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ON SOME RECENT RESULTS ON COMMON FIXED POINTS

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Common fixed point theorems for four self-mappings P, Q, S and T on a metric space X satisfying $d(Px, Qy) \leqslant \phi(d(Sx, Ty), d(Sx, Qy), d(Px, Ty), d(Px, Sx), d(Ty, Qy))$ for all x, y in X are established, where $\phi: R_+^5 \to R_+$ is u.s.c. and non-decreasing in each coordinate variable and satisfies the condition $\phi(t, t, t, t, t) < t$ for any $t > 0$. Our results extend and unify a multitude of fixed point theorems in metric spaces.

§0. This work was done during September 1979 and that it was sent to Professor Shouro Kasahara, Department of Mathematics, Kobe University, Nada Kobe, Japan 657 for his approval. In his letter dated October 13, 1979 he replied, "I hope that I can add some thing to the paper". Further, in his letter dated December 10, 1979, he pointed out "a serious difficulty" in proofs of Theorems 1 and 2. In his letter dated February 13, 1981 the first author corrected the arguments and requested him to publish this paper in the Mathematics Seminar Notes Kobe University. There was no reply. The first author was shocked to know in May 1981 that Professor Kasahara, the Managing Editor of the Math. Sem. Notes Kobe University, died on September 6, 1980.

§1. Throughout this paper, N stands for the set of positive integers, ω for the set of nonnegative integers and R_+ for the set of all nonnegative real numbers.

Let P, Q, S and T be self-mappings on X, and $d: X \times X \to R_+$. Consider the following condition:

(A) There is an upper semicontinuous (u.s.c.) mapping $\phi: R_+^5 \to R_+$ which is non-decreasing in each coordinate variable and satisfies the condition

$\phi(t, t, t, t, t) < t$ for any $t > 0$, and the inequality

$$d(Px, Qy) \leqslant \phi(d(Sx, Ty), d(Sx, Qy)\, d(Px, Ty), d(Px, Sx), d(Ty, Qy)) \quad \ldots(*)$$

for all x, y in X.

Theorem 1 of Kasahara (1979) has been proved for mappings on L-spaces in the spirit of (*), and our results are also obtained in the spirit of (*) but for metric space-mappings and under different conditions. Moreover, the method of proof is entirely different from that of Kasahara (1979). Results of this paper unify a number of known results and bring out natural extensions of the results of Fisher (1979), Park (1979) and others.

§2. Common fixed point theorems.

Theorem 1—Let P, Q, S and T be mappings from a complete metric space (X,d) to itself satisfying the condition (A) and the following:

(1.1) $PS=SP$, $PT=TP$, $QS=SQ$, $QT=TQ$ and $ST=TS$;

(1.2) there exists a sequence $\{x_n\}_{n\in\omega}$ in X such that

$PTx_{2n} = TSx_{2n+1}$, $QSx_{2n+1} = TSx_{2n+2}$ for all $n\in\omega$ and

$\sup\{d(TSx_i, TSx_j): i,j\in N\} < \infty$;

(1.3) S and T are continuous.

Then P, Q, S and T have a unique common fixed point and $\{TSx_n\}$ converges to the common fixed point.

PROOF : Let $TS=ST=A$. Then $s_n=\sup\{d(Ax_i, Ax_j): i\geqslant n, j\geqslant n\}$ is finite for each $n\in N$. Since $s_{n+1}\leqslant s_n$ for any $n\in N$, $\{s_n\}_{n\in N}$ converges to some $s\geqslant 0$. Let, if possible, $s>0$. If $i\geqslant n+1$, $j\geqslant n+1$ and i is odd and j even, then from (*),

$$\begin{aligned} d(Ax_i,Ax_j)&=d(PTx_{i-1}, QSx_{j-1})\\ &\leqslant\phi(d(Ax_{i-1}, Ax_{j-1}), d(Ax_{i-1}, Ax_j), d(Ax_i, Ax_{j-1}),\\ &\qquad d(Ax_i, Ax_{i-1}), d(Ax_{j-1}, Ax_j)),\end{aligned}$$

that is $s_{n+1}\leqslant\phi(s_n, s_n, s_n, s_n, s_n)$ for all $n\ (>1)\in N$. So by the u.s. continuity of ϕ,

$$s\leqslant\phi(s, s, s, s, s)<s,$$

a contradiction to $s>0$. Thus $s=0$. Consequently $\{Ax_n\}$ is a Cauchy sequence. By the hypothesis, $\{Ax_n\}$ converges to some $z\in X$, and $\{PTx_{2n}\}$ and $\{QSx_{2n+1}\}$ being the subsequences converge to the same point z. By (1.1) and the continuity of S and T, $PTSx_{2n_i}=SPTx_{2n_i}\to Sz$, $STSx_{2n_i}\to Sz$, $QTSx_{2n_j+1}=TQSx_{2n_j+1}\to Tz$ and $TTSx_{2n_j+1}\to Tz$ for some subsequences $\{n_i\}_{i\in N}$ and $\{n_j\}_{j\in N}$ of $\{n\}_{n\in N}$.

