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ASYMPTOTIC RESULTS ON SUMS OF SOME MULTIPLICATIVE FUNCTIONS

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Let k be a fixed integer $\geqslant 2$ and u be a fixed integer. Suppose g_k is a multiplicative function satisfying (i) $g_k(p^j) = j+1$, if $1 \leqslant j \leqslant k-1$ (ii) $g_k(p^k) = 0$ for $k=2$ or 3, for all primes p and (iii) $|g_k(n)| \leqslant \tau(n)$ for all n, where $\tau(n)$ is the number of divisors of n. In this paper we establish an asymptotic formula for the sum $\sum_{\substack{n \leqslant x \\ (n,u)=1}} g_k(n)$ and deduce several known results as particular cases in addition to some new results.

1. INTRODUCTION

Throughout this paper k denotes a fixed integer $\geqslant 2$, m and n denote positive integral variables, u denotes a fixed positive integer, p denotes a prime and x denotes a sufficiently large real variable.

In this paper we prove the following:

Theorem—Suppose g_k is a multiplicative function satisfying

$$g_k(p^j) = j+1 \text{ if } 1 \leqslant j \leqslant k-1 \quad \ldots(1.1)$$

$$g_k(p^k) = 0 \text{ for } k = 2 \text{ or } 3 \quad \ldots(1.2)$$

for all primes p and

$$|g_k(n)| \leqslant \tau(n) \text{ for all } n \quad \ldots(1.3)$$

where $\tau(n)$ is the number of divisors of n. Then we have

$$\sum_{\substack{n \leqslant x \\ (n,u)=1}} g_k(n) = A_k(u)\, x \log x + x\, A_k(u) \left\{ 2\gamma - 1 + 2 \sum_{p \mid u} \frac{\log p}{p-1} \right.$$

$$\left. - k \sum_{p \mid u} \frac{\log p}{p^k - 1} - k \frac{\zeta'(k)}{\zeta(k)} \right\} - x\, A_k'(u) + \Delta_{k,u}(x)$$

where γ is Euler's constant, $\zeta(s)$ is the Riemann Zeta function, $\zeta'(s)$ its derivative and

$$\Delta_{k,u}(x) = \begin{cases} O(x^{1/2}\, \delta(x) \log^2 x \;\; \sigma^{*2}_{-\alpha}(u)), & \text{if } k = 2 \\ O(x^{1/3}\, \delta'(x) \log^3 x \;\; \sigma^{*2}_{-\alpha}(u)\, \sigma^{*2}_{-1+\epsilon_1}(u)) & \text{if } k = 3 \\ O(x^{\alpha} \;\; \sigma^{*2}_{-\alpha}(u)) & \text{if } k \geqslant 4, \end{cases} \quad \ldots(1.4)$$

for every $\epsilon_1 > 0$, where

$$\delta'(x) = \exp\{-c_k (\log\log x)^{3/5} (\log\log\log x)^{-1/5}\}, \quad \ldots(1.5)$$

c_k being positive constant; $\sigma_s^*(u)$ is the sum of the s-th powers of the square-free divisors of u and α is the number which appears in the Dirichlet divisor problem; namely,

$\sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\alpha})$. Further we have

$$A_k(u) = \frac{u^{k-2}\varphi^2(u)}{\zeta(k)J_k(u)} \sum_{\substack{m=1 \\ (m,u)=1}}^{\infty} f_k(m)\, m^{-1} \quad \ldots(1.6)$$

and

$$A'_k(u) = \frac{u^{k-2}\varphi^2(u)}{\zeta(k)J_k(u)} \sum_{\substack{m=1 \\ (m,u)=1}}^{\infty} f_k(m)\, m^{-1} \log m \quad \ldots(1.7)$$

where φ is the Euler totient function and J_k is the Jordan totient function of order k. In the above f_k is the function defined by

$$f_k(m) = \sum_{d^k | m} h_k(m/d^k) \quad \ldots(1.8)$$

where h_k is defined by

$$h_k(m) = \sum_{d | m} h(d)\, g_k(m/d) \quad \ldots(1.9)$$

and

$$h(m) = \sum_{d | m} \mu(d)\mu(m/d) \quad \ldots(1.10)$$

μ being the Möbius function.

Further the constants implied by the symbol O in (1.4) depend atmost on k, α and ϵ_1.

In section 2 we prove some lemmas which are required for the proof of the above theorem. In section 3 we give the proof of the theorem. Some known results in the literature as particular cases of the above theorem are deduced in section 4.

2. Preliminaries

First we prove

Lemma 2.1—Let f_k be the multiplicative function defined by (1.8). Then

$$\sum_{m \leq x} | f_k(m) | = O(x^{(1/k)+\epsilon}),$$

for every $\epsilon > 0$, where the O-constant depends only ϵ and k.

