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# ON AN EXTERNAL THERMAL CRACK PROBLEM IN A TRANSVERSELY ISOTROPIC ELASTIC MEDIUM

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This paper deals with a potential function approach to the solution of a thermoelastic external crack problem for an infinite transversely isotropic elastic medium. As a mathematical tool Hankel transform technique has been applied for the solution of the problem.

## 1. INTRODUCTION

A number of problems determining the thermal stresses and displacements in an infinite solid containing cracks have been solved earlier by many researchers, viz. Olesiak and Sneddon (1960), Kassir (1969), Kassir and Sih (1967). Deutsch (1965), Florance and Goodier (1963), and many others. Kassir (1969) presented a solution for the problem of thermal stresses arising out in an infinite isotropic elastic medium containing an insulated external crack occupying the plane outside a circular region when the surfaces of the crack are subjected to a temperature gradient.

In this paper an external thermal crack problem in an infinite transversely isotropic elastic medium has been considered when the faces of the crack are subjected to variable temperature gradient. This type of problem can be solved by different methods. Singh (1960) considered a stress-function approach for the solution of the axisymmetric steady-state thermoelastic problem for the transversely isotropic material. But in this paper we have adopted a potential function approach for the solution of the problem. The main object of this paper is to calculate the stress intensity factor which governs the stability behaviour of the crack in the material. The medium is assumed to be conducting heat under steady-state condition. The effects of inertia and coupling between the temperature and strain fields are neglected to aovid complexity. The solution of the problem is obtained with the aid of Hankel Transform technique. Due to the mixed boundary conditions of the problem, the solution involves dual integral equations which have been solved following the procedure of Kassir (1969). The temperature field, displacements and stresses have been obtained at any point in the meduim and also on the crack surface. In particular, when the variation of the temperature gradient is prescribed on the crack surface, the solution including the stress-intensity factor have been evaluated following the procedure of Kassir (1969).

Finally, in case of Magnesium, numerical values for the shearing stress and radial displacement on the crack surface $z = 0$ for different values of $r$ in two particular cases, viz. $n = 3$ and 4 have been shown graphically.

## 2. Statement of the Problem, Fundamental Equations and Boundary Conditions

We consider an infinite transversely isotropic elastic solid containing a flat circular crack of unit radius on the surface of the solid. In cylindrical polar co-ordinates $(r, \theta, z)$, let the faces of the crack be assumed as $r \geqslant 1$, $z = 0\mp$ the origin of the co-ordinates being coinciding with the centre of the crack. It is assumed that the crack is opened out symmetrically by the application of heat flux and shearing stress prescribed in the region $r > 1$. The displacement vector will have the components $(u, 0, w)$ and the non-vanishing components of the stress tensor will be $\sigma_{rr}$, $\sigma_{\theta\theta}$, $\sigma_{zz}$ and $\tau_{rz}$. As the thermal and mechanical conditions on the upper surface of the crack are equal and opposite to those on the lower surface, the problem of finding the stress field in the infinite solid is reduced to that of finding it for a half-space $z \geqslant 0$ when its bounding smooth plane surface $z = 0$ is subjected to mixed conditions.

In the steady-state, the temperature field $T = T(r, z)$ at any point satisfies the heat conduction equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \beta^2 \cdot \frac{\partial^2 T}{\partial z^2} = 0 \qquad \ldots(1)$$

where $\beta$ is a constant depending on the ratio of the coefficients of the thermal conductivity of the solid along $z$-axis and in $z$-plane.

The well-known equations of equilibrium in terms of displacements for a transversely isotropic material are given by

$$\left.\begin{aligned} & c_{11}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} - \frac{u}{r^2}\right) + c_{44} \cdot \frac{\partial^2 u}{\partial z^2} \\ & \qquad + \left(c_{13} + c_{44}\right)\frac{\partial^2 w}{\partial r \,.\, \partial z} = b_1 \frac{\partial T}{\partial r} \\ & c_{44}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial w}{\partial r}\right) + c_{33}\frac{\partial^2 w}{\partial z^2} \\ & \qquad + \left(c_{13} + c_{44}\right)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = b_2 \cdot \frac{\partial T}{\partial z} \end{aligned}\right\} \qquad \ldots(2)$$

where

$$b_1 = (c_{11} + c_{12})\,\alpha_1 + c_{13}\,\alpha_2\,, \quad b_2 = 2\,c_{13}\,\alpha_1 + c_{33}\,\alpha_2\,.$$

Equations (1) and (2) are to be solved subject to the following boundary conditions :

$$\left.\begin{aligned} T &= 0, \quad 0 \leqslant r < 1 \;, \quad z = 0 \\ \frac{\partial T}{\partial z} &= Q(r) \;, \quad r > 1 \;, \quad z = 0 \end{aligned}\right\} \qquad \ldots(3)$$

and

$$\left.\begin{aligned} u(r, 0) &= 0 \;, \quad 0 \leqslant r < 1 \\ \tau_{rz}(r, 0) &= \tau(r) \;, \quad r > 1 \\ \sigma_{zz}(r, 0) &= 0 \;, \quad 0 \leqslant r < \infty \end{aligned}\right\} \qquad \ldots(4)$$

the functions $Q(r)$ and $\tau(r)$ appearing respectively in (3) and (4) must be bounded at infinity.

