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A NEW TYPE OF (1,2)-OPTIMAL CODES OVER GF(2)*

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In this communication, we study $(n_1 + n_2, k)$ linear codes that are optimal in a specific sense.

1. INTRODUCTION

The problem of the existence of perfect codes has been a good exercise for mathematicians for the last several years. This problem was finally settled by Tietavanien and Perko (1971) by showing that there is no perfect code other than the single error correcting Hamming, double and triple error correcting Golay and the repetetive codes. Therefore, there have been attempts in searching out codes which are not perfect in the usual sense but are of the type that for a given set of error patterns, these correct all such errors and no more. An attempt in this direction was made by Sharma and Dass (1977) by finding out 'Adjacent Error Correcting Perfect Codes' in the binary case.

The authors obtained a lower bound over the necessary number of parity-check digits for an $(n = n_1+n_2, k)$ linear code that corrects bursts of length b_1(fixed) in the first block of length n_1 and bursts of length b_2 (fixed) in the second block of length n_2 (see Dass and Tyagi 1980). A burst of length b(fixed) has been considered as an n-tuple whose only non-zero components are confined to b consecutive positions, the first of which is non-zero and the number of its starting positions is $(n-b+1)$. This definition is a modification due to Dass (1980) over the definition of a burst considered by Chien and Tang (1965) and has been found useful in error analysis experiments on telephone lines (Alexander *et al.* 1960) and in some space channel models in which an amplitude modulated carrier is generated aboard a satellite and transmitted to an earth antenna. The result proved in Dass and Tyagi (1980) is as follows :

Theorem—The number of parity-check-digits in an (n,k) linear code correcting all bursts of length b_1(fixed) in the first block of length n_1 and all bursts of length b_2(fixed) in the second block of length $n_2 (n = n_1+n_2)$ is atleast

$$\text{Log}_q[1 + (n_1-b_1+1)q^{b_1-1} + (n_2-b_2+1)q^{b_2-1}] (q-1). \quad ...(1)$$

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Whenever one obtains a bound, it is desirable to examine as to for which values of the parameters, the bound is tight. In this communication, we shall show that if we fix up $b_1 = 1$ and $b_2 = 2$, we can obtain optimal codes. The codes are optimal in the sense that these can be used to correct all single errors in the first block of length n_1 and all bursts of length 2(fixed) in the second block of length n_2 and no more. It is shown that for $n_1+n_2 \leqslant 50$, such codes exist for all possible values of the parameters. We shall call such codes to be (1,2) optimal codes.

The codes considered are (n,k) linear codes over $GF(q)$; n is the length of the code and k is the number of information digits.

2. Optimal Codes

Consider the inequality in (1) for the binary case with equality for $b_1 = 1$ and $b_2 = 2$. We get

$$2^{n_1+n_2-k} = 2n_2 + n_1-1. \qquad \ldots(2)$$

We now examine the possibilities of the existence of codes for the values of n_1, n_2 and k satisfying (2) such that $n_1+n_2 \leqslant 50$.

For this, we first note that the values of n_1 satisfying (2) should always be odd. We shall find out all possible values of n_2 and k by assigning values to n_1 as 1,3,5,... successively.

Let $n_1 = 1$. The values of n_2 and k satisfying (2) such that $n_1 + n_2 \leqslant 50$ are $(n_2,k) = (2,1)$, $(4,2)$, $(8,5)$, $(16,12)$, and $(32,27)$ which shows the possibility of the existence of $(1+2, 1)$, $(1+4,2)$, $(1+8,5)$, $(1+16,12)$, $(1+32,27)$ codes which may be (1,2)-optimal. Consider the matrices in the following examples:

Examples

(i) $$\begin{bmatrix} 1 & 01 \\ 0 & 10 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 0 & 0010 \\ 1 & 0101 \\ 1 & 1001 \end{bmatrix}$$

(iii) $$\begin{bmatrix} 1 & 00010101 \\ 1 & 00101111 \\ 0 & 01001101 \\ 1 & 10001011 \end{bmatrix}$$

(iv) $$\begin{bmatrix} 0 & 1000100100101101 \\ 0 & 0001001101010100 \\ 0 & 0010011110101001 \\ 1 & 0100011001011100 \\ 1 & 1000010010010110 \end{bmatrix}$$

(v) $$\begin{bmatrix} 1 & 100100010001000000100101010101101 \\ 1 & 00010010001100101001110101011000 \\ 1 & 10000100011101010111010000101001 \\ 1 & 00001000111010101110100010011110 \\ 0 & 00110000110001011001001011101010 \\ 0 & 01100000100010010010010101010101 \end{bmatrix}$$

These matrices [Example (i)—(v)] considered as parity-check matrices for a code give rise to (1+2,1), (1+4,2), (1+8,5), (1+16,12) and (1+32,27) codes. Moreover, these codes can correct all bursts of length 1 in the first block of length 1 and all bursts of length 2 (fixed) in the second block of lengths 2,4,8,16 and 32 respectively and no other error-pattern. For the sake of illustration, we list in Table I the error patterns and their syndromes for the (1+8,5) code which is the null space of the matrix given in Example (iii). The syndromes being altogether different mean that the code under discussion is an (1,2)-optimal code.