Therefore it follows from

$$\begin{aligned} d(PTSx_{2n_i}, QTSx_{2n_j+1})&\leqslant\phi(d(STSx_{2n_i}, TTSx_{2n_j+1}),\\ &\quad d(STSx_{2n_i}, QTSx_{2n_j+1}), d(PTSx_{2n_i}, TTSx_{2n_j+1}),\\ &\quad d(PTSx_{n2_i}, STSx_{2n_i}), d(TTSx_{2n_j+1}, QTSx_{2n_j+1})).\end{aligned}$$

and the u.s. continuity of ϕ that

$$d(Sz, Tz)\leqslant\phi(d(Sz, Tz), d(Sz, Tz), d(Sz, Tz), 0, 0).$$

Consequently $d(Sz, Tz)=0$, that is $Sz=Tz$. Similarly from

$$\begin{aligned} d(PTSx_{2n_i}, Qz)&\leqslant \phi(d(STSx_{2n_i}, Tz), d(STSx_{2n_i}, Qz)\\ &\quad d(PTSx_{2n_i}, Tz), d(PTSx_{2n_i}, STSx_{2n_i}), d(Tz, Qz)),\end{aligned}$$

we obtain $d(Sz, Qz)\leqslant\phi(0,d(Sz, Qz), 0, 0, d(Sz, Qz))$.

Consequently $Sz=Qz$. Similarly $Tz=Pz$. Thus

$$Sz=Tz=Pz=Qz.$$

Therefore from u.s. continuity of ϕ and

$$d(PTx_{2n}, Qz) \leqslant \phi(d(STx_{2n}, Tz), d(STx_{2n}, Qz), d(PTx_{2n}, Tz), d(PTx_{2n}, STx_{2n}), d(Tz, Qz)),$$

we obtain $d(z, Qz) \leqslant \phi(d(z, Qz), d(z, Qz), d(z, Qz), 0, 0)$, yielding $d(z, Qz)=0$. Hence $z=Qz=Pz=Tz=Sz$.

The uniqueness of the common fixed point z follows easily.

Remark 1 : If $Q=P$ and $S=T$ then (1.1) says that P and T are commuting, and (1.2) may be replaced by (1.2′) there exists a sequence $\{x_n\}_{n\in\omega}$ in X such that

$P(x_n)=T(x_{n+1})$ for all $n \in \omega$ and $\sup\{d(T(x_i), T(x_j)) : i, j \in N\} < \infty$.

In view of this remark, Theorem 1 of Kasahara (1978) is obtained as a corollary to the above theorem.

Now we present an extension of Theorem 1 to a non-complete metric space.

Theorem 2—Let P, Q, S and T be mappings from a metric space (X, d) to itself satisfying the conditions (A), (1.1), (1.2) and the following :

(2.1) the sequence $\{TSx_n\}$ has subsequences converging to a point $z \in X$;

(2.2) S and T are continuous at z.

Then z is the unique common fixed point of P, Q, S and T.

The proof of Theorem 1 works for this theorem. In view of the above remark, Theorem 2 presents an extension of the Theorem of Park (1979).

We remark that in the above theorems if P and Q are commuting then the commutativity of S and T can be avoided. Now we do this.

Theorem 3—Let P, Q, S and T be mappings from a complete metric space (X, d) to itself satisfying the condition (A) and the following :

(3.1) $PS=SP$, $PT=TP$, $QS=SQ$, $QT=TQ$ and $PQ=QP$;

(3.2) there exists a sequence $\{x_n\}_{n\in\omega}$ in X such that

$$SQx_{2n+1}=QPx_{2n},\ TPx_{2n+2} = QPx_{2n+1} \text{ for all } n\in\omega \text{ and}$$
$$\sup\{d(QPx_i, QPx_j) : i, j \in \omega\} < \infty;$$

(3.3) S and T are continuous.

Then P, Q, S and T have a unique common fixed point.

Theorem 4—Let P, Q, S and T be mappings from a metric space (X, d) to itself satisfying the conditions (A), (3.1), (3.2) and the following :

(4.1) the sequence $\{QPx_n\}$ has subsequences converging to a point $z \in X$;

(4.2) S and T are continuous at z.

Then z is the unique common fixed point of P, Q, S and T.

Proof of Theorem 3 will also work for Theorem 4. We prove Theorem 3.

