pi_+$ and $\sigma(B)$ is connected.

PROOF : If $\sigma(\text{Re } B) \subseteq \pi_+$, then Re B is an invertible positive operator. As such $0 \notin W(\text{Re } B)$, and the proof of (i) follows from Theorem 2. If (ii) holds, then (see Berberian 1964) 0 does not belong to the closure of $W(B)$. The proof in this case follows from Theorem 2 of Duggal and Khalagai (1981). If either of the conditions (iii)–(v) holds, then (see Kato and Moriya 1977) $\text{Re } \sigma(B) = \sigma(\text{Re } B)$. Hence the proof follows from case (i).

4. Existence — Necessary Conditions

Let B be a solution of (1) such that $[B, A]=0$. Then $(\text{Re } A)\, B = B\, (\text{Re } A) = \frac{1}{2} I$, and so both Re A and Re B are invertible. The example $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ shows that the condition $[B, A] = 0$ is not necessary to the validity of this result. What if we drop the hypothesis $[B, A] = 0$? The following theorem provides an answer to this question.

Theorem 3 — (i) If $AB + BA^* = I$ has a solution B, then $0 \notin \sigma_\delta(A)$ and $0 \in \rho(\text{Re } B)$. Furthermore $\| (\text{Re } B)^{-1} \| \leqslant 2 \| A \|$.

(ii) If (1) has a solution B, then $0 \in \rho(\text{Re } A)$ and $0 \in \rho(\text{Re } B)$.

PROOF : Let $\sigma_\pi(A)$ denote the approximate point spectrum of A; then $\sigma_\pi(A^*) = \overline{\sigma_\delta(A)}$. ('Bar' denotes complex conjugate.) Suppose that $0 \in \sigma_\delta(A)$. Then $0 \in \sigma_\pi(A^*)$, and so there exists a sequence of unit vectors $\{x_n\} \in H$ such that $A^*x_n \to 0$ as $n \to \infty$. Assuming $AB + BA^* = I$, we now have that

$$1 = (x_n, x_n) = (ABx_n, x_n) + (BA^* x_n, x_n)$$
$$= (Bx_n, A^*x_n) + (A^*x_n, B^*x_n) \to 0.$$

This is a contradiction; hence $0 \notin \sigma_\delta(A)$.

Since $AB + BA^* = I$, $AB^* = B^*A^* = I$, and so $A\, (\text{Re } B) + (\text{Re } B)\, A^* = I$. An argument similar to that above now shows that $0 \notin \sigma_\delta(\text{Re } B)$. But Re B being self-adjoint, $\sigma_\delta(\text{Re } B) = \sigma(\text{Re } B)$. Hence $0 \in \rho(\text{Re } B)$. To complete the proof, we note that for any $x \in H$,

$$\| x \|^2 = ((\text{Re } B)\, x, A^*x) + (A^*x, (\text{Re } B)\, x) \leqslant 2 \| (\text{Re } B)\, x \| \| A^*x \|.$$

Hence

$$\| (\text{Re } B)^{-1} \| \leqslant 2 \| A \|.$$

(ii) Proceeding as in (i), we have in this case that if $0 \in \sigma_\delta$ (Re A), then

$$2 = ((AB + BA^* + A^*B + BA)\, x_n, x_n)$$
$$= 2\,\{((\mathrm{Re}\, A)\, B\, x_n, x_n) + (B\,(\mathrm{Re}\, A)\, x_n, x_n)\} \to 0.$$

The contradiction implies that $0 \notin \sigma_\delta$ (Re A), and so, since Re A is self-adjoint, $0 \in \rho$ (Re A). Clearly, $0 \in \rho$ (Re B).

Theorem 4 — If (1) has a solution $B > 0$ (i. e., $(Bx, x) > 0$ for all $x \in H/\{0\}$) then there is an inner product on H, equivalent to the inner product (. , .), such that Re $A > 0$ with respect to it. In particular, $\sigma(A) \subseteq \pi_+$.