TABLE I

Error Pattern	*Syndrome*
100000000	1101
011000000	0011
010000000	0001
001100000	0110
001000000	0010
000110000	1100
000100000	0100
000011000	1111
000010000	1000
000001100	1001
000001000	0111
000000110	1011
000000100	1110
000000011	1010
000000010	0101

Let $n_1 = 3$. We see that the values of n_2 and k such that $n_1 + n_2 \leqslant 50$ and satisfying (2) may give rise to (3+3,3), (3+7,6), (3+15,13) and (3+31,28) codes. We give below the parity-check matrix for the (3+31,28) code and the matrices for the remaining three cases can be written down by a procedure to be given later.

Example (vi)—For a (3+31,28) binary code, the following can be regarded as parity-check matrix.

$$\begin{bmatrix} 110 & 1001000100010000010010101010110 \\ 100 & 0001001000110010100111010101100 \\ 111 & 1000010001110101011101000100111 \\ 011 & 0011000011000101100100101110101 \\ 010 & 0110000010001001001001010101010 \end{bmatrix}$$

As before, one can easily verify that the codes which are null spaces of these matrices correct all bursts of length 1 in the first block of length 3 and all bursts of length 2(fixed) in the second block of length 3,7,15 and 31 respectively and no other error-pattern.

Again $n_1 = 5$, the corresponding values of n_2 and k are given in Table II.

TABLE II

n_2	k
2	4
6	7
14	14
30	29

We give below the parity-check matrix for the $(5+30,29)$ code. The matrices for the remaining three cases can be constructed by the procedure to be given later.

Example (viii)—For $n_1 = 5$, $n_2 = 30$ and $k = 29$, the requisite code is $(5+30, 29)$ whose parity-check matrix may be considered as:

$$\begin{bmatrix} 11011 & 100100010001000000100101010101011 \\ 10000 & 000100100011001010011101010110 \\ 11000 & 100001000111010101110100001010 \\ 11101 & 000010001110101011101000100111 \\ 01110 & 001100001100010110010010111010 \\ 01011 & 011000001000100100100101010101 \end{bmatrix}$$

and as pointed out earlier, one can easily verify that the codes which are null spaces of these matrices correct all bursts of length 1 in the first block and all burst of length 2 (fixed) in the second block.

We now give in Table III all possible suitable values of n_2 and k for fixed value of n_1.

TABLE III

n_1	n_2	k
7	5	8
	13	15
	29	30
9	4	9
	12	16
	28	31
11	3	10
	11	17
	27	32
13	2	11
	10	18
	26	33
15	9	19
	25	34
17	8	20
	24	35
19	7	21
	23	36
21	6	22
	22	37
23	5	23
	21	38
25	4	24
	20	39
27	3	25
	19	40
29	2	26
	18	41
31	17	42
33	16	43
35	15	44

The parity-check matrices of (7+29,30), (9+28,31), (11+27,32), (13+26,33), (15+25,34), (17+24,35), (19+23,36), (21+22,37) and (23+21,38) codes are given below as these do not follow the procedure to be given later.

For (7+29,30) code, the parity-check matrix is

$$\begin{bmatrix} 1001011 & 00100010001000100100101010101 \\ 0010101 & 00100100011001001001110101011 \\ 1011101 & 00001000111010110111010000101 \\ 0111011 & 00010001110101001110100010011 \\ 0111101 & 01100001100010011001001011101 \\ 1011110 & 11000001000100010010010101010 \end{bmatrix}$$

For (9+28,31) code, the parity-check matrix is

$$\begin{bmatrix} 100101110 & 0010001000100010010010101010 \\ 001010101 & 0010010001100100100111010101 \\ 101110110 & 0000100011101011011101000010 \\ 011101101 & 0001000111010100111010001001 \\ 011110110 & 0110000110001001100100101110 \\ 101111011 & 1100000100010001001001010101 \end{bmatrix}$$

For (11+27,32) code, the parity-check matrix is

$$\begin{bmatrix} 10010111011 & 001000100010001001001010101 \\ 00101010101 & 001001000110010010011101010 \\ 10111011011 & 000010001110101101110100001 \\ 01110110101 & 000100011101010011101000100 \\ 01111011011 & 011000011000100110010010111 \\ 10111101101 & 110000010001000100100101010 \end{bmatrix}$$

For (13+26,33) code, the parity-check matrix is

$$\begin{bmatrix} 0111111110011 & 00000100010001001001010101 \\ 1000111101110 & 00001000110010010011101010 \\ 1011000111011 & 00010001110101101110100001 \\ 0101001101110 & 00100011101010011101000100 \\ 1001011111011 & 01000011000100110010010111 \\ 1111100011110 & 10000010001000100100101010 \end{bmatrix}$$

For (15+25,34) code, the parity-check matrix is

$$\begin{bmatrix} 011111011101101 & 0000010001000100100101010 \\ 100011111011011 & 0000100011001001001110101 \\ 101100001111101 & 0001000111010110111010000 \\ 010100111011000 & 0010001110101001000100010 \\ 100101011111110 & 0100001100010011001001011 \\ 111110100111011 & 1000001000100010010010101 \end{bmatrix}$$

