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GENERATING FUNCTIONS FOR THE KONHAUSER POLYNOMIALS $Y_n^\alpha(x;k)$

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The object of the present note is to derive some generating functions for the Konhauser polynomials $Y_n^\alpha(x;k)$ by applying a class of bilateral generating functions for these polynomials, given recently by Srivastava (1980).

§1. Konhauser (1967) discussed two polynomial sets $\{Y_n^\alpha(x;k)\}$ and $\{Z_n^\alpha(x;k)\}$, where $\alpha > -1$ and k is a positive integer. These polynomials are biorthogonal with respect to the weight function $x^\alpha e^{-x}$ over the interval $(0,\infty)$. In section 2 of the present note, we demonstrate some applications of a class of bilateral generating functions for the polynomials $Y_n^\alpha(x;k)$ due to Srivastava (1980) in obtaining some bilateral and trilateral generating relations for these polynomials and in section 3, we derive a bilinear generating relation by employing certain operational technique. We also mention some particular forms of our formulas. All the results obtained are believed to be new.

§2. Recently, Srivastava obtained a class of bilateral generating functions for the Konhauser polynomials $Y_n^\alpha(x;k)$ as a special case of his general theorem for obtaining bilinear, bilateral or mixed multilateral generating functions for a certain class of special functions. We recall it here as the following :

Theorem (Srivastava 1980, p. 241, Corollary 18)—Let

$$\Delta_{m,q}[x;\, y_1,\ldots,y_s;t] = \sum_{n=0}^{\infty} a_n\, Y_{m+qn}^\alpha(x;k)\,\Omega_{\mu+pn}(y_1,\ldots,y_s)\,t^n \qquad \ldots(2.1)$$

where $a_n \neq 0$ and $\Omega_\mu(y_1,\ldots,y_s)$ is a non-vanishing function of $y_1,\ldots,y_s$. Then, for every non-negative integer m,

$$\sum_{n=0}^{\infty} Y_{m+n}^\alpha(x;k)\; N_{n,m,q}^{p,\mu}(y_1,\ldots,y_s;z)\,t^n = (1-t)^{-m-(\alpha+1)/k}$$
$$\cdot \exp[x\{1-(1-t)^{-1/k}\}]\,\Delta_{m,q}[x(1-t)^{-1/k};\, y_1,\ldots,y_s;\, zt^q/(1-t)^q] \qquad \ldots(2.2)$$

where

$$N_{n,m,q}^{p,\mu}(y_1,\ldots,y_s;z) = \sum_{r=0}^{[n/p]} \binom{m+n}{n-qr} a_r\,\Omega_{\mu+pr}(y_1,\ldots,y_s)\,z^r, \qquad \ldots(2.3)$$

μ is an arbitrary complex number, p and q are positive integers.

We shall demonstrate some interesting applications of (2.2) when $m=0$ and $\Omega_\mu \equiv 1$.

Case I : When $q=1$—At the outset, we recall that the polynomials $\{f_n(y)\}$ form an Appell set provided

$$\frac{df_n(y)}{dy} = nf_{n-1}(y) \quad (n=0, 1, 2, \ldots).$$

It follows from the definition that

$$f_n(y) = \sum_{r=0}^{n} \binom{n}{r} C_r y^{n-r} \qquad \ldots(2.4)$$

for some sequence $\{C_r\}$. Comparing (2.3) and (2.4), we obtain the following bilateral relation from (2.2):

$$\sum_{n=0}^{\infty} Y_n^\alpha (x;k) f_n(y) t^n = (1-yt)^{-(\alpha+1)/k} \exp [x\{1-(1-yt)^{-1/k}\}] \qquad \ldots(2.5)$$

$$\times\ F^{(1)}[x(1-yt)^{-1/k}, \quad t/(1-yt)]$$

where

$$F^{(1)}[x,t] = \sum_{n=0}^{\infty} C_n Y_n^\alpha (x;k)\ t^n. \qquad \ldots(2.6)$$