PROOF : Define a new equivalent inner product on H by

$$\langle x, y \rangle := (x, B^{-1} y), \text{ for all } x, y \in H.$$

Since $[B, A - A^*] = 0$, we have

$$(B^{-1}(A - A^*)\, x, x) = ((A - A^*)\, B^{-1} x, x)$$

or

$$\langle (A - A^*)\, x, x \rangle = \langle \overline{(A^* - A)\, x, x} \rangle$$

for all $x \in H$. Hence

$$\mathrm{Re} \langle A\, x, x \rangle = \mathrm{Re} \langle A^*\, x, x \rangle \quad \text{.. (6)}$$

for all $x \in H$. By equations (1) and (6),

$$2 \langle B^{-1}\, x, x \rangle = 2\, \| B^{-1} x \|^2$$
$$= (B^{-1}(A + A^*)\, x, x) + ((A + A^*\ B^{-1} x, x)$$
$$= \langle (A^* + A)\, x, x \rangle + \langle \overline{(A^* + A)\, x, x} \rangle$$
$$= 4\ \mathrm{Re} \langle A\, x, x \rangle.$$

Hence

$$\mathrm{Re}\ \langle Ax, x \rangle = \tfrac{1}{2} \langle B^{-1} x, x \rangle$$

for all $x \in H$. This implies that Re $A > 0$.

Theorem 5 — If there exists a solution A to (1) such that the eigen-vectors of A^* span H, then $W(B) \subseteq \mathrm{R}/\{0\}$.

PROOF : Let $\lambda \in \sigma_p(A^*)$ (= point spectrum of A^*), and let $x \in H$, $x \neq 0$, be an eigen-vector corresponding to λ. Then

$$0 = ((A - A^*)\, Bx, x) + (B\,(A^* - A)\, x, x)$$
$$= (Bx, \lambda x - Ax) + (\lambda x - Ax, B^*x)$$

and so

$$(Bx, \lambda x) + (\lambda x, B^*x) = \bar{\lambda}\,(Bx, x) + \lambda\,(x, B^*x)$$
$$= 2\,(\mathrm{Re}\, \lambda)\,(Bx, x) = (Bx, Ax) + (Ax, B^*x).$$

Again,

$$2\|x\|^2 = ((A + A^*)Bx, x) + (B(A^* + A)x, x)$$
$$= (Bx, \lambda x + Ax) + (\lambda x + Ax, B^*x) = 4(\text{Re}\,\lambda)(Bx, x).$$

Hence, $2(\text{Re}\,\lambda)(By, y) = 1$, where we have set $x/\|x\| = y$. This clearly implies that $\text{Re}\,\lambda \neq 0$. Since the eigen-vectors of A span H, we have that $W(B) \subseteq \mathbb{R}/\{0\}$.

The following theorem is immediate from the proof above.

Theorem 6 — If there exists a solution $B \geqslant 0$ (i. e. $(Bx, x) \geqslant 0$ for all $x \in H$) to (1), then $\sigma_p(A^*) \subseteq \pi_+$.

5. Existence — Sufficient Conditions

In the following we give some sufficient conditions for the existence of positive definite solutions to eqns. (1). The argument that we use in the proof of our main result below is similar to that used by Stetkaer (1979) to prove the existence of Hermitian solutions to $AX + XA^* = -C$.

Theorem 7 — If $\sigma(A) \subseteq \pi_+$, then there exists a positive definite solution B to (1) such that

(i) $B = DU$ for some positive definite operators D and U;

(ii) $[U, A] = 0$.

Proof : Choose a positive definite operator V such that $[A, V] = 0$; then $[A^*, V] = 0$. Set $X = VB$. We have from $AB + BA^* = I$ that $VAB + VBA^* = V$, i. e. $AX + XA^* = V$. Define

$$D = \int_0^\infty e^{-tA}\, V\, e^{-tA^*}\, dt.$$

Since $\sigma(A) \subseteq \pi_+$, there exist scalars $M, \mu > 0$ such that

$$\|e^{-tA}\| \leqslant M e^{-\mu t} \text{ for all } t \geqslant 0.$$

This implies that

$$\|e^{-tA}\, V\, e^{-tA^*}\| \leqslant M^2 \|V\|\, e^{-2\mu t}$$

and hence that the integral D converges absolutely with respect to the operator norm on $B(H)$. We show that D is a solution of $AX + XA^* = V$.

For any $x \in H$, a simple calculation shows that

$$((AD + DA^*)x, x) = -\lim_{t\to\infty} \int_0^t (d/ds)\,(e^{-sA}\, V\, e^{-sA^*}x, x)\, ds$$
$$= -\lim_{t\to\infty} (e^{-tA}\, Ve^{-tA^*}x, x) + (Vx, x).$$

Since

$$\int_0^\infty (Ve^{-tA^*}x, e^{-tA^*}x)\, dt = (Dx, x) < \infty$$

we see that limit is zero. Hence $AD + DA^* = V$.