## 3. Solution of the Problem

The solution of equation (1) in the form of Hankel integral is assumed as

$$T(r, z) = \int_0^\infty A(\xi) \,.\, e^{-\xi z/\beta} \,.\, J_0(\xi r)\, d\xi \;, \qquad \ldots(5)$$

where $J_0(x)$ is the Bessel function of first kind of order zero and $A(\xi)$ is a function of $\xi$ only to be determined from the boundary condition (3).

Equation (5) with the aid of (3) gives a set of dual integral equations, viz.

$$\left.\begin{aligned} &\int_0^\infty A(\xi) \,.\, J_0(\xi r)\, d\xi = 0, \; 0 \leqslant r < 1 \\ &\int_0^\infty \xi \,.\, A(\xi) \,.\, J_0(\xi r)\, d\xi = -\beta \,.\, Q(r), \; r > 1 \end{aligned}\right\} . \qquad \ldots(6)$$

The solution of the above set of dual integral equations is obtained (vide, Kassir 1969) as

$$A(\xi) = -\frac{2}{\pi} \int_1^\infty \Omega(t) \,.\, \cos \xi t\, dt \qquad \ldots(7)$$

where

$$\Omega(t) = \beta \int_t^\infty \frac{rQ(r)\, dr}{\sqrt{r^2 - t^2}} \,. \qquad \ldots(8)$$

Thus the temperature field is completely determined with the aid of (7) and (8), and is given by

$$T(r, z) = -\frac{2}{\pi} \int_1^\infty \Omega(t)\, dt \int_0^\infty e^{-\xi z/\beta} \,.\, \cos \xi t \,.\, J_0(\xi r)\, d\xi. \qquad \ldots(9)$$

From eqn. (9) the temperature distribution on the plane of the crack surface is obtained as

$$T(r, 0) = -\frac{2}{\pi}\int_1^r \frac{\Omega(t)\,.\,dt}{\sqrt{r^2 - t^2}}\ , \quad r > 1 \qquad ...(10)$$

and also the temperature gradient is given by

$$\frac{\partial T(r, 0)}{\partial z} = -\frac{2}{\pi}\left[\frac{\Omega(1)}{\sqrt{1 - r^2}} + \int_1^\infty \frac{\Omega'(t)\,dt}{\sqrt{t^2 - r^2}}\right], \ 0 \leqslant r < 1. \qquad ...(11)$$

It is observed from eqns. (10) and (11) that when $\beta \to 1$ these results agree with those of isotropic material, (vide, Kassir 1969).

Next to find out the displacements and stresses, let us take

$$\left.\begin{aligned} u(r, z) &= \frac{\partial}{\partial r}\left(\phi + \mu_1 \psi\right) \\ w(r, z) &= \frac{\partial}{\partial z}\left(\lambda \phi + \mu_2 \psi\right) \end{aligned}\right\}, \qquad ...(12)$$

where $\phi$ and $\psi$ are functions of $r$ and $z$ only ; $\lambda$, $\mu_1$, $\mu_2$ are material constants. Now substituting (12) into eqns. (2), we find that eqns. (2) will be satisfied if

$$\left.\begin{aligned} & c_{11}\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\,.\,\frac{\partial\phi}{\partial r}\right) + \left[c_{44} + \lambda(c_{13} + c_{44})\right]\frac{\partial^2\phi}{\partial z^2} = 0 \\ & \mu_1\,.\,c_{11}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\,.\,\frac{\partial\psi}{\partial r}\right) \\ & \qquad + \left[\mu_1\,.\,c_{44} + \mu_2(c_{13} + c_{44})\right]\frac{\partial^2\psi}{\partial z^2} = b_1 T \end{aligned}\right\} \qquad ...(13)$$

and

$$\left.\begin{aligned} & \left(c_{13} + c_{44} + \lambda\,.\,c_{44}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \lambda\,.\,c_{33}\frac{\partial^2\phi}{\partial z^2} = 0 \\ & \left[\mu_1(c_{13} + c_{44}) + \mu_2\,.\,c_{44}\right]\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\,.\,\frac{\partial\psi}{\partial r}\right) \\ & \qquad + \mu_2\,.\,c_{33}\,.\,\frac{\partial^2\psi}{\partial z^2} = b_2 T\,. \end{aligned}\right\} \qquad ...(14)$$

To find out the solutions of the equations involving $\psi(r, z)$ and $T(r, z)$, let us assume

$$\psi(r, z) = \int_0^\infty \xi^{-2}\,.\,A(\xi)\,.\,e^{-\xi z/\beta}\,.\,J_0(\xi r)\,d\xi\,. \qquad ...(15)$$

