Indian J. pure appl. Math., 13(7): 732–740, July 1982

SUBGROUP THEORETICAL PROPERTIES DETERMINING CLASSES OF GENERALIZED NILPOTENT GROUPS*

T. E. STANLEY AND R. BHATTACHARYYA

Department of Mathematics, The City University, Northampton Square, London ECI, U. K.

(*Received* 10 *June* 1980; *after revision* 26 *September* 1981)

1. INTRODUCTION

Robinson (1972) has formally defined the concept of a subgroup theoretical property χ as follows : χ is a property pertaining to subgroups and if G is a group with a subgroup H having the property χ with respect to G we write $H \chi G$. χ is a subgroup theoretical property if $1 \chi G$ is always valid and whenever $H \chi G$ and θ is an isomorphism out of G then $H^{\theta} \chi G^{\theta}$. Examples of such properties are 'is a normal subgroup of' and 'is a central subgroup of'.

Subgroup theoretical properties may be used to define special sorts of series of subgroups or coverings in groups. Examples of these uses may be found in Petty (1976) and Phillips and Plotkin (1972). A normal series in a group G is called a χ-series if for each factor A/B of the series we have $(A/B) \chi (G/B)$. We denote the class of all groups with a χ-series by $\hat{\chi}$. The classes of groups with ascending χ-series, descending χ-series and χ-series of finite length are denoted by $\acute{\chi}$, $\grave{\chi}$ and $\mathring{\chi}$ respectively. Consequently

$$\mathring{\chi} \leqslant \acute{\chi} \cap \grave{\chi} \quad \text{and} \quad \acute{\chi} \cup \grave{\chi} \leqslant \hat{\chi} \quad .$$

The following definitions are made in Robinson (1972). The χ-centre of a group G is $\zeta^{\chi}(G) = \langle H \leqslant G ; H \chi G \rangle$.

It is possible, in the usual way, to define the upper χ-central series $\{ \zeta^{\chi}_{\alpha} (G) \}$ of G, and the terminal member of this series is called the χ-hypercentre of G, denoted by $\overline{\zeta^{\chi} (G)}$. If $N \lhd G$, let

$$\gamma^{\chi} (N, G) = \cap \{ M \lhd G;\ M \leqslant N \text{ and } (N/M) \chi (G/M) \}.$$

The terms of the lower χ-central series of G are defined by the rules :

$$\gamma^{\chi}_1 (G) = G,\ \gamma^{\chi}_{\alpha+1} (G) = \gamma^{\chi} (\gamma^{\chi}_{\alpha} (G), G), \gamma^{\chi}_{\lambda} (G) = \bigcap_{\beta<\lambda} \gamma^{\chi}_{\beta} (G)$$

where α is an ordinal number and λ is a limit ordinal number. The terminal member of this series is the χ-hypocentre of G, denoted by $\overline{\gamma^{\chi} (G)}$.

---

*Some of the contents of this paper formed part of the second author's thesis submitted to The City University, London, in 1977.

Sometimes it is necessary to impose extra conditions on a subgroup theoretical property χ, two of which we now describe.

I. χ is inherited by homomorphic images if $H^\theta \ \chi \ G^\theta$ whenever $H \ \chi \ G$ holds and θ is a homomorphism out of G. Consider the following statements:

(i) G is a group such that every non-trivial homomorphic image Q of G contains a non-trivial normal subgroup P such that $P \ \chi \ Q$.

(ii) $G \in \acute{\chi}$.

(iii) $G = \overline{\zeta^\chi (G)}$.

Then (i) implies (ii), and if χ is inherited by homomorphic images then (ii) implies both (i) and (iii). Also if $\zeta^\chi (X) \ \chi \ X$ is valid for any group X then the three statements are equivalent. Moreover, if χ is inherited by homomorphic images we see that $\zeta^\chi (G)^\theta \leqslant \zeta^\chi (G^\theta)$ if θ is a homomorphism out of G, and an alternative characterization of the χ-hypercentre is

$$\overline{\zeta^\chi (G)} = \cap \{N \lhd G ; \ \zeta^\chi (G/N) = 1\}.$$

II. We will say that χ satisfies condition (*) if whenever X, Y and N are normal subgroups of a group G with $Y \leqslant X$ and $(X/Y) \ \chi \ (G/Y)$ then $(N \cap X/N \cap Y) \ \chi \ (G/N \cap Y)$. Consider the following statements:

(i) Each non-trivial normal subgroup S of the group G has a proper subgroup T, normal in G, such that $(S/T) \ \chi \ (G/T)$.