Clearly, D is positive definite. Since $D = VB$, $B = V^{-1} D$ is a solution of $AB + BA^* = I$. Now consider the equation $A^*B + BA = I$. Since $[A, V] = 0$ and $[A^*, V] = 0$, $[V, \text{Re } A] = 0$. Hence

$$D = \int_0^\infty e^{-tA} \; Ve^{-tA^*} dt \;=\; \int_0^\infty Ve^{-2t \text{ Re } A} dt \;=\; \int_0^\infty Ve^{-tA^*} e^{-tA} dt$$

$$= \int_0^\infty e^{-tA^*} Ve^{-tA} dt.$$

An argument similar to that above shows that D is a solution of the equation $A^* BV + BVA = V$, or equivalently that $B = D V^{-1}$ is a solution of $AB + BA^* = I$. Hence, upon setting $V^{-1} = U$, $B = UD = DU$ is a solution of (1). Since U and D are positive definite, B is positive definite.

It is clear from the preceding argument that the hypothesis $\sigma(A) \subseteq \pi_+$ in the statement of Theorem 7 may be replaced by the hypothesis that $\sigma(\text{Re } A) \subseteq \pi_+$. Hence :

Corollary 2 —If Re $A > 0$, then there exists a positive definite solution B of (1).

Corollary 3 — If A is an M-hyponormal operator (i. e., there exists a real number M such that $\| (T - zI)^* x \| \leqslant M \| (T - zI) x \|$ for all x in H and for every $(z \in \mathbb{C})$ with $\sigma_\pi (A^*) \subseteq \pi_+$, then there exists a positive definite solution B to (1).

PROOF : Since A is M-hyponormal, $\sigma_\pi (A^*) = \sigma (A^*)$ (see Phadke and Thakare 1977). Since $\sigma (A^*) \subseteq \pi_+$ implies $\sigma (A) \subseteq \pi_+$, the proof follows from Theorem 7.

The example of the operator $A = 0$ shows that the condition Re $A \geqslant 0$ (similarly, $\sigma (A) \subseteq \pi_+ \cup \pi_0$) is not enough to ensure the existence of a solution B to equation (1), much less a positive definite solution. However, if (1) has a solution X, then the condition Re $A \geqslant 0$ (similarly, $\sigma (A) \subseteq \pi_+ \cup \pi_0$) is sufficient to guarantee the existence of a positive definite solution, as the following argument shows.

Let X be a solution of $AB + BA^* = I$. Choose a positive definite V such that $[V, A] = 0$. Set $Y = VX$. Then

$$Y - e^{-tA} \, Ye^{-sA^*} = \int_0^t e^{-sA} \, Ve^{-sA^*} ds \qquad \ldots(7)$$

for all $t > 0$. Since Re $A \geqslant 0$, $\| e^{-tA} \| \leqslant 1$, and so the left-hand side of (7), hence also the right-hand-side of (7), is bounded. Since an increasing symmetric operator-valued function converges strongly (see Halmos 1967, Solution 94),

$$D = \lim_{t \to \infty} \int_0^t e^{-sA} \, Ve^{-sA^*} ds$$

exists as a strong limit in $B(H)$. As already seen, $B = DU$ $(U^{-1} = V)$ is a positive definite solution of (1).

Corollary 4 — Let $\sigma (A)$ be the disjoint union of the spectral sets σ_1 and σ_2 such that $\sigma_1 \subseteq \pi_+$. (For definition of spectral set, see Taylor 1958, p. 298.) Let $H = H_1$

$\oplus H_2$ where $\sigma(A/H_i) = \pi_i$ $(i = 1, 2)$. Then the restriction of each solution B of (1) to $H_2^{\perp}$ is positive definite.

PROOF : Let P be the parallel projection of H onto H_1 along H_2. Set $A_1 = A/H_1$, $B_1 = B/H_1$ and $I_1 = I/H_1$. Then B_1 is a solution of the equation $A_1 B_1 + B_1 A_1^* = I_1 = A_1^* B_1 + B_1 A_1$. Since $\sigma(A_1) \subseteq \pi_+$, the proof follows as is Theorem 7.

6. UNIQUENESS

Assuming that solutions B to (1) exist, we consider now the problem of the uniqueness of these solutions. An important role here is played by the homogeneous form

$$AX + XA^* = 0; \qquad A^*X + XA = 0 \qquad \ldots(8)$$

of eqns. (1).

Lemma 1 — If $0 \notin W(\text{Re } A)$, then $X = 0$ is the only solution of eqns. (8).