Proof : Let G_k be the multiplicative function defined at prime powers $p^j, j \geq 1$ by

$$G_k(p^j) = \begin{cases} 0 \text{ if } 1 \leq j \leq k-1 \\ 3\,(j+1)^2 \text{ if } j \geq k. \end{cases}$$

From (1.10), (1.9) and (1.1), it follows that

$$h_k(p^j)=\begin{cases} 0, \text{ if } 1 \leqslant j \leqslant k-1 \\ g_k(p^j)-2g_k(p^{j-1})+g_k(p^{j-2}), \text{ if } j \geqslant k. \end{cases} \qquad \ldots(2.1)$$

Also, from (1.8), it follows that

$$f_k(p^j) = \sum_{i=0}^{t} h_k(p^{r+ik}) \qquad \ldots(2.2)$$

if $j=kt+r$, $0 \leqslant r < k$ and $t \geqslant 0$ is an integer.
From (2.1), (2.2), (1.1) and (1.3), we have

$$| f_k(p^j) | \leqslant G_k(p^j).$$

Since g_k is multiplicative, it follows from (1.10), (1.9) and (1.8) that f_k is multiplicative. Since G_k is also multiplicative the above inequality shows that

$$| f_k(n) | \leqslant G_k(n)$$

for all n; Lemma 2.1 follows from this and the fact that

$$\sum_{m=1}^{\infty} G_k(m)m^{-s} \text{ converges for } s>1/k.$$

Lemma 2.2 (cf. Suryanarayana and Prasad 1973, Lemma 3.5)—We have

$$M_u(x) \equiv \sum_{\substack{m \leqslant x \\ (m,u)=1}} \mu(m) = O(x\delta(x)\sigma^*_{-1+\epsilon}(u))$$

where

$$\delta(x) = \exp(-c(\log x)^{3/5} (\log \log x)^{-1/5}) \qquad \ldots(2.3)$$

c being a positive constant.

Lemma 2.3—Let $h_2'(m) \equiv h(m)$ be given by (1.10) and

$$h_3'(m) = \sum_{d|m} h(d)\, \mu(m/d) \qquad \ldots(2.4)$$

for all m. Then

$$\sum_{\substack{m \leqslant x \\ (m,u)=1}} h_k'(m) = \begin{cases} O(x\delta(x)\ \sigma^{*2}_{-1+\epsilon}(u)) \text{ if } k=2 \\ O(x\delta(x)\ \sigma^{*2}_{-1+\epsilon}(u)\sigma^{*2}_{-1+\epsilon_1}(u)) \text{ if } k=3, \end{cases}$$

for every positive numbers ϵ and ϵ_1, where the corresponding O-constants depend at most on ϵ and ϵ_1.

PROOF: By Lemma 2.2, we have

$$\sum_{\substack{m \leqslant x \\ (m,u)=1}} h_2'(m) = \sum_{\substack{dn \leqslant x \\ (d,u)=(n,u)=1}} \mu(d)\, \mu(n)$$

(*equation continued on p.* 775)

$$= 2 \sum_{\substack{d \leqslant \sqrt{x} \\ (d,u)=1}} \mu(d)\, M_u(x/d) - [M_u(\sqrt{x})]^2$$

$$= O(x\, \sigma^*_{-1+\epsilon}(u) \sum_{d \leqslant \sqrt{x}} d^{-1}\, \delta(x/d)) + O(x\delta^2(x) \sigma^{*2}_{-1+\epsilon}(u))$$

$$= O(x\delta(\sqrt{x})\sigma^*_{-1+\epsilon}(u) \sum_{d \leqslant \sqrt{x}} d^{-1}) + O(x\delta(x)\, \sigma^{*2}_{-1+\epsilon}(u))$$

$$= O(x\delta(x)\, \sigma^*_{-1+\epsilon}(u) \log x) + O(x\delta(x)\sigma^{*2}_{-1+\epsilon}(u))$$

$$= O(x\delta(x)\sigma^{*2}_{-1+\epsilon}(u)).$$

Note that, in the above, we have used the fact that $\delta(x)$ is decreasing for large x. The case $k = 3$ can be similarly established.

Lemma 2.4—Suppose $k=2$ or 3. Then

$$\sum_{\substack{m \leqslant x \\ (m,u)=1}} f_k(m) = \begin{cases} O(x^{1/2}\delta(x)\, \sigma^{*2}_{-1+\epsilon}(u)) \text{ if } k=2 \\ O(x^{1/3}\delta(x)\, \sigma^{*2}_{-1+\epsilon}(u))\, \sigma^{*2}_{-1+\epsilon_1}(u)) \text{ if } k=3 \end{cases} \qquad \ldots(2.5)$$

and

$$\sum_{m \leqslant x} | f_k(m) | = \begin{cases} O(x^{1/2} \log x) \text{ if } k=2 \\ O(x^{1/3} \log^2 x) \text{ if } k=3 \end{cases} \qquad \ldots(2.6)$$

for every positive numbers ϵ and ϵ_1, where the corresponding O-constants depend at most on ϵ and ϵ_1.