For (17+24,35) code, the parity-check matrix is

$$\begin{bmatrix} 01111111101100111 & 000001000100010010010101 \\ 10001111011011110 & 000010001100100100111010 \\ 10110001111100100 & 000100011101011011101000 \\ 01010011011010011 & 001000111010100111010001 \\ 10010111111101001 & 010000110001001100100101 \\ 11111000111011110 & 100000100010001001001010 \end{bmatrix}$$

For (19+23,36) code, the parity-check matrix is

$$\begin{bmatrix} 01111111100111011101 & 00000100010001001001010 \\ 10001111011101110111 & 00001000110010010011101 \\ 10110001110111000001 & 00010001110101101110100 \\ 01010011011100011101 & 00100011101010011101000 \\ 10010111110110101101 & 01000011000100110010010 \\ 11111000111101110111 & 10000010001000100100101 \end{bmatrix}$$

For (21+22,37) code, the parity-check matrix is

$$\begin{bmatrix} 011111111001110111011 & 0000010001000100100101 \\ 100011110111011101101 & 0000100011001001001110 \\ 101100011101110000000 & 0001000111010110111010 \\ 010100110111000111000 & 0010001110101001110100 \\ 100101111101101011011 & 0100001100010011001001 \\ 111110001111011101111 & 1000001000100010010010 \end{bmatrix}$$

For (23+21,38) code, the parity-check matrix is

$$\begin{bmatrix} 01111111100111011101111 & 000001000100010010010 \\ 10011111011101110110111 & 000010001100100100111 \\ 10110001110111000000011 & 000100011101011011101 \\ 01010011011100011100000 & 001000111010100111010 \\ 10010111110110101101110 & 010000110001001100100 \\ 11111000111101110110111 & 100000100010001001001 \end{bmatrix}$$

3. Construction of (1,2)-Optimal Codes

In the previous section, we have considered codes for all possible values of n_1, n_2 and k for which $n_1+n_2 \leqslant 50$ and we have seen that these codes correct all bursts of length 1 in the first block of length n_1 and all bursts of length 2(fixed) in the second block of length n_2 and no more and therefore, are called (1,2)-optimal (n_1+n_2, k) linear codes.

For the construction of the parity-check matrices of these codes, it is sufficient to construct the 2nd block because of the following reason :

Suppose the second block of length n_2 that corrects all bursts of length 2 (fixed) is constructed. This means that the syndromes of the first n_2-1 single error patterns and all double adjacent error patterns are different. (The last and the first component is not taken as an adjacent error pattern). If the number of all such syndromes is deleted from the total non-zero $(n-k)$-tuples, we remain with exactly $n_1(n-k)$-tuples. Since we are correcting only single errors in the first block, the remaining $(n-k)$-tuples can be taken as the columns of the first block irrespective of their order.

The construction of the 2nd block is as follows: Select any non-zero $(n-k)$-tuple as the first column of the parity-check-matrix H. Subsequent columns are added to H such that after having selected n_2-1 columns $h_1, h_2, \ldots, h_{n_2}-1$, a column h_{n_2} is added provided that

$$h_{n_2} \neq u_{n_2}-1 \; h_{n_2}-1 + v_i h_i + v_{i+1} h_{i+1}$$

where either both v_i and v_{i+1} are zero or if v_s $(s = i$ or $i+1)$ is the last non-zero coefficient, then $2 \leqslant s \leqslant n_2 - 2$; and $u_{n_2}-1 \in GF(2)$.

We now propose a technique of constructing parity-check-matrices for such codes in some special cases: If we choose the first n_2-1 columns of the second block of the parity-check matrix H as the binary representation of the nos. 1,2,4,8,16,32,..., 7,14,28..., 5,10,20,..., 11,22,44,..., 13,26,52..., successively (whenever the lesser columns are required, we shall stop there at) then the last column can be chosen such that its linear combination with the n_2-1th columns is different from syndromes of all single and double error-patterns of the preceeding columns. It is by this technique that we can write down the parity-check matrices of the codes for the cases other than those whose parity-check matrices are given in the previous section. In fact we notice that this technique is applicable

when (i) The number of parity-checks is 3

(ii) The number of parity-checks is 4

(iii) The number of parity-checks is 5 except in examples nos. 4,5

(iv) The number of parity-checks is 6 except when $n_1 < n_2$.

We now propose some open problems:

Problem 1—The existence of such codes has been shown only in the binary case. The existence of such codes in the non-binary case is not known.

Problem 2—The existence of such codes has been shown only when the code length does not exceed 50. We conjecture that such codes exist for all possible values of the parameters satisfying (2).

Problem 3—Can there be a systematic way of constructing these codes for which the above mentioned rule is not applicable? It may be remarked that if we interchange the two blocks, i.e. if we consider (n_2+n_1, k) linear codes, these will form a class of (2,1) optimal codes.

It is hoped that the existence of such codes may prove to be fruitful for the development of the subject as well as from the application point of view.

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