(ii) $G \in \acute{\chi}$.

(iii) $\overline{\gamma^\chi (G)} = 1$.

Then (i) implies (ii), and if χ satisfies (*) then (ii) implies both (i) and (iii). Also if $(Y/\gamma^\chi (Y, X)) \ \chi \ (X/\gamma^\chi (Y, X))$ is valid for any group X with $Y \lhd X$, then the three statements are equivalent. Moreover, if χ satisfies (*) we see that $\gamma^\chi (K, G) \leqslant \gamma^\chi (L, G)$ whenever K and L are normal subgroups of G with $K \leqslant L$, and an alternative characterization of the χ-hypocentre is

$$\overline{\gamma^\chi (G)} = \pi \{N \lhd G \ ; \ \gamma^\chi (N, G) = N\}.$$

2. The Subgroup Theoretical Properties ϕ and ψ

For a class of groups $\mathfrak{X}$ and a group G we let

$$N (G : \mathfrak{X}) = \{N \lhd G; G/N \in \mathfrak{X}\}$$

$$H_1 (G : \mathfrak{X}) = \{x \in G \ ; \ [x, N] = 1 \text{ for some } N \in N (G : \mathfrak{X})\}.$$

It was shown in Stanley (1970) that if $\mathfrak{X}$ is closed with respect to the formation of subgroups, quotients and finite direct products then $H_1 (G : \mathfrak{X})$ is a subgroup of G, called its $\mathfrak{X}$-centre. Actually, $H_1 (G : \mathfrak{X})$ is a strictly characteristic subgroup of G. We say that a normal subgroup H of G is $\mathfrak{X}$-central in G if $H \leqslant H_1 (G : \mathfrak{X})$, and we write $H \ \phi \ G$ in this case. The following result relates ϕ to the above discussion.

Theorem 1— (i) ϕ is a subgroup theoretical property.

(ii) ϕ is inherited by homomorphic images.

(iii) ϕ satisfies (*).

(iv) $(Y/\gamma^\phi (Y, X)) \ \phi \ (X/\gamma^\phi (Y, X))$ is not true in general.

(v) $\zeta^\phi (X) \ \phi \ X$ is always valid.

PROOF : All parts of the Theorem except (iv) follow easily from the results in Stanley (1970) and the definitions. To prove (iv) we take $\mathfrak{X}$ to be $\mathscr{F}$, the class of finite groups, and

$$X = \langle x, a,b \,;\, [a,b] = 1,\ a^x = ab,\ b^x = a \rangle .$$

Then $H_1(X : \mathscr{F}) = 1$. Let $Y = \langle a, b \rangle$, a free abelian normal subgroup of X of rank 2. For any nonnegative integer r, Y^r is a normal subgroup of X and $(Y/Y^r)\ \phi\ (X/Y^r)$. Thus $\gamma^{\phi}(Y, X) \leqslant \bigcap_{r=1}^{\infty} Y^r = 1$. But $Y\ \phi\ X$ is not true.

The classes $\overset{\circ}{\phi}$ and $\overset{\prime}{\phi}$ have been discussed in Stanley (1970) where they were denoted by $\mathfrak{X}(\infty)$ and $\overline{\mathfrak{X}}$ respectively. Also the upper ϕ-central series was called the upper $\mathfrak{X}$-central series and its terms were denoted by $H_\alpha(G : \mathfrak{X})$ for an ordinal number α. $\mathfrak{X}(\infty)$ may be regarded as the union of all the classes $\mathfrak{X}(c)$ for positive integers c, where $G \in \mathfrak{X}(c)$ if $G = H_c(G : \mathfrak{X}) > H_{c-1}(G : \mathfrak{X})$.

Arrell (unpublished) has defined the following subgroups of a group G for a class of groups $\mathfrak{X}$.