Now inserting the values of $\psi(r, z)$ given by (15) and $T(r, z)$ given by (5) in the second equations of (13) and (14) we get

$$\mu_1(c_{44} - \beta^2\,.\,c_{11}) + \mu_2(c_{13} + c_{44}) = b_1\,\beta^2$$

$$\mu_2(c_{33} - \beta^2\,.\,c_{44}) - \mu_1(c_{13} + c_{44})\,.\,\beta^2 = b_2\,.\,\beta^2$$

which are satisfied if

$$\mu_1 = \frac{\beta^2 [b_1 (c_{33} - \beta^2 . c_{44}) - b_2 (c_{13} + c_{44})]}{(c_{44} - \beta^2 . c_{11}) (c_{33} - \beta^2 . c_{44}) + \beta^2 (c_{13}+c_{44})^2} \qquad ...(16)$$

$$\mu_2 = \frac{\beta^2 [b_1 (c_{13} + c_{44}) \beta^2 + b_2 (c_{44} - \beta^2 . c_{11})]}{(c_{44} - \beta^2 . c_{11}) (c_{33} - \beta^2 . c_{44}) + \beta^2 (c_{13} + c_{44})^2} . \qquad ...(17)$$

The first equations of (13) and (14) will give non-zero solutions if they are identical and this happens if

$$\frac{\lambda (c_{13} + c_{44}) + c_{44}}{c_{11}} = \frac{\lambda . c_{33}}{\lambda c_{44} + c_{13} + c_{44}} = m^2 \text{ (say)}$$

or,

$$c_{11} . c_{44} . m^4 + (c_{13}^2 + 2 . c_{13} . c_{44} - c_{13} . c_{33}) m^2 + c_{33} . c_{44} = 0 . \qquad ...(18)$$

The solutions of equations of equilibrium involving $\phi (r, z)$ given by (13) and (14) can be obtained in terms of two stress functions $\phi_1 (r, z)$ and $\phi_2 (r, z)$ and the expressions for the stresses and the displacements can also be written, vide, Sharma (1956) as

$$\sigma_{rr} = \left( c_{11} . \frac{\partial^2}{\partial r^2} + c_{12} . \frac{1}{r} . \frac{\partial}{\partial r} \right) \left( \phi_1 + \phi_2 + \mu_1 \psi \right) + c_{13} . \frac{\partial^2}{\partial z^2} \left( \lambda_1 \phi_1 + \lambda_2 \phi_2 + \mu_2 \psi \right) - b_1 T$$

$$\sigma_{\theta\theta} = \left( c_{12} . \frac{\partial^2}{\partial r^2} + c_{11} . \frac{1}{r} . \frac{\partial}{\partial r} \right) \left( \phi_1 + \phi_2 + \mu_1 \psi \right) + c_{13} . \frac{\partial^2}{\partial z^2} \left( \lambda_1 \phi_1 + \lambda_2 \phi_2 + \mu_2 \psi \right) - b_1 T$$

$$\sigma_{zz} = c_{13} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} . \frac{\partial}{\partial r} \right) \left( \phi_1 + \phi_2 + \mu_1 \psi \right) + c_{33} . \frac{\partial^2}{\partial z^2} \left( \lambda_1 \phi_1 + \lambda_2 \phi_2 + \mu_2 \psi \right) - b_2 T$$

$$\tau_{rz} = c_{44} . \frac{\partial^2}{\partial r . \partial z} \left[ (1 + \lambda_1) \phi_1 + (1 + \lambda_2) \phi_2 + (\mu_1 + \mu_2) \psi \right]$$

$$u = \frac{\partial}{\partial r} \left( \phi_1 + \phi_2 + \mu_1 \psi \right)$$

$$w = \frac{\partial}{\partial z} \left( \lambda_1 \phi_1 + \lambda_2 \phi_2 + \mu_2 \psi \right) \qquad ...(19)$$

where $\phi_1 (r, z)$ and $\phi_2 (r, z)$ are the solutions of the partial differential equations

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} . \frac{\partial}{\partial r} + m_i^2 \frac{\partial^2}{\partial z^2} \right) \phi_i = 0 , \quad i = 1, 2 \qquad ...(20)$$

where $m_1^2$ and $m_2^2$ are the roots of the biquadratic equation (18) and $\lambda_1$, $\lambda_2$ are the two values of $\lambda$ corresponding to $m_1^2$ and $m_2^2$ respectively.

As solutions of (20), let us take

$$\phi_1(r, z) = \int_0^\infty B(\xi) \cdot e^{-\xi z/m_1} \cdot J_0(\xi r)\, d\xi$$

$$\phi_2(r, z) = \int_0^\infty C(\xi) \cdot e^{-\xi z/m_2} \cdot J_0(\xi r)\, d\xi \qquad \ldots(21)$$

where $B(\xi)$ and $C(\xi)$ are both functions of $\xi$ only to be determined from the elastic boundary conditions (4).