PROOF OF THEOREM 3 : Let $PQ=QP=A$ and $s_n=\sup\{d(Ax_i, Ax_j) : i \geqslant n\}, j \geqslant n$. Then s_n is finite for each $n \in \omega$ and $s_{n+1} \leqslant s_n$ for any $n \in \omega$. So $\{s_n\}_{n\in\omega}$ converges to some $s \geqslant 0$. If $i \geqslant n+1$, $j \geqslant n+1$ and i is even and j odd, then from (*),

$$d(Ax_i; Ax_j)=d(PQx_j, QPx_i)$$
$$\leqslant \phi(d(Ax_{j-1}, Ax_{i-1}), d(Ax_{j-1}, Ax_i), d(Ax_j, Ax_{i-1}), d(Ax_j, Ax_{j-1}), d(Ax_{i-1}, Ax_i))$$

that is $s_{n+1} \leqslant \phi\,(s_n, s_n, s_n, s_n, s_n)$ for all $n \in \omega$.
The u.s. continuity of ϕ implies

$$s \leqslant \phi\,(s, s, s, s, s)$$

which yields $s=0$. Thus $\{Ax_n\}$ is a Cauchy sequence, and $\{Ax_n\}$ converges to some $z \in X$. So the subsequences $\{SQx_{2n+1}\}$ and $\{PTx_{2n+2}\}$ of $\{Ax_n\}$ also converge to the same point z. By (3.1) and the continuity of S and T,

$$PSQx_{2n_j+1} = SQPx_{2n_j+1} \to Sz, \quad SSQx_{2n_{j+1}} \to Sz,$$

$$QTPx_{2n_i} = TQPx_{2n_i} \to Tz \text{ and } TTPx_{2n_i} \to Tz$$

for some subsequences $\{n_j\}_{j \in \omega}$ and $\{n_i\}_{i \in \omega}$ of $\{n\}_{n \in \omega}$.
Therefore it follows from

$$d(PSQx_{2n_j+1},\ QTPx_{2n_i}) \leqslant \phi\ (d\ (SSQx_{2n_j+1},\ TTPx_{2n_i}),$$

$$d(SSQx_{2n_j+1},\ QTPx_{2n_i}),\ d(PSQx_{2n_j+1},\ TTPx_{2n_i}),$$

$$d(PSQx_{2n_j+1},\ SSQx_{2n_j+1}),\ d(TTPx_{3n_i},\ QTPx_{2n_i}))$$

and the u.s. continuity of ϕ that

$$d(Sz, Tz) \leqslant \phi(d(Sz), Tz),\ d(Sz, Tz),\ d(Sz, Tz),\ 0,\ 0).$$

So $Sz=Tz$. Similarly from

$$d(PSQx_{2n_j+1},\ Qz) \leqslant \phi(d(SSQx_{2n_j+1},\ Tz),\ d(SSQx_{2n_j+1},\ Qz),$$

$$d(PSQx_{2n_j+1},\ Tz),\ d(PSQx_{2n_j+1},\ SSQx_{2n_j+1}).\ d(Tz, Qz))$$

or $\quad d(Sz, Qz) \leqslant \phi\ (0,\ d(Sz, Qz),\ 0,\ 0,\ d(Sz, Qz))$

we obtain $Sz=Qz$. Similarly $Tz=Pz$. Therefore from

$$d(PQx_{2n+1},\ Qz) \leqslant \phi(d(SQx_{2n+1},\ Tz),\ d(SQx_{2n+1},\ Qz)),$$

$$d(PQx_{2n+1},\ Tz),\ d(PQx_{2n+1},\ SQx_{2n+1}),\ d(Tz,\ Qz))$$

we obtain $d(z, Qz) \leqslant \phi(d(z, Qz),\ d(z, Qz),\ d(z, Qz),\ 0,\ 0)$.
Consequently $z=Qz=Sz=Tz=Pz$. Uniqueness of z as a common fixed point of P, Q, S and T follows easily.

Remark 2 : If $Q=P$ then (3.1) says that P commutes with each of S and T, and (3.2) may be replaced by

(3.2′) there exists a sequence $\{x_n\}_{n \in \omega}$ in X such that
$Sx_{2n-1} = Px_{2n-2}$, $Tx_{2n} = Px_{2n-1}$ for all $n \in N$ and
$\sup\{d(Px_i, Px_j) : i, j \in \omega\} < \infty$.

Remark 3 : Theorem 4 extends Theorem 3 to non-complete spaces while Theorem 3 extends Theorem 2 of Fisher (1979). In fact, in view of Remark 2, conclusion of the essential part of Theorem 2 of Fisher (1979) is obtained under considerable weaker conditions by taking $Q=P$ in Theorem 4.

Remark 4 : If for some positive integers p, q, s and r mappings P, Q, S and T in Theorems 1-4 are replaced respectively by P^p, Q^q S^s and T^r then the conclusion that the mappings P, Q, S and T have a unique common fixed point remains true provided the commutativity conditions (1.1) or (3.1) hold. It can be proved in a way similar to the Corollary of Singh (1980).

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