PROOF : From (2.2), (2.1), (1.1) and (1.2), it follows that

$f_k(p^j) = 0$ for $1 \leqslant j \leqslant k-1$ and $f_k(p^k) = -k$. Hence for $s > 1/k$,

$$(\zeta(ks))^k \sum_{m=1}^{\infty} f_k(m)\, m^{-s} = \prod_p \left\{1 - \frac{k}{p^{ks}} + \frac{f_k(p^{k+1})}{p^{(k+1)s}} + \cdots\right\}$$

$$\times \prod_p \left\{1 - \frac{1}{p^{ks}}\right\}^{-k}$$

$$= \sum_{m=1}^{\infty} f^*_k(m)\, m^{-s}, \text{ say} \qquad \ldots(2.7)$$

which converges absolutely for $s > 1/(k+1)$. Also, it is not difficult to see that there exists a multiplicative function g^*_k which depends only on k (as G_k in the proof of Lemma 2.1) and such that

$$\left| f^*_k(m) \right| \leqslant g^*_k(m) \text{ for all } m \qquad \ldots(2.8)$$

and

$$\sum_{m\leqslant x} g_k^*(m)\, m^{-s} = O(1), \qquad \ldots(2.9)$$

if $s=1/(k+1)+\epsilon$, $\epsilon>0$, where the O-constant depends only on k and ϵ.

From (2.7), (2.4) and the fact $h_2'(m) = h(m)$, where $h(m)$ is given by (1.10) we have

$$f_k(m) = \sum_{d^k n=m} h_k'(d)\, f_k^*(n) \qquad \ldots(2.10)$$

so that by (2.8) and Lemma 2.3,

$$\Big|\sum_{\substack{m<x\\(m,u)=1}} f_k(m)\Big| \leqslant \sum_{n<x} g_k^*(n) \Big|\sum_{\substack{d\leqslant(x/n)^{1/k}\\(d,u)=1}} h_k'(d)\Big|$$

$$= \begin{cases} O(x^{1/k}\, \sigma^{*2}_{-1+\epsilon}(u)\, T_k(x)), & \text{if } k=2\\ O(x^{1/k}\, \sigma^{*2}_{-1+\epsilon}(u)\, \sigma^{*2}_{-1+\epsilon_1}(u)\, T_k(x)), & \text{if } k=3\end{cases}$$

where

$$T_k(x) = \sum_{n\leqslant x} g_k^*(n)\, n^{-1/k}\, \delta((x/n)^{1/k}).$$

Now, since $x^\beta\, \delta(x)$ is increasing for large x, for every $\beta>0$, we have for $0<\beta<1/k(k+1)$,

$$T_k(x) = O(\delta(x^{1/k}) \sum_{n\leqslant x} g_k^*(n)\, n^{-1/k+\beta})$$

$$= O(\delta(x^{1/k})) = O(\delta(x))$$

where we have used (2.9). Hence (2.5) follows.

Now we shall prove (2.6). Let $k = 2$. Since $|h_2'(m)| = |h(m)| \leqslant \tau(m)$ for all m and $\sum_{m\leqslant x} \tau(m) = O(x \log x)$, we have by (2.10), (2.8) and (2.9),

$$\sum_{m<x} \Big|f_2(m)\Big| \leqslant \sum_{n\leqslant x} g_2^*(m) \sum_{d\leqslant(x/n)^{1/k}} \tau(d)$$

$$= O(x^{1/2} \log x \sum_{n\leqslant x} g_2^*(n)\, n^{-1/2})$$

$$= O(x^{1/2} \log x).$$

Hence (2.6) follows, when $k=2$. When $k=3$, using

$$|h_3'(m)| \leqslant \sum_{d\mid m} \tau(d) \equiv \tau_3(m), \text{ and the fact } \sum_{m\leqslant x} \tau_3(m) = O(x \log^2 x),$$

(2.6) ($k=3$) follows from (2.10).

Hence Lemma 2.4 follows.

Lemma 2.5 (cf. Suryanarayana and Prasad 1975, Theorem 1)—Let $\tau_{(k)}(n)$ denote the number of k-free divisors of n. Then we have

$$\sum_{\substack{n\leq x\\(n,u)=1}} \tau_{(k)}(n) = \frac{u^{k-2}\varphi^2(u)}{J_k(u)\zeta(k)}\, x\left\{\log x+2\gamma-1+2\alpha(u) - k\,\frac{\zeta'(k)}{\zeta(k)} - \alpha_k(u)\right\} + E^{(u)}_{(k)}(x) \qquad \ldots(2.11)$$

where

$$\alpha(u) = \sum_{p|u} \frac{\log p}{p-1}, \quad \alpha_k(u) = \sum_{p|u} \frac{\log p}{p^k-1}$$

and

$$E^{(u)}_{(k)}(x) = O\left(x^{1/k}\,\delta(x)\sigma^{*2}_{-\alpha}(u)\right) \text{ or } O\left(x^{\alpha}\,\sigma^{*2}_{-\alpha}(u)\right) \qquad \ldots(2.12)$$

according as $k = 2$ or 3 or $k \geqslant 4$; the O-estimates being uniform in x and u, where α is the number which appears in the Dirichlet divisor problem.