$$D_1(G : \mathfrak{X}) = G,\ D_{\alpha+1}(G : \mathfrak{X}) = \bigcap \{[D_\alpha(G : \mathfrak{X}), N] \,;\, N \in N(G : \mathfrak{X})\},$$

$$D_\lambda(G : \mathfrak{X}) = \bigcap_{\beta<\lambda} D_\beta(G : \mathfrak{X}),$$

where α is an ordinal number and λ is a limit ordinal number. These subgroups form a descending chain of normal subgroups of G, and if $\mathfrak{X}$ is closed under the formation of subgroups and quotients they are actually fully invariant. Arrell calls this chain the lower $\mathfrak{X}_d$-central series, and a class of groups $\check{\mathfrak{X}}(c)$ for a positive integer c is defined by $G \in \check{\mathfrak{X}}(c)$ if $1 = D_{c+1}(G : \mathfrak{X}) < D_c(G : \mathfrak{X})$. We call a normal subgroup H of G an $\mathfrak{X}_d$-central subgroup if

$$\bigcap \{[H, N] \ ;\ N \in N(G : \mathfrak{X})\} = 1.$$

We write $H\ \psi\ G$ when this is the case. Notice that when $\mathfrak{X}$ is the trivial class then $\phi = \psi =$ 'is a central subgroup of'. Comparable with Theorem 1 we have

Theorem 2— (i) ψ is a subgroup theoretical property.

(ii) ψ is not inherited by homomorphic images.

(iii) ψ satisfies ($*$).

(iv) $(Y/\gamma^{\psi}(Y, X))\ \psi\ (X/\gamma^{\psi}(Y, X))$ always holds.

(v) $\zeta^{\psi}(X)\ \psi\ X$ is not true in general.

PROOF : (i) Clearly $1\ \psi\ G$. Let θ be an isomorphism out of G, and suppose $H\ \psi\ G$. Let $x \in [H^\theta, N]$ for each $N \in N(G^\theta : \mathfrak{X})$. Such an N has the form N_1^θ for some $N_1 \in N(G : \mathfrak{X})$. Thus $x \in [H, N_1]^\theta$ for each N_1, and since θ is one-to-one and $H\ \psi\ G$ we see that $x = 1$. Consequently $H^\theta\ \psi\ G^\theta$.

(ii) We take $\mathfrak{X} = \mathscr{F}$, the class of finite groups and let G be a non-abelian radicable group. Then $G \cong F/R$ for some free group F and a proper normal subgroup R of F. Now $F\ \psi\ F$ because free groups are residually finite. On the other hand, since a radicable group cannot have a proper normal subgroup of finite index, we have that $\bigcap \{[G, N] \,;\, N \in N(G : \mathscr{F})\} = G' \neq 1$, so that $G\ \psi\ G$ is not true.

(iii) Let N, X and Y be normal subgroups of G with $Y \leqslant X$ and $(X/Y)\psi(G/Y)$. Let $x\,(Y \cap N)$ be an element of

$$\cap\left\{\left[\frac{X\cap N}{Y\cap N}, \frac{K}{Y\cap N}\right];\ \frac{K}{Y\cap N} \in N\left(\frac{G}{Y\cap N} : \mathfrak{X}\right)\right\}.$$

Without loss of generality we may suppose that $x \in [X\cap N, K]$ for all such K, and so $x \in N \cap [X, K]$. Thus $x\, Y \in [X/Y, KY/Y]$ for all such K. Let $L/Y \in N\,(G/Y, \mathfrak{X})$. Then $L \lhd G$, $G/L \in \mathfrak{X}$ and $Y \leqslant L$, so that $Y\cap N \leqslant L$. Also $L/(Y\cap N) \in N\,(G/(Y\cap N) : \mathfrak{X})$ and so $x\, Y \in [X/Y, L/Y]$ for all such L. But $(X/Y)\,\psi\,(G/Y)$, which shows that $x \in Y$ and so $x \in Y\cap N$, as required.

(iv) Let $\Gamma = \gamma^{\psi}(Y, X)$ and choose $M \lhd X$ such that $M \leqslant Y$ and $(Y/M)\,\psi\,(X/M)$. We write $I = \cap\,\{[Y/\Gamma, K/\Gamma]\,;\, K/\Gamma \in N\,(X/\Gamma : \mathfrak{X})\}$, and select $x\Gamma \in I$. If $L/M \in N(X/M : \mathfrak{X})$ then $L/\Gamma \in N(X/\Gamma : \mathfrak{X})$, and so $x\,\Gamma \in [Y/\Gamma, L/\Gamma]$ for all such L. Thus $x\, M \in [Y/M, L/M]$ for all such L. Consequently, since $(Y/M)\,\psi\,(\mathfrak{X}\, Y/M)$ we have that $x \in M$. Thus $I \leqslant M/\Gamma$ for any $M \lhd X$ for which $M \leqslant Y$ and $(Y/M)\,\psi\,(X/M)$. This shows that $I = 1$, and so $(Y/\Gamma)\,\psi\,(X/\Gamma)$.