Inserting the values of $\phi_1(r, z)$, $\phi_2(r, z)$, $T(r, z)$ and $\psi(r, z)$ from (21), (5) and (15) in (19), we obtain the displacements and the stresses at any point as

$$u(r, z) = -\int_0^\infty [\xi \cdot B(\xi) \cdot e^{-\xi z/m_1} + \xi \cdot C(\xi) \cdot e^{-\xi z/m_2}$$

$$+ \mu_1 \xi^{-1} \cdot A(\xi) \cdot e^{-\xi z/\beta}] \times J_1(\xi r)\, d\xi$$

$$w(r, z) = -\int_0^\infty \left[ \frac{\lambda_1}{m_1} \cdot \xi \cdot B(\xi) \cdot e^{-\xi z/m_1} + \frac{\lambda_2}{m_2} \xi \cdot C(\xi) \cdot e^{-z/m_2}\right.$$

$$\left. + \frac{\mu_2}{\beta} \cdot \xi^{-1} \cdot A(\xi) \cdot e^{-\xi z/\beta} \right] J_0(\xi r)\, d\xi$$

$$\sigma_{rr}(r, z) = (c_{11} - c_{12}) \int_0^\infty [B(\xi) \cdot e^{-\xi z/m_1} + C(\xi) \cdot e^{-\xi z/m_2} + \mu_1 \xi^{-2} \cdot A(\xi) \cdot e^{-\xi z/\beta}]$$

$$\times \frac{\xi}{r} \cdot J_1(\xi r)\, d\xi + \int_0^\infty \left[ \left( \frac{\lambda_1}{m_1^2} \cdot c_{13} - c_{11} \right) \xi^2 \cdot B(\xi)\, e^{-\xi z/m_1} \right.$$

$$+ \left( \frac{\lambda_2}{m_2^2} \cdot c_{13} - c_{11} \right) \xi^2 \cdot C(\xi)\, e^{-\xi z/m_2}$$

$$\left. + \left( \frac{\mu^2}{\beta^2} \cdot c_{13} - \mu_1 \cdot c_{11} - b_1 \right) \times A(\xi) \cdot e^{-\xi z/\beta} \right] J_0(\xi r)\, d\xi$$

$$\sigma_{\theta\theta}(r, z) = \left( c_{12} - c_{11} \right) \int_0^\infty \left[ B(\xi)\, e^{-\xi z/m_1} + C(\xi) \cdot e^{-\xi z/m_2} \right.$$

$$\left. + \mu_1 \xi^{-2} \cdot A(\xi)\, e^{-\xi z/\beta} \right] \times \frac{\xi}{r} \cdot J_1(\xi r)\, d\xi$$

$$+ \int_0^\infty \left[ \left( \frac{\lambda_1}{m_1^2} \cdot c_{13} - c_{12} \right) \xi^2 \cdot B(\xi) \cdot e^{-\xi z/m_1} \right.$$

$$+ \left( \frac{\lambda_2}{m_2^2} \cdot c_{13} - c_{12} \right) \xi^2 \cdot C(\xi) \cdot e^{-\xi z/m_2}$$

(*equation continued on p.* **1399**)

$$+\left(\frac{\mu_2}{\beta^2}\cdot c_{13}-\mu_1 c_{12}-b_1\right)\times A(\xi)\, e^{-\xi z/\beta}\Big]\, J_0(\xi r)\, d\xi$$

$$\sigma_{zz}(r,z)=\int_0^{\infty}\Big[\left(\frac{\lambda_1}{m_1^2}\cdot c_{33}-c_{13}\right)\xi^2\cdot B(\xi)\cdot e^{-\xi z/m_1}$$

$$+\left(\frac{\lambda_2}{m_2^2}\cdot c_{33}-c_{13}\right)\xi^2\cdot C(\xi)\cdot e^{-\xi z/m_2}$$

$$+\left(\frac{\mu_2}{\beta^2}\cdot c_{33}-\mu_1\cdot c_{13}-b_2\right)\cdot A(\xi)\cdot e^{-\xi z/\beta}\Big]\, J_0(\xi r)\, d\xi$$

$$\tau_{rz}(r,z)=c_{44}\int_0^{\infty}\Big[\frac{1+\lambda_1}{m_1}\cdot\xi^2\cdot B(\xi)\cdot e^{-\xi z/m_1}+\frac{1+\lambda_2}{m_2}\,\xi^2\cdot C(\xi)$$

$$\times e^{-\xi z/m_2}+\frac{\mu_1+\mu_2}{\beta}\cdot A(\xi)\cdot e^{-\xi z/\beta}\Big]\, J_1(\xi r)\, d\xi \qquad ...(22)$$

where the unknown functions $B(\xi)$ and $C(\xi)$ are to be determined from the elastic boundary conditions (4).