Remark 2.1: If α is the number which appears in the Dirichlet divisor problem (cf. Hardy and Wright 1960, p. 272) then it is known that $1/4 < \alpha < 1/2$. The best known result till now is due to Kolesnik (1973/74) who proved that the order of the error term in the Dirichlet Divisor problem is $O(x^{346/1067}(\log x)^{211/100})$. Also, there is a conjecture that $\alpha = 1/4+\epsilon$, for every $\epsilon>0$.

Lemma 2.7—We have

$$\sum\nolimits_k(x) \equiv \sum_{m\leqslant x} \left| f_k(m)\, E^{(u)}_{(k)}\,(x/m)\right|$$

$$= \begin{cases} O(x^{1/2}\,\delta'(x)\,\log^2 x\,\sigma^{*2}_{-\alpha}(x)) & \text{if } k=2\\ O\left(x^{1/3}\,\delta'(x)\,\log^3 x\,\sigma^{*2}_{-\alpha}(u)\right) & \text{if } k=3\\ O\left(x^{\alpha}\,\sigma^{*2}_{-\alpha}(u)\right), & \text{if } k\geqslant 4,\end{cases}$$

where the O-constants depend only on k and α.

PROOF: First suppose the $k = 2$ or 3. By (2.12) we have

$$\Sigma_k(x) = O\left(x^{1/k}\,\sigma^{*2}_{-\alpha}(u)\;T'_k(x)\right)$$

where

$$T'_k(x) = \sum_{m\leqslant x} |f_k(m)|\, m^{-1/k}\,\delta(x/m).$$

Now writing $z = \log x$, we have

$$T'_k(x) = \sum_{m\leqslant x/z} |f_k(m)|\, m^{-1/k}\,\delta(x/m) + \sum_{x/z<m\leqslant x} |f_k(m)|\, m^{-1/k}\,\delta(x/m)$$

$$= S_1 + S_2, \text{ say.}$$

Since $\delta(x)$ is decreasing for large x, we have

$$S_1 = O\left(\delta(\log x)\sum_{m\leqslant x} |f_k(m)|\, m^{-1/k}\right) = O\left(\delta'(x)\sum_{m\leqslant x} |f_k(m)|\, m^{-1/k}\right)$$

since $\delta(\log x) = \delta'(x)$ for large x.

By (2.6) and partial summation, we have

$$\sum_{m\leqslant x} |f_k(m)|\, m^{-1/k} = \begin{cases} O(\log^2 x) \text{ if } k=2\\ O(\log^3 x) \text{ if } k=3.\end{cases}$$

Hence

$$S_1 = \begin{cases} O(\delta'(x)\log^2 x) & \text{if } k = 2, \\ O(\delta'(x)\log^3 x) & \text{if } k = 3. \end{cases}$$

Now by (2.6), we have

$$S_2 = O\Big((z/x)^{1/k} \sum_{m \leqslant x} |f_k(m)|\Big) = \begin{cases} O(\log x)^{1+(1/k)} & \text{if } k = 2 \\ O(\log x)^{2+(1/k)} & \text{if } k = 3. \end{cases}$$

Hence Lemma 2.7 follows in case $k = 2$ or 3.

Now suppose $k \geqslant 4$. If G_k is the multiplicative function given in the proof of Lemma 2.1, we have by (2.12),

$$\Sigma_k(x) = O\Big(x^{\alpha} \, \sigma^{*2}_{-\alpha}(u) \sum_{m \leqslant x} G_k(m)\, m^{-\alpha}\Big)$$

$$= O\Big(x^{\alpha} \, \sigma^{*2}_{-\alpha}(u)\Big),$$

since $\alpha > 1/4 \geqslant 1/k$ and $\sum_{m=1}^{\infty} G_k(m) m^{-s}$ converges for $s > 1/k$. Hence Lemma 2.7 follows.

Lemma 2.8—We have

$$\sum_{\substack{m > x \\ (m,u)=1}} f_k(m)\, m^{-1} = \begin{cases} O\Big(x^{-1/2}\,\delta(x)\,\sigma^{*2}_{-1+\epsilon}(u)\Big) & \text{if } k = 2 \\ O\Big(x^{-2/3}\,\delta(x)\,\sigma^{*2}_{-1+\epsilon}(u)\,\sigma^{*2}_{-1+\epsilon_1}(u)\Big) & \text{if } k = 3, \end{cases}$$