(v) Let A be a group of type 2^{∞} generated by $a_1, a_2, \ldots$ where $a_n^2 = a_{n-1}$ $(n = 1, 2, \ldots)$ and where $a_0 = 1$. An automorphism t of A is defined by $a^t = a^3$ $(a \in A)$. Let $T = \langle t \rangle$ and let G be the split extension of A by T. Take $\mathfrak{X} = \mathfrak{F}$ and suppose $H\,\psi\, G$. For a normal subgroup N of finite index in G we have $A \leqslant N$, so that $[H, A] = 1$. Thus $H \leqslant C_G\,(A) = A$, which shows that $\zeta^{\psi}\,(G) \leqslant A$. Now for any r we have $[a_r, t^n] = a_r^{3n-1}$. But the order of a_r is 2^r and there exists an integer $n \geqslant 1$ such that 2^r divides $3^n - 1$. Thus $[a_r, t^n] = 1$ for some n and so $[a_r, AT^n] = 1$. But AT^n has finite index in G, so that $\langle a_r \rangle\,\psi\, G$ for each r. Thus $A \leqslant \zeta^{\psi}\,(G)$ and so $\zeta^{\psi}(G) = A$. But $[A, N] = A$ for any normal subgroup N of finite index in G, because A is radicable. Thus $A\,\psi\, G$ is not true.

By an induction on α we see that $D_{\alpha}\,(G : \mathfrak{X}) = \gamma_{\alpha}^{\psi}(G)$. The classes $\overset{\circ}{\psi}$ and $\acute{\psi}$ have been discussed by Arrell (unpublished) when they were denoted by $\check{\mathfrak{X}}\,(\infty)$ and $\check{\mathfrak{X}}$ respectively.

Although sometimes unnecessary, we will invariably assume henceforth that the class of groups $\mathfrak{X}$ relative to which ϕ and ψ are defined is $\langle S, Q, D_0 \rangle$- closed.

A subgroup theoretical property χ gives rise to another class of groups, the class of residually χ-groups, $\overset{*}{\chi}$, where $G \in \overset{*}{\chi}$ if given $1 \neq x \in G$ there exists $N \lhd G$ such that $x \notin N$ and $(x^G N/N)\,\chi\,(G/N)$. The class $\overset{*}{\phi}$ was considered by Durbin (1968) and Ayoub (1969) when $\mathfrak{X} = 1$, and for an arbitrary $\langle S, Q, D_0 \rangle$-closed class $\mathfrak{X}$ by Stanley (1972), where it was denoted by $\overset{*}{\mathfrak{X}}$. The class $\overset{*}{\phi}$ has also been considered by Newell (1970).

Now if χ is a subgroup theoretical property that satisfies (*) then $\hat{\chi} \leqslant \overset{*}{\chi}$. For, let $G \in \chi'$ and suppose that A/B is a factor of a χ-series of G. If $x \in A - B$ then

$x^G B \lhd G$ and $x^G B \leqslant A$. Then, using (*) we have that $(x^G B/B)\, \chi\, (G/B)$. Consequently $G \in \overset{*}{\chi}$.

We are concerned with the classes $\overset{\circ}{\phi}$, $\overset{\circ}{\psi}$; $\dot{\phi}$, $\dot{\psi}$; $\dot{\phi}$, $\dot{\psi}$; $\hat{\phi}$, $\hat{\psi}$; $\overset{*}{\phi}$, $\overset{*}{\psi}$, which when $\mathfrak{X} = 1$ coincide in pairs with the classes of nilpotent, hypercentral, hypocentral, Z- and residually central groups respectively. The next two Theorems show that coincidence in two cases takes place for a general $\langle S, Q, D_0 \rangle$ - closed class $\mathfrak{X}$.

Theorem 3—$\overset{*}{\phi} = \overset{*}{\psi}$.

PROOF: $G \in \overset{*}{\phi}$ and $1 \neq x \in G$ then $x^G N/N \leqslant H_1(G/N : \mathfrak{X})$ for some $N \lhd G$ with $x \not\in N$. Thus there exists $M \lhd G$ with $N \leqslant M$ such that $[x^G, M] \leqslant N$ and $G/M \in \mathfrak{X}$. Then $[x^G N/N, M/N]=1$ and $M/N \in N(G/N : \mathfrak{X})$, and so we have $(x^G N/N)\, \psi\, (G/N)$. This shows that $G \in \overset{*}{\psi}$.