Next, on taking

$$a_n = \prod_{j=1}^{p} (b_j)_n / \prod_{j=1}^{s} (c_j)_n$$

and replacing y by $-y$, the polynomials

$$\sigma_n^1 (y) \equiv N_{n,0,1}(y) = \sum_{r=0}^{n} \binom{n}{r} a_r y^r \qquad \ldots(2.7)$$

become identical with the extended Laguerre polynomials $\mathscr{L}_n (y;b_1, \ldots ,b_p;c_1, \ldots ,c_s)$ defined by [Srivastava and Panda 1976, p. 420, (11)]

$$\mathscr{L}_n(y;b_1, \ldots ,b_p;c_1, \ldots ,c_s) = {}_{p+1}F_s \left[\begin{matrix} -n, b_1, \ldots, b_p; \\ \\ c_1, \ldots, c_s; \end{matrix} \; y \right]. \qquad \ldots(2.8)$$

Thus, we obtain from (2.2)

$$\sum_{n=0}^{\infty} Y_n^\alpha (x;k)\ \mathscr{L}_n(y;b_1,\ldots,b_p;c_1,\ldots,c_s) t^n = (1-t)^{-(\alpha+1)/k}$$

$$. \exp [x\{1-(1-t)^{-1/k}\}]\ F^{(2)} [x(1-t)^{-1/k}, -yt/(1-t)] \qquad \ldots(2.9)$$

where

$$F^{(2)}[x,t]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(b_j)_n t^n}{\prod_{j=1}^{s}(c_j)_n} Y_n^{\alpha}(x;k) . \qquad \ldots(2.10)$$

Taking $p=1$, $s=3$, $b_1=(\alpha+n+1)$, $c_1=(\alpha+1)$, $c_2=\frac{1}{2}(\alpha+1)$, $c_3=\frac{1}{2}(\alpha+2)$ and using the result [Magnus *et al.* 1966, p.63]

${}_1F_1(\alpha;\beta;z)\,{}_1F_1(\alpha;\beta;-z)={}_2F_3(\alpha,\beta-\alpha;\beta,\frac{1}{2}\beta,\frac{1}{2}(\beta+1);z^2/4)$, we record from (2.9), the following trilateral relation

$$\sum_{n=0}^{\infty}\{n!/(1+\alpha)_n\}^2\, Y_n^{\alpha}(x;k)L_n^{(\alpha)}(2\sqrt{y})\, L_n^{(\alpha)}(-2\sqrt{y})t^n$$

$$=(1-t)^{-(\alpha+1)/k}\exp[x\{1-(1-t)^{-1/k}\}]\,.$$
$$\times F^{(3)}[x(1-t)^{-1/k},\,-yt/(1-t)], \qquad \ldots(2.11)$$

where $L_n^{(\alpha)}(z)$ are the classical Laguerre polynomials and

$$F^{(3)}[x,t]=\sum_{n=0}^{\infty}\frac{(\alpha+n+1)_n\, t^n}{(\alpha+1)_n((\alpha+1)/2)_n((\alpha+2)/2)_n} Y_n^{\alpha}(x;k). \qquad \ldots(2.12)$$

For $k=1$, above reduces to the interesting form

$$\sum_{n=0}^{\infty}\{n!/(1+\alpha)_n\}^2\, L_n^{(\alpha)}(x)L_n^{(\alpha)}(2\sqrt{y})\, L_n^{(\alpha)}(-2\sqrt{y})t^n$$

$$=(1-t)^{-(\alpha+1)}\exp(-xt)\,F^{(4)}[x/(1-t),\,-yt/(1-t)] \qquad \ldots(2.13)$$

where

$$F^{(4)}[x,t]=\sum_{n=0}^{\infty}\frac{(\alpha+n+1)_n\, t^n}{(\alpha+1)_n((\alpha+1)/2)_n((\alpha+2)/2)_n} L_n^{(\alpha)}(x). \qquad \ldots(2.14)$$