$$\sum_{\substack{m > x \\ (m,u)=1}} f_k(m)\, m^{-1}\log m = \begin{cases} O\Big(x^{-1/2}\,\delta(x)\,\sigma^{*2}_{-1+\epsilon}(u)\Big) & \text{if } k = 2 \\ O\Big(x^{-2/3}\,\delta(x)\,\sigma^{*2}_{-1+\epsilon}(u)\,\sigma^{*2}_{-1+\epsilon_1}(u)\Big) & \text{if } k = 3, \end{cases}$$

for every positive numbers ϵ and ϵ_1,

$$\sum_{m > x} |f_k(m)|\, m^{-1} = O(x^{-1+1/k+\epsilon_2})$$

and

$$\sum_{m > x} |f_k(m)|\, m^{-1}\log m = O(x^{-1+1/k+\epsilon_2}),$$

for every $\epsilon_2 > 0$. All the O-constants implied by the corresponding O symbols above depend at most on ϵ, ϵ_1 and ϵ_2.

PROOF : Follows from (2.5), Lemma 2.1 and partial summation.

Lemma 2.9—If g_k is the multiplicative function given in the theorem, then

$$g_k(n) = \sum_{dm=n} \tau_{(k)}(d)\, f_k(m) \qquad \ldots(2.13)$$

for all m.

PROOF : We have by (1.8), (1.9) and (1.10), for $s > 1$,

$$\sum_{m=1}^{\infty} f_k(m)\, m^{-s} = \zeta(ks) \sum_{m=1}^{\infty} h_k(m)\, m^{-s}$$

(*equation continued on p.* 779)

$$\sum_{m=1}^{\infty} h_k(m)\, m^{-s} = \sum_{m=1}^{\infty} h(m)\, m^{-s} \sum_{m=1}^{\infty} g_k(m)\, m^{-s}$$

and

$$\sum_{m=1}^{\infty} h(m)\, m^{-s} = \left(\zeta(s)\right)^{-2};$$

hence

$$\frac{\zeta^2(s)}{\zeta(ks)} \sum_{m=1}^{\infty} f_k(m)\, m^{-s} = \sum_{m=1}^{\infty} g_k(m)\, m^{-s}.$$

Now (2.13) follows from the fact that

$$\sum_{m=1}^{\infty} \tau_{(k)}(m)\, m^{-s} = \zeta^2(s)/\zeta(ks).$$

3. Proof of the Theorem

By (2.13) and Lemma 2.5 we have

$$\sum_{\substack{m \leqslant x \\ (m,u)=1}} g_k(m) = \sum_{\substack{m \leqslant x \\ (m,u)=1}} f_k(m) \sum_{\substack{d < x/m \\ (d,u)=1}} \tau_{(k)}(d)$$

$$= \frac{u^{k-2}\varphi^2(u)}{\zeta(k)J_k(u)}\, x \left(\log x + 2\gamma - 1 + 2\alpha(u) - k\ \alpha_k(u) - k\frac{\zeta'(k)}{\zeta(k)}\right) \sum_{\substack{m \leqslant x \\ (m,u)=1}} f_k(m) m^{-1}$$

$$- x\frac{u^{k-2}\varphi^2(u)}{\zeta(k)J_k(u)} \sum_{\substack{m < x \\ (m,u)=1}} f_k(m)\, m^{-1} \log m + \sum_{\substack{m < x \\ (m,u)=1}} f_k(m) E^{(u)}_{(k)}(x/m).$$

Now by (1.6), (1.7) and Lemma 2.8, we have

$$\sum_{\substack{m \leqslant x \\ (m,u)=1}} g_k(m) = A_k(u)\, x \log x + x\, A_k(u) \left\{2\gamma - 1 + 2\alpha(u) - k\ \alpha_k(u) - k\ \frac{\zeta'(k)}{\zeta(k)}\right\}$$

$$- x\ A'_k(u) + \Delta'_{k,u}(x) + O\left(\Sigma_k(x)\right) \qquad \ldots(3.1)$$

where $\Sigma_k(x)$ is as given in Lemma 2.7 and

$$\Delta'_{k,u}(x) = \begin{cases} O\left(x^{1/2}\delta(x)\, \sigma^{*2}_{-1+\epsilon}(u)\right) \text{ if } k = 2 \\ O\left(x^{1/3}\delta(x)\, \sigma^{*2}_{-1+\epsilon}(u) \sigma^{*2}_{-1+\epsilon_1}(u)\right) \text{ if } k = 3 \\ O\left(x^{1/k+\epsilon_2}\ \sigma^{*2}_{-1+\epsilon}(u)\right) \text{ if } k \geqslant 4. \end{cases} \qquad \ldots(3.2)$$

for every positive numbers ϵ, ϵ_1 and ϵ_2. In the above we have also used the results $u^{k-2}\varphi^2(u)/J_k(u) = O(1)$, $\alpha_k(u) = O(1)$ and that (cf. Suryanarayana and Prasad 1973, Remark 3.1) $\alpha(m)\varphi(m)\, m^{-1} = O\left(\sigma^*_{-1+\epsilon}(m) \right)$, for every $\epsilon > 0$.