PROOF : Clearly, $X = 0$ is a solution of (8). Let U be another solution of (8). Then $AU + UA^* = 0 = A^*U + UA$, and so $(\text{Re } A)\, U = -U(\text{Re } A)$ and $(\text{Re } A)\, U^* = -U^*(\text{Re } A)$. Set $U - U^* = V$ and $\text{Re } A = T$. Then, this is easily verified, $[V, T^*T] = 0$ and $[V, T^*VT] = 0$. It follows from Theorem N that $[V, T] = 0$. Since $TV = -VT$, this implies that $VT = 0$, and so since $0 \notin W(T)$ that $V = 0$, i. e. U is self-adjoint. Again, since $[U, T^*T] = 0$ and $[U, T^*UT] = 0$, an application of Theorem N implies that $[U, T] = 0$. Since $UT = -TU$, this implies that $UT = 0$, and hence that $U = 0$.

Remark : In view of the preceding lemma, it is now clear from the proof of Theorem 1 that if $0 \notin W(\text{Re } B)$, then the solution A to (1) is normal : this provides an alternative proof of Theorem 2 (i).

Lemma 2 — If $X = 0$ is the only solution of (8), then the solution B of (1) is unique.

PROOF : Suppose that B_1 and B_2 are two distinct solutions of (1). Then

$$A(B_1 - B_2) + (B_1 - B_2)A^* = 0 = A^*(B_1 - B_2) + (B_1 - B_2)A.$$

But this implies that (8) has a non-zero solution $B_1 - B_2$, contrary to our hypothesis. Hence $B_1 = B_2$.

Theorem 8 — If $0 \notin W(\text{Re } A)$, then the solution B of (1) is unique. Moreover, $B^* = B$.

PROOF : If B is a solution to (1), then a simple manipulation shows that

$$A(B^* - B) + (B^* - B)A^* = 0 = (B^* - B)A + A^*(B^* - B)$$

and hence that

$$(B^* - B)(A^* + A) = -(A^* + A)(B^* - B).$$

Set $A^* + A = T$ and $B^* - B = V$; then $VT = -TV$. It is easily verified that $[V, T^*T] = 0$ and $[V, T^*VT] = 0$, and so it follows from Theorem N that $(T, V] = 0$, and hence that $VT = 0$. Since $0 \notin W(T)$, this implies that $V = 0$, i. e. $B^* = B$. The proof, in view of Lemmas 1 and 2, is now complete.

As final result we show that if Re A is injective, and if the solutions X to (8) are of a certain type, then the solutions B to (1) are unique in as much as that they are self-adjoint.

Theorem 9 — Let Re A be injective. If for each solution X of (8) the unique positive square root Y of $X^* X$ is also a solution of (8), then the solutions B of (1) are self-adjoint.

PROOF : Let B be a solution of (1). Then, upon setting $X = B^* - B$ and following the argument of the proof of the preceding theorem, we have that A and X satisfy equations (8). Hence

$$AX^2 = -XA^* X = X^2 A.$$

Since $-X^2 \geqslant 0$, there exists a (unique) $Y \geqslant 0$ such that $Y^2 = -X^2$.Since $[A, -X^2] = 0$, $[A, Y] = 0$. By hypothesis, Y is also a solution of (8). We have that

$$AY + YA^* = (A + A^*) Y = 0,$$

and hence that Ran $(Y) \subseteq$ Ker (Re A). But Re A being injective Ker (Re A) $= \{0\}$. Hence $Y = 0$, and so, since $Y^2 = X^* X$, $X = 0$. This completes the proof.

REFERENCES

Berberian, S. K. (1964). The numerical range of a normal operator. *Duke Math. J.*, **31**, 479–83.

Duggal, B. P., and Khalagai, J. M. (1981). On the operator equation $AB+BA^*=A^*B + BA = I$. *Math. Japon.*, 26.

Halmos, P. R. (1967). A Hilbert Space Problem Book. Van Nostrand, Princeton, N. J.

Kamei, E., and Kato, Y. (1978). On operator equations $AB + BA^* = A^* B + B A = I$. *Math. Japon.*, **23**, 129–30.

Kato, Y., and Moriya, N. (1977). On operators considered by Campbell. *Math. Japon.*, **22**, 233–38.

Nakamoto, R. (1977). On a problem of Duggal. *Math. Japon.*, 22, 441–42.

Phadke, S. V., and Thakare, N. K. (1977). M-hyponormal operators invariant subspaces and spectral localization. *Indian. J. pure. appl. Math.*, **8**, 487–96.

Rosenblum, M. (1956). On operator equation $BX - XA = Q$. *Duke Math. J.*, 23, 263–69.

Stetkaer, H. (1979). On positive semi-definite solutions of the operator Lyapunov equations. *J. Math. Anal. Appl.*, **69**, 153–70.

Taylor, A. E. (1958). Introduction Functional Analysis. Wiley, New York.