Case II : $q>1$—Wright's (1935 & 1940) generalized hypergeometric function is defined by

$${}_p\Psi_s\left[\begin{matrix}(a_1,\alpha_1)\ldots,(a_p,\alpha_p);\\ & z\\ (b_1,\beta_1),\ \ ,(b_s,\beta_s);\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n\alpha_j}z^n}{\prod_{j=1}^{s}(b_j)_{n\beta_j}\, n!}, \qquad \ldots(2.15)$$

where the variable z and the various parameters are such that the series converges. On setting

$$a_n=\frac{\prod_{j=1}^{m}(b_j)_{n\mu_j}\prod_{j=1}^{s}(e_j)_{qn\xi_j}\,(qn)!}{\prod_{j=1}^{p}(c_j)_{n\nu_j}\prod_{j=1}^{l}(d_j)_{qn\eta_j}} \qquad \ldots(2.16)$$

in the polynomials

$$\sigma_n^q(y) \equiv N_{n,0,q}(y) = \sum_{r=0}^{[n/q]} \binom{n}{qr} a_r y^r \qquad \ldots(2.17)$$

and replacing y by $(-1)^q y$, we arrive at the following bilateral relation:

$$\sum_{n=0}^{\infty} Y_n^\alpha(x;k)\,{}_{m+s+1}\Psi_{p+l}\left[\begin{matrix}(-n,q),\{(b_m,\mu_m)\},\{(e_s,q\xi_s)\};\\ \{(c_p,\nu_p)\},\ \{(d_l,q\eta_l)\};\end{matrix}\ y\right]$$

$$=(1-t)^{-(\alpha+1)/k}\exp[x\{1-(1-t)^{-1/k}\}]$$

$$\times\ F^{(5)}[x(1-t)^{-1/k}, y\{-t/(1-t)\}^q], \qquad \ldots(2.18)$$

where $\{(a_p,\alpha_p)\}$ abbreviate the array of p parameter-pairs $(a_1,\alpha_1),\ldots,(a_p,\alpha_p)$ and

$$F^{(5)}[x,t] = \sum_{n=0}^{\infty} a_n Y_{qn}^{\alpha}(x;k)\,t^n,$$

a_n is as given by (2.16).

Two special forms of the generating relation (2.18) are worthy of note. If in (2.18), we take $q=1$, $m=s=l=0$, $p=1$, $c_1=\gamma+1$, $\nu_1=\beta$ and then apply the definition

$$L_n^{(\beta,\gamma)}(x) = \frac{(\gamma+1)_{n\beta}}{n!}\,{}_1\Psi_1\,[(-n,1);(\gamma+1,\beta);x] \qquad \ldots(2.19)$$

$L_n^{(\beta,\gamma)}(x)$ being a generalization of the Konhauser polynomials $Z_n^\alpha(x;k)$ studied by Prabhakar and Rekha (1972) and Srivastava and Panda (1979), we obtain

$$\sum_{n=0}^{\infty}\{n!/(\gamma+1)_{n\beta}\}\,Y_n^\alpha(x;k)\,L_n^{(\beta,\gamma)}(y)t^n = (1-t)^{-(\alpha+1)/k}$$

$$\times\exp[x\{1-(1-t)^{-1/k}\}]\,F^{(6)}[x(1-t)^{-1/k}, -yt/(1-t)] \qquad \ldots(2.20)$$

where

$$F^{(6)}[x,t] = \sum_{n=0}^{\infty}\frac{(qn)!\,t^n}{(\gamma+1)_{n\beta}}\,Y_{qn}^{\alpha}(x;k). \qquad \ldots(2.21)$$

In view of the relation Prabhakar and Rekha (1972).

$$L_n^{k,\beta}(x^k) = Z_n^\beta(x;k),$$

we obtain from (2.20) a known bilateral generating relation due to Srivastava, 1973 p. 491).

