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BEST APPROXIMATION IN ULTRAMETRIC SPACES

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The existence of best approximation has been established in spherically complete ultrametric spaces in this paper. Regarding the uniqueness of best approximation, it has been shown that in ultrametric spaces, the best approximation may be unique which is in contrast with the corresponding result for non-archimedean normed spaces where best approximation is never uniquely determined except in the trivial case.

1. INTRODUCTION

Let E be a non-archimedean (n.a.) normed space over a field K with a non-trivial n.a. valuation which is supposed to be complete. Let V be a closed linear subspace of E. For a given $x \in E$, a best approximation of x in V is defined to be an element $\xi \in V$ s.t.

$$\| x-\xi \| = \inf_{g \in V} \| x-g \| .$$

Regarding the existence of best approximation, it was shown by Monna (1956) that if V is a spherically complete normed space then every $x \in E$ has a best approximation in V. We have shown that this result of Monna holds when the underlying space is an ultrametric space. Moreover, we have shown that the condition of spherical completeness is not necessary for the existence of best approximation.

Regarding the uniqueness of best approximation, it was shown by Monna (1968) that 'a best approximation of $x \in E$, $x \notin V$ (V, a closed subspace of a n.a. normed space E) when it exists, is never uniquely determined unless $V = \{0\}$'. It is a consequence of this result that the problem of best approximation in n.a. normed spaces leads essentially to the problem of existence. In contrast to this we have shown that best approximation of $x \in E$, $x \notin V$ (V, a closed subspace of an ultrametric space E) in V when it exists, may be unique i.e. V may be a Chebyshev set.

2. EXISTENCE OF BEST APPROXIMATION

In order to establish the existence of best approximation in ultrametric spaces we introduce few definitions and establish certain lemmas.

Definition 2.1—Let X be a non empty set. A metric d on $X \times X$ is called an 'ultrametric' if it satisfies the strong triangle inequality

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$$d(x, y) \leqslant \max \{d(x,z), d(z,y)\}$$

for every x,y,z in X.

The following lemma is an easy consequence of this definition.

Lemma 2.2—A metric d on a set X is an ultrametric if and only if for all $x, y, z \in X$

$$d(x, y) > d(y, z) \Longrightarrow d(x,z) = d(x,y).$$

Next lemma shows that in ultrametric spaces every point in a spherical neighbourhood of a point is a centre.

Lemma 2.3—Every point in the open spherical neighbourhood of x of radius ϵ, $S_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$, is a centre, i.e., if $y \in S_\epsilon(x)$, then $S_\epsilon(x) = S_\epsilon(y)$ Same is true for closed spheres

$$C_\epsilon(x) = \{y \in X \mid d(x, y) \leqslant \epsilon\}.$$

Corollary 2.4—If $S_{\epsilon_1}(x) \cap S_{\epsilon_2}(y) \neq \phi$ and $\epsilon_1 \leqslant \epsilon_2$,

then $S_{\epsilon_1}(x) \subset S_{\epsilon_2}(x) = S_{\epsilon_2}(y)$.

The proofs of Lemma 2.3 and Corollary 2.4 are minor modifications of the corresponding results given in Narici *et al.* (1971).

Next we consider a notion stronger than that of completeness of a space.

Definition 2.5—An ultrametric space X is said to be spherically complete (Van Rooij 1978) if every nest of closed spheres has non empty intersection.

The following lemma shows that we can rephrase spherical completeness by a weaker looking condition.

Lemma 2.6—An ultrametric space X is spherically complete if and only if every countable nest of closed spheres $C_{\epsilon_i}(x_i)$, $i = 1,2,\ldots$, has non empty intersection. Also X is spherically complete if and only if every nested sequence of closed spheres has non-empty intersection.

The proof of this is exactly similar to that given in Narici *et al.* (1971).

Next, we introduce the notions of pseudo Cauchy sequence and pseudo limit in ultrametric spaces.

Definition 2.7—A sequence $\langle x_n \rangle$ in an ultrametric space (X, d) is said to be a pseudo Cauchy sequence or a PC-sequence if there exist some n_0 such that whenever

$$n_0 \leqslant n_1 < n_2 < n_3$$

then $d(x_{n_3}, x_{n_2}) < d(x_{n_2}, x_{n_1})$.

Definition 2.8—In an ultrametric space (X, d), an element x is called a pseudo limit or p-limit of $\langle x_n \rangle$ if for all n and m sufficiently large, $n > m$ implies $d(x_n, x) < d(x_m, x)$, i.e., if ultimately the sequence $\langle d(x_n, x) \rangle$ decreases monotonically. We denote this by $x_n \rightarrow_p x$.

The following lemma, whose proof is an obvious modification of the one given in Narici *et al.* (1971) for n.a. normed spaces relates p-limit and PC-sequences.

Lemma 2.9—In an ultrametric space (X, d), if x is a p-limit of $\langle x_n \rangle$, then $\langle x_n \rangle$ is a PC-sequence.

Next we define a pseudo complete space.

Definition 2.10—An ultrametric space (X, d) is said to be pseudo complete or p-complete if every PC-sequence in X has a p-limit.

As is well known that ordinary Cauchy sequences are bounded, for PC-sequence we have the following result :

Lemma 2.11—If $\langle x_n \rangle$ is PC sequence in an ultrametric space (X,d), then either $\langle d(x_n, p) \rangle$ is ultimately strictly decreasing or the set $\{d(x_n, p) \mid n = 1,2,...\}$ is finite; where p is an arbitrary but fixed element of the space X.

The proof of Lemma 2.11 is an obvious modification of the one given in Narici *et al.* (1971) for n.a. normed spaces.