Now suppose that $G \in \overset{*}{\psi}$ and let $1 \neq x \in G$. Then for some $N \lhd G$ with $x \not\in N$ we have $(x^G N/N)\, \psi\, (G/N)$. So for some $K/N \in N(G/N ; \mathfrak{X})$ we have $xN \not\in [x^G N/N, K/N]$. Let $M = N[x^G N, K] \lhd G$. Then $N \leqslant M \leqslant K$ and $x^G M = x^G N$. Also $[x^G M/M, K/M]=1$ and $K/M \in N(G/M : \mathfrak{X})$ since $K \in N(G : \mathfrak{X})$. This shows that $(x^G M/M) \phi (G/M)$. Finally, if $x \in M$ then $xN \in M/N = [x^G N/N, K/N]$, which is a contradiction. Thus $x \not\in M$ and this argument shows that $G \in \overset{*}{\phi}$, completing the proof.

Theorem 4— $\hat{\phi} = \hat{\psi}$.

PROOF: Firstly we note the following : for normal subgroups X, Y and Z of a group G with $Y \leqslant Z \leqslant X$ then if $(X/Y)\, \psi\, (G/Y)$ then $(Z/Y)\, \psi\, (G/Y)$, while if $(X/Y)\, \phi\, (G/Y)$ then $(X/Z)\, \phi\, (G/Z)$.

Let $G \in \hat{\phi}$ and let A/B be a factor of a ϕ-series of G. We define an ascending ψ-series $\{G_\alpha\}$ of normal subgroups of G between B and A such that $(A/G_\alpha)\, \phi\, (G/G_\alpha)$ for all ordinals α. Let $G_0 = B$ and suppose G_β has been defined for all ordinals $\beta < \alpha$ satisfying this condition. If α is a limit ordinal, let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta \lhd G$. Then $(A/G_\alpha)\, \phi\, (G/G_\alpha)$. If $\alpha-1$ exists, let $x \in A - G_{\alpha-1}$, if possible. Now $(A/G_{\alpha-1})\, \phi\, (G/G_{\alpha-1})$ and so there exists $N \lhd G$ with $G_{\alpha-1} \leqslant N$, $[x^G, N] \leqslant G_{\alpha-1}$ and $G/N \in \mathfrak{X}$. Let $G_\alpha = x^G G_{\alpha-1}$. Then $G_\alpha \lhd G$, $G_{\alpha-1} < G_\alpha$ and $[G_\alpha, N] \leqslant G_{\alpha-1}$. Thus, since $N/G_{\alpha-1} \in N(G/G_{\alpha-1} : \mathfrak{X})$ we have $(G_\alpha/G_{\alpha-1})\, \psi\, (G/G_{\alpha-1})$. Also $(A/G_\alpha)\, \phi\, (G/G_\alpha)$. Consequently G has a ψ-series and so $\hat{\phi} \leqslant \hat{\psi}$.

Let $G \in \hat{\psi}$ and let A/B be a factor of a ψ-series of G. We define a descending ϕ-series $\{G_\alpha\}$ of normal subgroups of G between A and B such that $(G_\alpha/B)\, \psi\, (G/B)$ for all ordinals α. Let $G_0 = A$ and suppose G_β has been defined for all ordinals $\beta < \alpha$ satisfying this condition. If α is a limit ordinal, let $G_\alpha = \bigcap_{\beta<\alpha} G_\beta \lhd G$. Then $(G_\alpha/B)\, \psi\, (G/B)$. If $\alpha-1$ exists, let $x \in G_{\alpha-1} - B$, if possible. Then $xB \not\in [G_{\alpha-1}/B, N/B]$ for some $N/B \in N(G/B : \mathfrak{X})$. Let $G_\alpha = B[G_{\alpha-1}, N]$.

Then $G_\alpha \lhd G$ and $B \leqslant G_\alpha \leqslant G_{\alpha-1}$. If $G_{\alpha-1} = G_\alpha$ then $xB \in G_\alpha/B = [G_{\alpha-1}/B, N/B]$, which is a contradiction. Thus $G_\alpha < G_{\alpha-1}$. Now $[G_{\alpha-1}/G_\alpha, N/G_\alpha] = 1$ and $N/G_\alpha \in N(G/G_\alpha : \mathscr{X})$, so that $(G_{\alpha-1}/G_\alpha)\,\phi\,(G/G_\alpha)$. Also $(G_\alpha/B)\,\psi\,(G/B)$. Consequently G has a ϕ-series and so $\hat{\psi} \leqslant \hat{\phi}$. This establishes the result.