Taking $\epsilon = 1-\alpha$ and when $k \geqslant 4$ choosing ϵ_2 to be $0 < \epsilon_2 < \alpha-(1/k)$ in (3.2), the theorem follows from Lemma 2.7 and (3.1).

4. Special Cases

Let q_k be the characteristic function of the k-free integers (that is, those natural numbers which are not divisible by the k-th power of any prime). Then we have

Theorem 4.1— $$\sum_{\substack{n\leqslant x\\(n,u)=1}} \tau(n)\, q_k(n) = a_k(u)\, x \log x + x\, b_k(u) + \Delta_k(x;u) \qquad \ldots(4.1)$$

where

$$\Delta_k(x;u) = \begin{cases} O\left(x^{1/2}\,\delta'(x)\log^2 x\, \sigma^{*2}_{-\alpha}(u)\right) \text{ if } k=2, \\ O\left(x^{1/3}\delta'(x)\log^3 x\;\sigma^{*2}_{-\alpha}(u)\,\sigma^{*2}_{-1+\epsilon_1}(u)\right) \text{if } k=3 \\ O\left(x^{\alpha}\sigma^{*2}_{-\alpha}(u)\right) \text{ if } k \geqslant 4, \end{cases} \qquad \ldots(4.2)$$

for every $\epsilon_1 > 0$, where the O-estimates are uniform in x and u; further we have

$$a_k(u) = \prod_p p^{-k-1}\,(p^{k+1}-(k+1)p+k) \prod_{p|u} \frac{(p-1)^2\, p^{k-1}}{(p^{k+1}-(k+1)p+k)}$$

and

$$b_k(u) = a_k(u)\left\{2\gamma-1+2\sum_{p|u}\frac{\log p}{p-1}+k(k+1)\sum_{p|u}\frac{(p-1)\log p}{(p^{k+1}-(k+1)p+k)}\right\}.$$

Proof: Taking $g_k(n) = \tau(n)\, q_k(n)$ in the Theorem, we obtain Theorem 4.1.

Corollary 4.1 (Theorem 4.1, $k=2$)—We have

$$\sum_{\substack{m\leqslant x\\(m,u)=1}} |\mu(m)|\,\tau(m) = A\, x\log x + Bx + \Delta_u(x) \qquad \ldots(4.3)$$

where

$$\Delta_u(x) = O\left(x^{1/2}\,\delta'(x)\log^2 x\,\sigma^{*2}_{-\alpha}(u)\right) \qquad \ldots(4.4)$$

$$A = \prod_p \{p^{-3}(p-1)^2(p+2)\} \prod_{p|u} (p/p+2)) \qquad \ldots(4.5)$$

and

$$B = A\left\{2\gamma-1+2\sum_{p|u}\frac{\log p}{p-1}+6\sum_{p|u}\frac{\log p}{(p-1)(p+2)}\right\}. \qquad \ldots(4.6)$$

Remark 4.1: Formula (4.3) in case $u = 1$ was originally established by Wigert (1934) with a weaker O-estimate of the error term viz., $\Delta_1(x) = O(x^{3/4+\epsilon})$, using analytic methods.

Later, it has been established by Gordon and Rogers (1964) by using elementary methods that

$$\sum_{\substack{n<x \\ (n,u)=1}} |\mu(n)|\,\tau(n) = Ax\log x + B'x + \Delta_u(x)\,, \qquad \ldots(4.7)$$

where A is given by (4.5),

$$B' = A\left\{2\gamma - 1 + 2\sum_{p|u}\frac{\log p}{p-1} + 6\sum_{p|u}\frac{(p-1)\log p}{p^2(p+2)}\right\}. \qquad \ldots(4.8)$$

and

$$\Delta_u(x) = O(x^{1/2+\epsilon}\,S(u))$$

where

$$S(u) \equiv \sum_{d|u} \mu^2(d)\,3^{\omega(d)}\,d^{-1/2} = O\left(\exp\left(\frac{c\sqrt{\log 3u}}{\log\log 3u}\right)\right) \qquad \ldots(4.9)$$

for some constant $c > 0$. In fact, Gordon and Rogers established (4.7) for any square-free u; the same method works out to establish (4.7) for any u. Further, if we carefully follow their method we get that $\Delta_u(x) = O(x^{1/2}\log^5 x\, S(u))$. We remark here that the main term in (4.7) with B' given by (4.8) is incorrect. We get the correct result only if we replace B' by B, where B is given by (4.6) and then it is clear that main terms in (4.3) and (4.7) coincide. Kátai (1969) by a different method (he did not notice the mistake in the main term of (4.7) improved the order estimates of $\Delta_u(x)$ to $O(x^{1/2}\log^3 x\, S(u))$, where $S(u)$ is given by (4.9). Except for the factor $\sigma^{*2}_{-\alpha}(u)$, our order estimate of $\Delta_u(x)$ is better than of Kátai. In case $u = 1$, Sita Rama Chandra Rao (1980) recently announced a result viz., $\Delta_1(x) = O(x^{1/2}\log^2 x)\log\log x^{(q)}$ for every real q, which turns out to be weaker than that given in (4.4) ($u=1$). One of the authors (Sita Ramaiah 1978) had earlier announced a much weaker result viz., $\Delta_u(x) = O(x^{1/2}\log^2 x\;\sigma^{*2}_{-\alpha}(u))$.