On the other hand, in terms of the Brafmann polynomials defined by (Brafmann 1957, p. 186).

$$B_n^q[a_1,\ldots,a_p;b_1,\ldots,b_s;y] = {}_{p+q}F_s\left[\begin{matrix}\Delta(q;-n),a_1,\ldots,a_p;\\ b_1,\quad,b_s;\end{matrix}\ y\right] \qquad \ldots(2.22)$$

we deduce from (2.18), the following bilateral relation

$$\sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)\, B_n^q[a_1,\ldots,a_p;b_1,\ldots,b_s:y]t^n = (1-t)^{-(\alpha+1)/k}$$

$$\times \exp[x\{1-(1-t)^{-1/k}\}]\, F^{(7)}[x(1-t)^{-1/k}, y\{-t/q(1-t)\}^q], \qquad \ldots(2.23)$$

where

$$F^{(7)}[x,t] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\,(qn)!\,t^n}{\prod_{j=1}^{s}(b_j)_n}\, Y_{qn}^{\alpha}(x;k). \qquad \ldots(2.24)$$

By suitably specializing the arbitrary parameters involved in (2.22), the Brafmann polynomials can be reduced to a number of familar polynomials including, for example, the Gould-Hopper generalized Hermite polynomials. Thus, the relation (2.23) may be employed in deriving a number of bilateral relations for $Y_n^{\alpha}(x;k)$.

§3. In this section, we shall prove the following bilinear relation :

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n (n+m)!\,(n+l)!\,t^n}{\prod_{j=1}^{r}(b_j)_n\, n!}\, Y_{n+m}^{\alpha}(x;k)$$

$$\cdot Y_{n+l}^{\beta}(y;k) = e^{x+y} \sum_{i,j=0}^{\infty} \frac{(-x)^i}{i!}\frac{(-y)^j}{j!}$$

$$\cdot\left(\frac{\alpha+1+i}{k}\right)_m \left(\frac{\beta+1+j}{k}\right)_l\, {}_{p+2}F_r\left[\begin{matrix}\frac{\alpha+1+i}{k}+m,\ \frac{\beta+1+j}{k}+l,\ (a_p); \\ (b_r);\end{matrix}\; t\right] \qquad \ldots(3.1)$$

where m and l are non-negative integers.

The method of proof employs the Rodrigues formula

$$Y_n^{\alpha+s}(x;k) = \frac{x^{-(\alpha+1+kn)}}{k^n\, n!} e^x\, \theta_x^n \left[e^{-x}x^{\alpha+1}\right] \qquad \ldots(3.2)$$

$\theta_x \equiv x^k(s+xD_x)$, $\quad D_x \equiv d/dx$.

PROOF OF (3.1) : We start by considering the sum

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n\,(n+m)!\,(n+l)!}{\prod_{j=1}^{r}(b_j)_n\, n!}\{(xy)^k t\}^n\, Y_{n+m}^{\alpha+s}(x;k)\quad Y_{n+l}^{\beta+s}(y;k),$$

make use of (3.2) and then apply the known result

$$\theta_x^n(x^b) = x^{b+kn}\, k^n \left(\frac{b+s}{k}\right)_n .$$

We thus obtain

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, (n+m)! \, (n+l)!}{\prod_{j=1}^{r} (b_j)_n \, n!} \{(xy)^k t\}^n \, Y_{n+m}^{\alpha+s}(x;k) \, Y_{n+l}^{\beta+s}(y;k)$$

$$= e^{x+y} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, t^n}{\prod_{j=1}^{r} (b_j)_n \, n!} \sum_{i=0}^{\infty} \frac{(-1)^i}{i} x^{kn+i}$$

$$\cdot \left(\frac{\alpha+i+1+s}{k}\right)_{n+m} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} y^{kn+j} \left(\frac{\beta+j+1+s}{k}\right)_{n+l}$$

$$= e^{x+y} \sum_{i,j=0}^{\infty} \frac{(-x)^i}{i!} \frac{(-y)^j}{j!} \left(\frac{\alpha 1++i+s}{k}\right)_m \left(\frac{\beta+1+j+s}{k}\right)_l$$

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{r} (b_j)_n \, n!} (x^k y^k t)^n \left(\frac{\alpha+1+i+s}{k} + m\right)_n$$

$$\cdot \left(\frac{\beta+1+j+s}{k} + l\right)_n$$

where at the last step, we have used the identity

$$(c)_{n+k} = (c+k)_n \, (c)_k.$$

Now, the bilinear relation in (3.1) would follow on replacing t by $t/x^k y^k$, α by $\alpha - s$ and β by $\beta - s$.

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