Notice that if we restrict the argument of the first part of the proof of Theorem 4 to $\acute{\phi}$ then we obtain $\acute{\phi} \leqslant \acute{\psi}$. Restricting the second part of the proof to $\acute{\psi}$ yields $\acute{\psi} \leqslant \acute{\phi}$. Also, using Theorems 1 and 2, we have $\hat{\phi} \leqslant \overset{*}{\phi}$. The classes of groups under consideration are shown in Fig. 1.

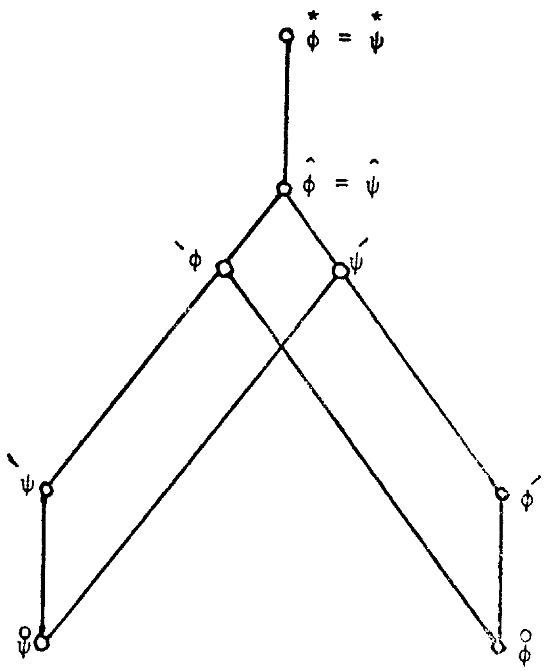

Fig. 1.

Taking $\mathscr{X} = 1$ illustrates the following proper inclusions:

$$\mathring{\psi} < \acute{\psi}, \ < \mathring{\phi} < \acute{\phi}, \acute{\phi} < \hat{\phi}, \acute{\psi} < \hat{\psi}, \ \hat{\phi} < \overset{*}{\phi},$$

the last one following because Phillips and Roseblade (1978) have found recently an example of a residually central group possessing no central series. We produce two examples to illustrate that all the other inclusions in the diagram are proper (which cannot be deduced by taking $\mathscr{X} = 1$), and that there are no other inclusions.

Firstly, taking X to be the group of Theorem 1 (iv), then X is a polycyclic group and so is residually finite. Thus with $\mathscr{X} = \mathscr{F}$ we see that $X \in \mathring{\psi}$. But $H_1(X : \mathscr{F}) = 1$ and so $X \notin \acute{\phi}$. Secondly, taking G to be the group of Theorem 2 (v) we have that $A \leqslant H_1(G : \mathscr{F})$, so that $G = H_2(G : \mathscr{F})$ and $G \in \mathring{\phi}$. Now if N is a normal subgroup of finite index in G then $A = [A, N] \leqslant D_2\,(G : \mathscr{F})$. Consequently $A \leqslant D_\alpha$ $(G : \mathscr{F})$ for any ordinal number α. Thus $G \notin \acute{\psi}$.

3. Finiteness Conditions

We consider now the various classes of groups defined above subject to the minimal and maximal conditions on normal subgroups, Min-n and Max-n respectively.

If G is a group satisfying Min-n and R is its $\mathscr{X}$-residual then G/R is an $\mathscr{X}$-group. Arrell (unpublished) has shown that if $G/R \in \mathscr{X}$ then $G \in \mathring{\phi}$ if and only if $G \in \mathring{\psi}$. Thus $\mathring{\phi} = \mathring{\psi}$ for groups satisfying Min-n, and this class coincides with $\mathfrak{N}\mathscr{X}$, where $\mathfrak{N}$ is the class of nilpotent groups, as is shown in Stanley (1970).

We now prove

Theorem 5—For groups satisfying Min-n we have that

(i) a ψ-series is a ϕ-series.

(ii) $\dot{\phi} = \acute{\psi} = \mathfrak{N}\mathscr{X} = \mathring{\phi} = \mathring{\psi} < \acute{\phi} = \dot{\psi} = \hat{\phi} = \hat{\psi} = \overset{*}{\phi} = \overset{*}{\psi}$.

Proof : (i) Let A/B be a factor of a ψ-series of the group G satisfying Min-n. Suppose $x \in A - B$. Then $[x, R] \leqslant [A, R] \leqslant B$ because $(A/B)\,\psi\,(G/B)$. This means that $x\,B \in H_1\,(G/B : \mathscr{X}$ and shows that $A/B\ \phi\ G/B$.