Remark 4.2: It has been mentioned by Gordon and Rogers (1964) at the end of their paper, that their method could be applied to estimate the sum in (4.7) when $|\mu(n)|$ is replaced by $q_k(n)$; that is, an asymptotic formula for the sum in (4.1) could be established by the methods of their paper. Although they did not mention anything about the order estimates of the error term $\Delta_k(x;u)$ in the formula (4.1), it is possible by their method to establish (4.1) for $k \geqslant 3$ also, with $\Delta_k(x;u) = O(x^{1/3}\log^6 x S_\alpha(u))$ or $O(x^\alpha S_\alpha(u))$, according as $k = 3$ or $k \geqslant 4$, where $S_\alpha(u) = \sum_{d|u}\mu^2(d)\,3^{\omega(d)}\,d^{-\alpha}$. Further if we make use of Lemma 2.6 of Suryanarayana and Prasad (1975) instead of Lemma 3 of Gordon and Rogers (1964) and adopt their method, it is possible to establish the formula (4.1) with slightly better O-estimates of the error term than those mentioned above viz., $\Delta_k(x;u) = O(x^{1/3}\log^6 x\;\sigma^{*2}_{-\alpha}(u))$ or $O(x^\alpha\,\sigma^{*2}_{-\alpha}(u))$ according as $k = 3$ or $k \geqslant 4$. In case $k = 3$, it is clear that the order estimate of the error term given in (4.2) ($k = 3$) is better than the above (except for the factor involving u).

$n > 1$ is called semi-k-free if in the canonical factorization of n no exponent is equal to k. We shall also consider the integer 1 to be semi-k-free. The concept of semi-k-free integers is due to Suryanarayana (1971). Let q_k^{s*} denote the characteristic function of semi-k-free integers.

Taking $g_k(n) = \tau(n)\ q_k^{s*}(n)$ in the Theorem, we obtain the following asymptotic formula:

$$\sum_{\substack{n\leqslant x\\(n,u)=1}} \tau(n)\ q_k^{s*}(n) = A_k\, x \log x + B_k x + \Delta_{k,u}^{s*}(x) \qquad \ldots(4.10)$$

where A_k and B_k are constants (depending on u) as given in Theorem 1 of Subbarao and Suryanarayana (1978), and

$$\Delta_{k,u}^{s*}(x) = \begin{cases} O\Big(x^{1/2}\delta'(x)\log^2 x\ \sigma^{*2}_{-\alpha}(u)\Big) \text{ if } k = 2, \\ O\Big(x^{1/3}\delta'(x)\log^3 x\ \sigma^{*2}_{-\alpha}(u)\sigma^{*2}_{-1+\epsilon_1}(u)\Big) \text{ if } k = 3, \\ O\Big(x^{\alpha}\ \sigma^{*2}_{-\alpha}(u)\Big) \text{ if } k \geqslant 4, \end{cases} \qquad \ldots(4.11)$$

for every $\epsilon_1 > 0$, the O-estimates being uniform in x and u.

Remark 4.3: Formula (4.10) has been established by Subbarao and Suryanarayana (cf. Subbarao and Suryanarayana 1978, Theorem 1) with an error term $\Delta_{k,u}^{s*}(x)$; where $\Delta_{k,u}^{s*}(x) = O(x^{1/2}\log^3 x\ S'(u))$ or $O(x^{1/2}S'(u))$, according as $k = 2$ or $k \geqslant 3$, where $S'(u) = \sum_{d|u} 3^{\omega(d)} d^{1/2}$. It is clear that the order estimates of the error term $\Delta_{k,u}^{s*}(x)$ given by (4.11) are better than the order estimates of $\Delta_{k,u}^{s*}(x)$ obtained by Subbarao and Suryanarayana (except for the factor involving u).

$n > 1$ is called unitarily k-free (Cohen 1961) if in the canonical representation of n no exponent is a multiple of k. We shall also consider the integer 1 to be a unitarily k-free integer. Let q_k^* denote the characteristic function of the unitarily k-free integers.