Now the last boundary condition of (4) with the aid of (21) yields

$$B(\xi)=\frac{\left(b_2+\mu_1\cdot c_{13}-\frac{\mu_2}{\beta^2}\,c_{33}\right)\xi^{-2}\cdot A(\xi)+\left(c_{13}-\frac{\lambda_2}{m_2^2}\,c_{33}\right)\cdot C(\xi)}{\frac{\lambda_1}{m_2^2}\cdot c_{33}-c_{13}}$$

...(23)

Again the first two boundary conditions of (4) with the aid of (21) and (23) yield a set of dual integral equations, viz.

$$\left.\begin{aligned}&\int_0^{\infty}[K_0\cdot\xi^{-1}\cdot A(\xi)+K_1\cdot\xi\cdot C(\xi)]\, J_1(\xi r)\, d\xi=0, && 0\leqslant r<1\\ &\int_0^{\infty}[K_2\cdot A(\xi)+K_3\cdot\xi^2\cdot C(\xi)]\, J_1(\xi r)\, d\xi=\tau(r), && r>1\end{aligned}\right\}\qquad ...(24)$$

where

$$K_0=\frac{b_2\, m_1^2\cdot\beta^2+c_{33}\,(\mu_1\,\lambda_1\cdot\beta^2-m_1^2\,\mu_2)}{\beta^2\,(\lambda_1\cdot c_{33}-m_1^2\cdot c_{13})},\quad K_1=\frac{c_{33}\,(\lambda_1\, m_2^2-\lambda_2\, m_1^2)}{(\lambda_1\, c_{33}-m_1^2\, c_{13})m_2^2}$$

$$K_2=c_{44}\left[\frac{\mu_1+\mu_2}{\beta}+\frac{m_1\,(1+\lambda_1)\,(b_2\cdot\beta^2+\mu_1\cdot\beta^2\, c_{13}-\mu_2\cdot c_{33})}{B\;(\lambda_1\, c_{33}-m_1^2\, c_{13}}\right]$$

(*equation continued on p.* 1400)

$$K_3 = c_{44}\left[\frac{1+\lambda_2}{m_2} + \frac{m_1(1+\lambda_1)(m_2^2 . c_{13} - \lambda_2 . c_{33})}{m_2^2(\lambda_1 . c_{33} - m_1^2 c_{13})}\right]$$

and $A(\xi)$ is known and is given by (7) and (8) and $C(\xi)$ is unknown to be determined from the above set of dual integral equations.

On further straight-forward simplifications of (24), we arrive at the following set of dual integral equations, viz.

$$\left.\begin{aligned} &\int_0^\infty D(\xi) . J_1(\xi r)\, d\xi = F(r), \quad 0 \leqslant r < 1 \\ &\int_0^\infty \xi . D(\xi) . J_1(\xi r)\, d\xi = G(r), \quad r > 1 \end{aligned}\right\}, \quad \ldots(25)$$

where

$$D(\xi) = \xi . C(\xi)$$

$$F(r) = \frac{2K_0}{\pi K_1} . \int_1^\infty \Omega(t)\, dt . \int_0^\infty \xi^{-1} . \cos \xi t . J_1(\xi r)\, d\xi = 0,$$

for $t^2 > r^2$, (vide Erdelyi 1953)

$$G(r) = \frac{1}{K_3}\left[\tau(r) + \frac{2K_2}{\pi}\int_1^\infty \Omega(t)\, dt \int_0^\infty \cos \xi t . J_1(\xi r)\, d\xi\right]$$

$$= \frac{1}{K_3}\left[\tau(r) + \frac{2K_2}{\pi r}\int_1^\infty \Omega(t)\, dt\right], \text{ for } t^2 < r^2, \text{ (vide Erdelyi 1953).} \quad \ldots(26)$$

Since $F(r) = 0$ for $t^2 > r^2$, the above set of dual integral equations reduces to the following form

$$\left.\begin{aligned} &\int_0^\infty D(\xi) . J_1(\xi r)\, d\xi = 0, \quad 0 \leqslant r < 1 \\ &\int_0^\infty \xi . D(\xi) . J_1(\xi r)\, d\xi = G(r), \quad r > 1 . \end{aligned}\right\} \quad \ldots(27)$$

Now the solution of (27) is obtained by using the standard results, vide Kassir (1969) as

$$D(\xi) = \frac{2}{\pi}\int_0^\infty t . \sin \xi t . \bar{\Omega}(t)\, dt, \quad \ldots(28)$$

where

$$\overline{\Omega}(t) = \int_t^{\infty} \frac{G(r)\, dr}{\sqrt{r^2 - t^2}} \qquad \ldots(29)$$

and $G(r)$ is given by (26).

Now $C(\xi)$ is known from (26) with the aid of (28) and (29). Thus, all the stresses and displacements are completely determined.

On the plane surface $z = 0$, the following expressions for the displacements and the shear stress are obtained as

$$u(r, 0) = -\frac{2K_1}{\pi \cdot c_{33}} \cdot \int_1^{\infty} \frac{t^2 \cdot \overline{\Omega}(t)}{r \cdot \sqrt{r^2 - t^2}}\, dt \;, \; r > t$$

$$w(r, 0) = -\frac{2}{\pi}\left\{ \frac{\lambda_1 m_1 (m_2^2 \cdot c_{13} - \lambda_2 \cdot c_{33})}{m_2^2 (\lambda_1 c_{33} - m_1^2 \cdot c_{13}} + \frac{\lambda_2}{m_2} \right\} \int_1^{\infty} \frac{t \cdot \overline{\Omega}(t)}{\sqrt{t^2 - r^2}}\, dt, \; r > t$$

$$\tau_{rz}(r, 0) = \begin{cases} -\dfrac{2K_2}{\pi r} \cdot \displaystyle\int_1^{\infty} \Omega(t)\, dt - \dfrac{\sqrt{2} \cdot K_3 \cdot r}{\pi} \displaystyle\int_1^{\infty} \frac{t \cdot \overline{\Omega}(t)}{(t^2 - r^2)^{3/2}}\, dt \;, 0<r<t \\ -\dfrac{2K_2}{\pi r} \displaystyle\int_1^{\infty} \Omega(t)\, dt \;, \; 0 < t < r. \end{cases} \qquad \ldots(30)$$