(ii) We see by (i) that $\acute{\phi} = \acute{\psi}$. Stanley (1973) has shown that $\overset{*}{\phi}$-groups satisfying Min-n, are precisely ZA-by-$\mathscr{X}$-groups satisfying Min-n, so that $\overset{*}{\phi} \leqslant \acute{\phi}$. It remains to observe that $\mathring{\phi} < \acute{\phi}$ is possible for groups with Min-n because of the existence of the locally dihedral 2-group.

We now turn to groups satisfying Max-n. We need a preliminary result, and for brevity we write $D_\alpha(X) = D_\alpha(X : \mathscr{X})$ for a group X and $N(X) = N(X : \mathscr{X})$

Lemma—If θ is a homomorphism out of G then $D_\alpha(G)^\theta \leqslant D_\alpha(G^\theta)$ for each ordinal number α.

Proof: The inclusion is clear if $\alpha = 1$. Suppose it is true for all ordinals $\beta < \alpha$. Again, the result is clear if α is a limit ordinal, so suppose that $\alpha - 1$ exists. Let $y \in D_\alpha(G)^\theta$, so that $y = x^\theta$ for some $x \in D_\alpha(G)$. Thus $x \in [D_{\alpha-1}(G), N]$ for all $N \in N(G)$. Choose $N_1 \in N(G^\theta)$. Then $N_1{}^{\theta-1} \in N(G)$, so that $x \in [D_{\alpha-1}(G), N_1{}^{\theta-1}]$. Thus we see that $y \in [D_{\alpha-1}(G)^\theta, (N_1{}^{\theta-1})^\theta] \leqslant [D_{\alpha-1}(G^\theta), N_1]$. This is true for all $N_1 \in N(G^\theta)$ and so shows that $y \in D_\alpha(G^\theta)$, as required.

It follows from the Lemma that for $N \lhd G$ we have $D_\alpha(G)N/N \leqslant D_\alpha(G/N)$. Also that each $D_\alpha(G)$ is a fully invariant subgroup of G because an induction shows that $D_\alpha(Y) \leqslant D_\alpha(X)$ whenever $Y \leqslant X$.

Theorem 6—For groups satisfying Max-n we have that

$$\acute{\phi} = \mathfrak{N}\mathscr{X} = \mathring{\phi} < \mathring{\psi} = \acute{\psi}.$$

Proof: Suppose $G \in \mathring{\phi}$ and G satisfies Max-n. We use the notation of Arrell (unpublished) and Stanley (1970). Thus $G \in X(c)$ for some $c \geqslant 0$. We will show that

$G \in \check{\mathfrak{X}}(c)$. If $c=1$ we have, using Max-n, that $[G,N]=1$ for some $N \in N(G)$. Consequently $D_2(G)=1$ and $G \in \check{\mathfrak{X}}(1)$ Suppose $c>1$. Now $G/H_1(G) \in \mathfrak{X}(c-1)$, so that $D_c(G/H_1(G))=1$. Thus $D_c(G) \leqslant H_1(G)$ by the Lemma. Since G satisfies Max-n we know that $D_c(G)$ can be generated by a finite number of its elements and their conjugates in G, so that $[D_c(G),K]=1$ for some $K \in N(G)$. Consequently $D_{c+1}(G)=1$ and $G \in \check{\mathfrak{X}}(c)$. Thus $\overset{0}{\phi}=\overset{0}{\psi}$.

It is known that $\overset{0}{\phi}=\mathfrak{N}\mathfrak{X}$ for groups with Max-n see Stanley (1970). Thus it remains to show that $\overset{0}{\phi}<\overset{0}{\psi}$ is possible. An example is provided by the group $G=\langle a,b;b^a=b^2\rangle$. G is a finitely generated metabelian group which is thus residually finite and satisfies Max-n, by results of Hall (1954, 1959). Thus, with $\mathfrak{X}=\mathfrak{F}$, we have that $G \in \overset{0}{\psi}$. But if $G \in \overset{0}{\phi}$ then G would be a finite extension of a finitely generated and nilpotent group. Such a group satisfies the maximal condition on all subgroups, which is not the case with G.