Lemma 2.12—If $\langle x_n \rangle$ is PC-sequence, then, for some n_0 and all $k > n \geqslant n_0$ $(x_k, x_n) = d(x_{n+1}, x_n)$.

PROOF : For $k > n+1$, we have $d(x_k, x_{n+1}) < d(x_{n+1}, x_n)$. Therefore, $d(x_k, x_n) = d(x_{n+1}, x_n)$.

Remark 2.13—Above lemma implies that the terms $d(x_k, x_n)$ are all equal for $k > n \geqslant n_0$. We denote the common value $d(x_{n+1}, x_n)$ by μ_n. Furthermore, since $d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n)$, we see that μ_n is ultimately a monotonically decreasing sequence. Also x is a p-limit of $\langle x_n \rangle$ if $\langle d(x, x_n) \rangle$ is ultimately a decreasing sequence. Using this we get that $d(x, x_m) = \mu_m$ for sufficiently large m. To see this consider $k > m$. Then

$$d(x, x_k) < d(x, x_m),$$

so $$d(x_k, x_m) = d(x, x_m).$$

Lemma 2.12 and Remark 2.13 yield the following :

Lemma 2.14—An element x is a p-limit of the pC-sequence $\langle x_n \rangle$ if and only if $d(x, x_n) = \mu_n = d(x_{n+1}, x_n)$ for sufficiently large n.

Remark 2.15 : The condition in Lemma 2.14 that $d(x, x_n) = \mu_n$ may be weakened to $d(x, x_n) \leqslant \mu_n$. To see this, suppose that the latter condition holds. Suppose further that $d(x, x_n) < \mu_n$ for some n. By definition of μ_n, this means that $d(x,x_n) < d(x_{n+1}, x_n)$. This however implies that $d(x, x_{n+1}) = d(x_{n+1}, x_n) = \mu_n$. By assumption, $d(x, x_{n+1}) \leqslant \mu_{n+1}$. So $\mu_n \leqslant \mu_{n+1}$ contradicting the monotonic decreasing character of μ_n.

The following lemma whose proof is an obvious modification of that given in Narici *et al.* (1971) n.a. normed spaces, proves the equivalence of pseudo and spherical completeness.

Lemma 2.16—An ultrametric space (X,d) is spherically complete if and only if it is pseudo-complete.

The following theorem establishes the existence of best approximation for spherically complete ultrametric spaces :

Theorem 2.17—Let Y be a subspace of the ultrametric space X. If Y is spherically complete, then, for any $x \in X$, there is some $y \in Y$ such that $d(x, y) = d(x, Y)$, where $d(x, Y)$ denotes the distance from x to Y, namely $\inf_{y \in Y} d(x, y)$.

The proof of Theorem 2.17 which makes use of Lemma 2.2 and Lemma 2.16, is an obvious modification of the one given in Narici *et al.* (1971) for n.a. normed spaces.

The following example illustrates that spherical completeness is not a necessary condition for the existence of a best approximation.

Example 2.18—Let $X = \mathbb{N}$, the set of natural numbers. Define a function d from $\mathbb{N} \times \mathbb{N}$ to the set of nonnegative reals by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ \max(1+m^{-1}, 1+n^{-1}) & \text{if } m \neq n. \end{cases}$$

It is easy to verify that this d is an ultrametric and so (X, d) is an ultrametric space.

Let n_0 be any fixed natural number greater than one. Let

$$Y = \{n_0 + 1, n_0 + 2, n_0 + 3, \ldots\}.$$

Firstly we show that Y is not spherically complete. Let $\{C_{1+(1/n)}(n) \,;\, n > n_0\}$ be a family of closed spheres in Y. Denote $C_{1+(1/n)}(n)$ by $C(n)_4$. It is easy to see that $C(n) = \{n, n+1, n+2, \ldots\}$ and therefore $\{C(n) \,;\, n > n_0\}$ is a nest. To show that this nest has empty intersections, it is sufficient to show that for any $p > n_0$ there is some member of the family which does not contain p. Take $m > p$. Then $d(p, m) = 1+(1/p) > 1+(1/m)$ of the and so $p \notin C(m)$. Hence the above nest has empty intersection.

Now we show that every $x \in X$ has a best approximation in Y. If x is greater than n_0, the result is trivial. So suppose $x \leqslant n_0$. Then for $k \in Y$; $d(x, k) = 1+x^{-1}$. This means that every element of Y is best approximation to x.

Thus for the existence of a best approximation in ultrametric spaces, spherical completeness is not a necessary condition.

3. Uniqueness of Best Approximation

Regarding the problem of uniqueness of best approximation, the following result was proved by Monna (1968).

Theorem 3.1—Let V be a closed linear subspace of a n.a. normed linear space E. A best approximation of $x \in E$, $x \notin V$, in V when it exists is never uniquely determined unless $V = \{0\}$, i.e. V is not a Chebyshev set unless $V = \{0\}$.

The following example shows that in ultrametric spaces, best approximation may be unique.

Example 3.2— Let X be as in Example 2.18. Let $Z = \{1, 2, 3, \ldots n \mid n > 1\}$ and take $n_0 \in X$, $n_0 \notin Z$, Then it is easy to see that n is best approximation for n_0 and it is unique. It is interesting to note that every element of X which is not in Z has n as the best approximation in Z. Clearly for any $z \in Z$

$$d(z, n_0) = 1+z^{-1} > 1+n^{-1} = d(n_0, n).$$

Note : Example 2.18 shows that every element of Y is a best approximation for every $y \in X$, $y \notin Y$; i.e. best approximation is not unique.

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