Now as in Kassir (1969), the stress-intensity factor $K$ is obtained as

$$K = \frac{2K_2}{\pi} \cdot \Omega(1)$$

$$= \frac{2K_2}{\pi} \cdot \beta \cdot \int_1^{\infty} \frac{r \cdot Q(r)}{\sqrt{r^2 - 1}}\, dr \,. \qquad \ldots(31)$$

This factor has been known to control the forces which motivate and produce crack extension due to thermal stress.

*Particular Case*

When the variation of the temperature gradient on the crack surface is given by

$$Q(r) = Q_0\, r^{-n} \;;\; n > 1 \;,\; r > 1$$

where $Q_0$ is a constant, then

$$\Omega(t) = \beta \cdot Q_0 \cdot \int_t^{\infty} \frac{r^{1-n}}{\sqrt{r^2 - t^2}} \cdot dr$$

*(equation continued on p. 1402)*

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}\, n\right)} \cdot \beta \cdot Q_0 \cdot t^{1-n}, \qquad \ldots(32)$$

where $\Gamma(x)$ is the Gamma Function. Equations (10) and (11) with the aid of (32) give the following expressions for the variations of temperature and temperature gradient on the plane containing the crack :

$$T(r, 0) = -\frac{2}{\pi} \int_1^r \frac{\Omega(t)}{\sqrt{r^2 - t^2}}\, dt, \quad r > 1$$

$$= \begin{cases} -\dfrac{Q_0 \cdot \beta}{2\sqrt{\pi}} \cdot \dfrac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}\, n\right)} \cdot r^{1-n} \cdot B_{(1-r^{-2})}\left(\frac{1}{2}, 1-\frac{n}{2}\right), & r > 1 \\ 0, \quad 0 \leqslant r < 1 \end{cases} \qquad \ldots(33)$$

and

$$\frac{\partial T(r, 0)}{\partial z} = -\frac{2}{\pi}\left[\frac{\Omega(1)}{\sqrt{1 - r^2}} + \int_1^\infty \frac{\Omega'(t)}{\sqrt{t^2 - r^2}}\, dt\right], \quad 0 \leqslant r < 1$$

$$= \begin{cases} -\dfrac{Q_0\, \beta}{\sqrt{\pi}} \cdot \dfrac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}\, n\right)} \cdot \left[(1 - r^2)^{-1/2} + \dfrac{1-n}{2r^n} \cdot \beta_{r^2}\left(\dfrac{n}{2}, \dfrac{1}{2}\right)\right], & 0 \leqslant r < 1 \\ Q_0 r^{-n}, \quad r > 1 \end{cases} \qquad \ldots(34)$$

where $B_x(p, q)$ is the incomplete Beta function defined by

$$B_x(p, q) = \int_0^x y^{p-1} (1-y)^{q-1}\, dy, \quad \text{Re } |p| > 0, \quad \text{Re } |q| > 0.$$

For $n = 3$, the expression (33) and (34) reduces to

$$T(r, 0) = \begin{cases} 0, \quad 0 \leqslant r < 1 \\ -\dfrac{2Q_0 \cdot \beta}{\pi} \cdot \dfrac{\sqrt{r^2 - 1}}{r}, \quad r > 1 \end{cases}$$

and

$$\frac{\partial T(r, 0)}{\partial z} = \frac{Q_0 \cdot \beta}{r^3} \cdot \begin{bmatrix} -\dfrac{2r}{\pi}\left\{(1 - r^2)^{-1/2} - \dfrac{1}{r} \cdot \sin^{-1} r\right\}, & 0 \leqslant r < 1 \\ \dfrac{1}{\beta}, \quad r > 1. \end{bmatrix}$$

If there is no shear stress on the surface containing the crack i.e. $\tau(r) = 0$ and the temperature gradient is $Q(r) = Q_0 . r^{-n}$, $n > 1$, $r > 1$ then from equation (28) we obtain

$$G(r) = \frac{2K_2}{\pi K_3 r} . \int_1^\infty \Omega(t) . dt$$

$$= \frac{K_2 . Q_0}{(n-2) . K_3 . \sqrt{\pi}} . \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} . \beta r^{-1}, \text{ for } n > 2$$

and

$$\overline{\Omega} t = \frac{K_2 . \beta . Q_0 . \sqrt{\pi}}{2(n-2) . K_3} . \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} . t^{-1} . \quad \text{...(35)}$$