Finally, we consider the question of the relationship between $\acute{\phi}$ and $L\overset{0}{\phi}$, the class of locally $\overset{0}{\phi}$-groups. It is known that ZA-groups are locally nilpotent and that FC-hypercentral groups are locally FC-nilpotent (McLain 1956). These results show that $\acute{\phi} \leqslant L\overset{0}{\phi}$ when $\mathfrak{X}=1$ and $\mathfrak{X}=\mathfrak{F}$ respectively. We have not been able to decide in general whether $\acute{\phi} \leqslant L\overset{0}{\phi}$, but we have found a result which includes the two already mentioned, and others. This is Proposition 7 below.

Now it is not usually true for an $\langle S, Q, D_0\rangle$-closed class $\mathfrak{X}$ of groups that if finitely generated $\mathfrak{X}$-groups are finitely presented then finitely generated $\mathfrak{X}(\infty)$-groups are finitely presented, as the example $C_\infty wr C_\infty$ shows when $\mathfrak{X}$ is taken to be the class of abelian groups this is a metabelian group with Max-n which is not finitely presented (see Hall 1954).

Proposition 7—If $\mathfrak{X}$ is an $\langle S,Q,D_0\rangle$-closed class of groups such that finitely generated $\mathfrak{X}(\infty)$-groups are finitely presented, then $\overline{\mathfrak{X}} \leqslant L(\overline{\mathfrak{X}}(\infty))$.

PROOF: Suppose $G \in \overline{\mathfrak{X}}$ and let H be a finitely generated subgroup of G. Now there exists ordinal numbers α for which $H/(H \cap H_\alpha(G:X)) \in \mathfrak{X}(\infty)$ because $G=H_\delta(G:X)$ for some δ. Let $\alpha \geqslant 0$ be least with the former property. Suppose that $\alpha \neq 0$ and $\alpha-1$ exists. Then

$$\frac{H \cap H_\alpha(G:\mathfrak{X})}{H \cap H_{\alpha-1}(G:\mathfrak{X})} \leqslant H_1\left(\frac{H}{H \cap H_{\alpha-1}(G:\mathfrak{X})}:\mathfrak{X}\right),$$

and so $H/(H \cap H_{\alpha-1}(G:\mathfrak{X})) \in \mathfrak{X}(\infty)$ by Corollary 6 of Stanley (1970). This contradicts the choice of α. So α must be a limit ordinal. Since H is finitely generated and $H/(H \cap H_\alpha(G:\mathfrak{X}))$ is finitely presented, $H \cap H_\alpha(G:\mathfrak{X})$ can be generated by a finite number of elements and their conjugates in H. Thus for some $\beta < \alpha$ we have

$H \cap H_\alpha(G:\mathfrak{X}) = H \cap H_\beta(G:\mathfrak{X})$, a contradiction. The only remaining possibility is that $\alpha = 0$ and $H \in \mathfrak{X}(\infty)$. It follows that $G \in L(\mathfrak{X}(\infty))$.

References

Arrell, D.G. (unpublished). Some remarks on $\mathfrak{X}$-central series in groups.

Ayoub, C. (1969). On properties possessed by solvable and nilpotent groups. *J. Aust. Math. Soc.*, **9** 218–27.

Durbin, J. (1968). Residually central elements in groups. *J. Algebra*, **9**, 408–13.

Hall, P. (1954). Finiteness conditions for soluble groups. *Proc. Lond. Math. Soc.* (3), **4**, 419–36.

——(1959). On the finiteness of certain soluble groups. *Proc. Lond. Math. Soc.* (3), **9**, 595–622.

McLain, D.H. (1956). Remarks on the upper central series of a group. *Proc. Glasgow Math. Assoc.*, **3**, 38–44.

Newell, M.L. (1970). On normal coverings of groups. *Arch. Math. (Basel)*, **21**, 337–43.

Petty, J. V. (1976). Classes of groups defined by series or normal factor coverings. *J. Algebra*, **40**, 610–17.

Phillips, R. E., and Plotkin, J. M. (1972). On factor coverings of groups. *Coll. Math.*, **33**, 175–87.

Phillips, R. E., and Roseblade, J.E. (1978). A residually central group that is not a Z-group. *Mich. Math. J.*, **25**, 233–34.

Robinson, D.J.S. (1972). Finiteness Conditions and Generalized Soluble Groups. Springer-Verlag, Berlin.

Stanley, T.E. (1970). Generalizations of the classes of nilpotent and hypercentral groups. *Math. Z.*, **118**, 180–90.

——(1972). Residual $\mathfrak{X}$-centrality in groups. *Math. Z*, **126**, 1–5.