Taking $g_k(n) = \tau(n)\ q_k^*(n)$ in the theorem, we obtain the following:

Theorem 4.2—We have

$$\sum_{\substack{n\leqslant x\\(n,u)=1}} \tau(n)\ q_k^*(n) = a_k^*(u)\, x \log x + x\, b_k^*(u) + \Delta_k^*(x;u) \qquad \ldots(4.12)$$

where

$$\Delta_k^*(x;u) = \begin{cases} O\Big(x^{1/2}\delta'(x)\log^2 x\ \sigma^{*2}_{-\alpha}(u)\Big) \text{ if } k = 2 \\ O\Big(x^{1/3}\delta'(x)\log^3 x\ \sigma^{*2}_{-\alpha}(u)\,\sigma^{*2}_{-1+\epsilon_1}(u)\Big) \text{ if } k = 3 \\ O\Big(x^{\alpha}\ \sigma^{*2}_{-\alpha}(u)\Big) \text{ if } k \geqslant 4. \end{cases}$$

for every $\epsilon_1 > 0$, the O-estimates being uniform in x and u. Further we have

$$a_k^*(u) = \prod_p \left\{1 - \frac{((k+1)p^k-1)(p-1)^2}{p^2(p^k-1)^2}\right.$$

$$\times \prod_{p|u} \left(\frac{p-1}{p}\right)^2 \left\{1 - \frac{((k+1)p^k-1)(p-1)^2}{p^2(p^k-1)^2}\right\}^{-1}$$

and

$$b_k^*(u) = a_k^*(u) \left\{2\gamma - 1 + 2\sum_{p|u} \frac{\log p}{p-1}\right.$$

$$\left. + \sum_{p|u} \frac{(p-1)(k(k+1)p^{2k+1}-(k+1)(k+2)p^{2k}+(3k+4)p^k-kp^{k+1}-2)\log p}{(p^k-1)(p^{2k+2}-(k+3)p^{k+2}+2(k+1)p^{k+1}-(k+1)p^k+2p^2-2p-1)}\right\}.$$

Remark 4.4: Let r be fixed positive integer. $n > 1$ is called a k-skew integer of rank r (Cohen 1961) if no exponent in the canonical representation is equal to jk where $1 \leqslant j \leqslant r$. We shall also consider the integer 1 to be a k-skew integer of rank r. Let $q_{k,r}^*$ denote the characteristic function of the set of k-skew integers of rank r, Taking $g_k(n) = \tau(n)\, q_{k,r}^*(n)$ in the Theorem, we obtain an asymptotic formula for the sum

$$\sum_{\substack{n \leqslant x \\ (n,u)=1}} \tau(n)\, q_{k,r}^*(n).$$

Remark 4.5: Let $1 < r < k$ be integers. $n > 1$ is called a (k,r)-free integer (Hardy 1976) if in the canonical factorization of n no exponent belongs to the interval $[r, k-1]$. We shall also consider the integer 1 to be a (k,r)-free integer. These integers include as special cases the r-free integer ($k = \infty$) and semi-r-free integers ($k = r+1$). n is called a (k,r)-integer is n can be represented as $n = n_1^k n_2$ where n_1 is a positive integer and n_2 if an r-free integer. The concept of (k,r)-integers has been introduced independently (using different notations) by Cohen (1963) and Subbarao and Harris (1966). Let $q_{(k,r)}$ denote the characteristic function of the set of (k,r)-free integers and $q_{k,r}$ denote the characteristic function of the set of (k,r)-integers. Taking $g_r(n) = \tau(n)\, q_{(k,r)}(n)$ and $g_r(n) = \tau(n)\, q_{k,r}(n)$ in the theorem successively, we obtain asymptotic formulas for the sums

$$\sum_{\substack{n < x \\ (n,u)=1}} \tau(n)\, q_{(k,r)}(n) \quad \text{and} \quad \sum_{\substack{n \leqslant x \\ (n,u)=1}} \tau(n)\, q_{k,r}(n).$$

Remark 4.6: Let K be a non-empty set of natural numbers > 1. $m > 1$ is called a K-void integer if in the canonical factorization m no exponent belongs to the set K. We shall also consider the integer 1 to be a K-void integer. The concept of K-void integers is due to Rieger (1973). Let q_K denote the characteristic function of the set of K-void integers. It is interesting to note that q_K coincides with the functions

q_k, q_k^{s*}, q_k^*, $q_{k,r}^*$, $q_{(k,r)}$ and $q_{k,r}$, if K is suitably chosen. Let $k = \min K$. Taking $g_k(n) = \tau(n)\, q_k(n)$ in the Theorem, we can establish an asymptotic formula for the sum $\sum_{\substack{n \leq x \\ (n, u)=1}} \tau(n)\, q_K(n)$.

Remark 4.7: Let K and K' be two non-empty subsets of natural numbers > 1 with $k = \min K = \min K'$. A positive divisor of d of n is called a K-void divisor if d is a K-void integer. Let $\tau_K(n)$ denote the number of K-void divisors of n. Let $q_{K'}$ be the characteristic function of the K'-void integers. We note that by taking $g_k(n) = \tau_K(n)\, q_{K'}(n)$ in the theorem we can establish an asymptotic formula for the sum $\sum_{\substack{n \leq x \\ (n,u)=1}} \tau_K(n)\, q_{K'}(n)$.

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