Equations (30) with the aid of (32) and (35) give the following expressions for the displacements and the shearing stress on the surface z = 0 containing the crack:

$$u(r, 0) = \frac{K_1 . K_2 . \beta . Q_0}{\sqrt{\pi} . K_3 (n-2) . c_{33}} . \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} \times \begin{bmatrix} 0, & r < 1 \\ \left\{1 - \left(1 - \frac{1}{r^2}\right)^{1/2}\right\}, & r \geqslant 1 \end{bmatrix} \quad \text{...(36)}$$

$$w(r, 0) = \frac{K_2 . \beta . Q_0}{K_3 (n-2) . \sqrt{\pi}} \left\{ \frac{\lambda_1 m_1 (m_2^2 . c_{13} - \lambda_2 . c_{33})}{m_2^2 (\lambda_1 c_{33} - m_1^2 . c_{13})} + \frac{\lambda_2}{m_2} \right\} \times \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} . \log(1 + \sqrt{1 - r^2}) \quad \text{.. (37)}$$

$$\tau_{rz}(r, 0) = -\frac{K_2 . \beta . Q_0}{(n-2)\sqrt{\pi}} . \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} \begin{bmatrix} \frac{1}{r} + \frac{1}{r\sqrt{2}}\{(1-r^2)^{-1/2} - 1\}, & 0 < r < 1 \\ \frac{1}{r}, & r > 1. \end{bmatrix} \quad \text{...(38)}$$

As in Kassir (1969), the stress-intensity factor in this case also is obtained as

$$K = \frac{K_2 . \beta . Q_0}{(n-2)\sqrt{\pi}} . \frac{\Gamma\left(\frac{1}{2}(n-1)\right)}{\Gamma\left(\frac{1}{2}n\right)} . \quad \text{...(39)}$$

For $n = 3$, the expressions (38), (39) and (40) reduce to

$$u(r, 0) = \frac{2 K_1 . K_2 . \beta . Q_0}{\pi . K_3 . c_{33}} \begin{cases} 0, & r < 1 \\ 1 - \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}}, & r \geqslant 1 \end{cases} \quad ...(40)$$

$$w(r, 0) = \frac{2 K_2 . \beta . Q_0}{\sqrt{\pi}\, K_3} \left\{ \frac{\lambda_1 m_1 (m_2^2 . c_{13} - \lambda_2 . c_{33})}{m_2^2 (\lambda_1 . c_{33} - m_1^2 . c_{13})} + \frac{\lambda_2}{m_2} \right\} . \log (1 + \sqrt{1 - r^2}$$

$$\tau_{rz}(r, 0) = - \frac{2 K_2 . \beta . Q_0}{\pi} \begin{bmatrix} \frac{1}{r} + \frac{1}{r\sqrt{2}} \{(1 - r^2)^{-1/2} - 1\}, & 0 < r < 1 \\ \frac{1}{r}, & r > 1. \end{bmatrix} \quad ...(41)$$

The stress-intensity factor is then given by

$$K = \frac{2 K_2 \ \beta \ Q_0}{\pi} .$$

For $n = 4$, the expressions (36), (37) and (38) similarly reduce to

$$u(r, 0) = \frac{K_1 . K_2 . \beta Q_0}{4 K_3 . c_{33}} . \begin{cases} 0, & r < 1 \\ 1 - \left(1 - \frac{1}{r^2}\right)^{\frac{1}{2}}, & r \geqslant 1 \end{cases} \quad ...(42)$$

$$w(r, 0) = \frac{K_2 . \beta . Q_0}{4 K_3} \left\{ \frac{\lambda_1 m_1 (m_2^2 . c_{13} - \lambda_2 . c_{33})}{m_2^2 (\lambda_1 \ c_{33} - m_1^2 . c_{13})} + \frac{\lambda_2}{m_2} \right\} \log (1 + \sqrt{1 - r^2})$$

$$\tau_{rz}(r, 0) = - \frac{K_2 . \beta . Q_0}{4} \begin{bmatrix} \frac{1}{r} + \frac{1}{r\sqrt{2}} \{(1 - r^2)^{-1/2} - 1\}, & 0 < r < 1 \\ \frac{1}{r}, & r > 1. \end{bmatrix} \quad ...(43)$$

The stress-intensity factor in this case is given by

$$K = \frac{K_2 . \beta . Q_0}{4} .$$

Thus, we observe that in all cases the stress-intensity factor $K$ depends on the thermal conductivity of the material and also on the material constants.

## 4. Numerical Results

For the purpose of numerical calculations of shearing stress given by (41) and (43) and the radial displacement given by (40) and (42), let us consider the case of Magnesium for which the roots of the equation (18) are real. The elastic constants in the case of Magnesium, (vide Dey and Das 1972), are given by

$c_{11} = 0.595 \times 10^{12}$ dyn/cm$^2$, $c_{33} = 0.587 \times 10^{12}$ dyn/cm$^2$

$c_{12} = 0.232 \times 10^{12}$ dyn/cm$^2$, $c_{44} = 0.168 \times 10^{12}$ dyn/cm$^2$.

$c_{13} = 0.181 \times 10^{12}$ dyn/cm$^2$

The coefficients of linear thermal expansion along and perpendicular to the $z$-axis are given by

$$\alpha_1 = 27.7 \times 10^{-6}/°C, \qquad \alpha_2 = 26.6 \times 10^{-6}/°C.$$

With the help of the above numerical values the roots of the equation (18) are found to be

$$m_1^2 \approx 1.986 \qquad m_2^2 \approx 0.524$$

and

$$b_1 \approx 26.8915 \times 10^6 \text{ dyns/cm}^2 \qquad b_2 \approx 25.6416 \times 10^6 \text{ dyns/cm}^2.$$

The values of $\lambda_1$ and $\lambda_2$ corresponding to $m_1^2$ and $m_2^2$ are

$$\lambda_1 \approx 2.733 \qquad \lambda_2 \approx 0.367.$$

Taking $\beta = 1.5$ the other required values of the constants are obtained as

$$\mu_1 = -172.275 \times 10^{-6}$$
$$\mu_2 = -474.745 \times 10^{-6}$$
$$K_1 = 0.633$$
$$K_2 = 11.589 \times 10^6$$
$$K_3 = 0.15372 \times 10^{12}$$

With these values of the constants we have calculated the numerical values of shearing stress $\frac{\tau_{rz}(r, 0)}{Q_0} \times 10^{-6}$ and the radial displacement $\frac{u(r, 0)}{Q_0} \times 10^{18}$ for different values of $r$ when $n = 3$ and 4 and their results are given in Tables I and II.

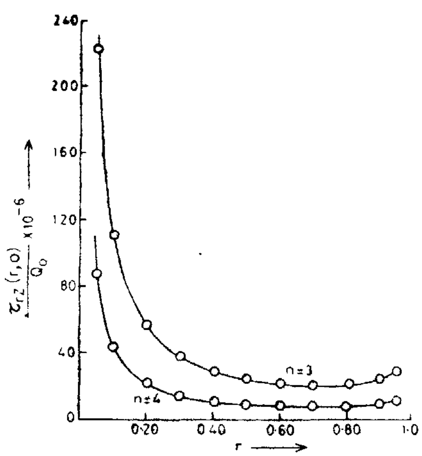


FIG. 1.

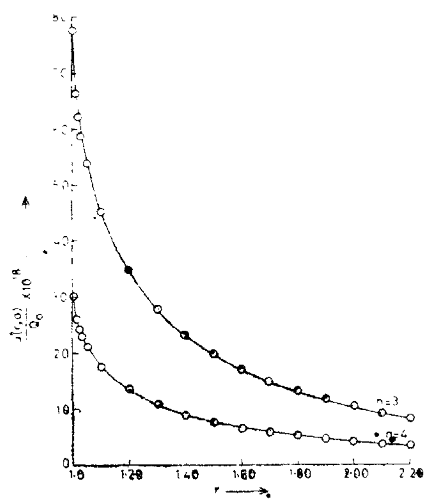


FIG. 2.

TABLE I (a)

*Shearing stress for $n = 3$*

| $r$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\tau_{rz}(r, 0)}{Q_0} \times 10^{-6}$ | 111.1026 | 56.1046 | 38.0670 | 29.4355 | 24.5665 | 21.6894 | 20.2807 | 20.3614 | 23.5706 |

TABLE I (b)

*Shearing stress for $n = 4$*

| $r$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{\tau_{rz}(r, 0)}{Q_0} \times 10^{-6}$ | 43.6134 | 22.0481 | 14.9668 | 11.5656 | 9.6433 | 8.5267 | 7 9649 | 7.9888 | 9.2465 |

TABLE II (a)

*Radial displacement for $n = 3$.*

| $r$ | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 | 1.60 | 1.70 | 1.80 | 1.90 | 2.00 | 2.10 | 2.20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{u(r, 0)}{Q_0} \times 10^{18}$ | 45.2843 | 34.7125 | 27.9898 | 23.2678 | 19.7829 | 17.0243 | 14.8577 | 13.1246 | 11.6118 | 10.4165 | 9.3553 | 8.4531 |

TABLE II (b)

*Radical displacement for $n = 4$.*

| $r$ | 1.10 | 1.20 | 1.30 | 1.40 | 1.50 | 1.60 | 1.70 | 1.80 | 1.90 | 2.00 | 2.10 | 2.20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{u(r, 0)}{Q_0} \times 10^{18}$ | 17.7788 | 13.6283 | 10.9889 | 9.1350 | 7.7668 | 6.6838 | 5.8322 | 5.1527 | 4.5588 | 4.0895 | 3.5729 | 3.3187 |

## DISCUSSIONS

It is observed that for Magnesium, shearing stress within the circular region is found to be of parabolic type for $n = 3$ and 4 as in Fig. 1. The behaviour of the radial displacement outside the circular region is also found to be of similar nature as in Fig. 2. But Elliot (1948) and Singh (1960) have shown that Magnesium is almost an isotropic material. So in case of Zinc which is very different from an isotropic material it is expected that the results will very appreciably as the roots of (20) in this case will not be real.

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