Indian J. pure appl. Math., **13**(7): 785–794, July 1982

ON THE ABSOLUTE ZERO ORDER SUMMABILITY OF A FOURIER SERIES AND ITS ALLIED SERIES

B. PATRA

Sinha Lodge, Emporium Lane, Cuttack 753007

(*Received* 10 *October* 1980; *after revision* 22 *January* 1982)

Zero order summability was first studied by Bosanquet and Linfoot (1931a). This method is obtained by merely putting $\alpha = 0$ in the generalized summability (i.e., (α, β)-summability) defined by them (Bosanquet and Linfoot 1931b). Boyer and Holder (1963) have studied the consistency of $|\alpha, \beta|$-summability method. In the present paper we study the 'C' independence and absolute regularity of absolute zero order summability method and its application to a Fourier series and its allied series.

§ 1. *Preliminaries*—Let $f(t)$ be integrable-L over $(-\pi, \pi)$ and periodic with period 2π. The Fourier series of $f(t)$ is

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n(t) \equiv \sum_{n=0}^{\infty} A_n(t). \qquad \ldots(1.1)$$

The series allied or conjugate to (1.1) is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) \equiv \sum_{n=1}^{\infty} B_n(t). \qquad \ldots(1.2)$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\}.$$

Throughout the paper by '$F(t) \in BV(h, k)$' we mean $F(t)$ is of bounded variation over (h, k).

2. *Definition*—Suppose that either $\alpha > 0$ (any real β) or $\alpha = 0, \beta \geqq 0$, then we say that a series Σa_n is summable (α, β) to s, if as $w \to \infty$ (Bosanquet and Linfoot 1931 b)

$$A_{\alpha,\beta}(w) = \sum_{n<w} B\left(1 - \frac{n}{w}\right)^{\alpha} \log^{-\beta} \frac{C}{1-nw^{-1}}\, a_n \to s \qquad \ldots(2.1)$$

where C is sufficiently large and

$$B = (\log C)^{\beta}.$$

A series Σa_n is said to be absolutely summable (α, β) or in short $|\alpha, \beta|$ if $A_{\alpha,\beta}(w) \in BV(k, \infty)$

$$\text{i.e. } \int_k^\infty \left| \frac{d}{dw} A_{\alpha,\beta}(w) \right| dw < \infty \qquad \text{.. (2.2)}$$

k being a positive constant.

For absolute zero order summability we will be concerned with case $\alpha = 0$ of (2.2). From (2.2) it is evident that a series Σa_n is summable $|0, \beta|$, $\beta > 0$ if,

$$\int_k^\infty \frac{dw}{w^2} \left| \sum_{n<w} (1-\frac{n}{w})^{-1} \log^{-\beta-1} \frac{C}{1-nw^{-1}}\, n\, a_n \right| < \infty\,. \qquad \text{.. (2.3)}$$

We suppose throughout in what follows that $\alpha > 0$ (any real β) or that $\alpha = 0$, $\beta > 0$.

§ 3. Bosanquet and Linfoot (1931 b, Theorem 3.2) have proved that for a given α and β, there is a C_0 such that if this definition of summability-(α, β) is satisfied with some $C > C_0$ then it is satisfied for all $C > 1$. Thus, provided we take $C > C_0$, the definition is independent of C. It is natural to conjecture that a corresponding result holds for absolute summability too. We have been able to solve it partially and our result reads as follows :

Theorem 3.1—To each $\beta > 0$, there corresponds a $C_0 \geqq 1$ such that the validity of (2.3) for some $C > C_0$ implies its validity for every $C > 1$.

To prove this we need the help of the following lemma.

Lemma 3.1.1 Boyer and Holder (1963),—Let $f(x)$, $k(u)$, $K(u)$ satisfy the following conditions.*

(i) For some $n \geqq 0$, $V_0^T(x^{-n} f(x)) < \infty$ for all $T > 0$, (It will be assumed throughout that for $x = 0$, the function $x^{-n} f(x)$ is replaced by $\lim_{x \to +0} x^{-n} f(x)$)

(ii) $k(u)$ is absolutely continuous in [0, 1]

(iii) $K(u)$ is positive, continuously differentiable in [0, 1), Lebesgue integrable over [0, 1],

$$\lim_{u-1-} K(u) = +\infty \text{ and } \frac{uK'(u)}{K(u)} \text{ is nondecreasing.}$$

$$\text{Let } F(x) = x^{-n} \int_0^1 K(u) f(xu)\, du$$

$$G(x) = x^{-n} \int_0^1 k(u)\, K(u) f(xu)\, du$$

$$\text{then } \mathop{V}_0^\infty G(x) \leqq \gamma \mathop{V}_0^\infty F(x), \text{ where } \gamma = \int_0^1 |k'(u)|\, du + k(1).$$

Proof of Theorem 3.1 : It is known that (Bosanquet and Linfoot 1931 b) $A_{\alpha,\beta}(w)$ has the following integral representation,

$$A_{\alpha,\beta}(w) = \int_0^1 \Phi'_{\alpha,\beta}(1-u)\, A(wu)\, du \qquad \text{...(3.1.2)}$$

where $A(x) = \sum_{n<x} a_n$ and $\Phi_{\alpha,\beta}(u) = \begin{cases} B\, u^{\alpha} \log^{-\beta} \frac{C}{u}, & \text{if } 0 < u \leqq 1 \\ 0, & \text{if } u \leqq 0. \end{cases}$

In the proof of theorem 3.1 we shall write $\Phi_{\alpha,\beta,C}(u)$ and $A_{\alpha,\beta,C}(w)$ in place of $\Phi_{\alpha,\beta}(u)$ and $A_{\alpha,\beta}(w)$ respectively since we intend to establish that C is a dummy parameter.

For $\beta > 0$, we write

$$\Phi_{0,\beta,C}(u) = \begin{cases} B \log^{-\beta} \frac{C}{u}, & 0 < u \leqq 1 \\ 0, & u \leqq 0 \,. \end{cases}$$

Let C_0 be such that for $C > C_0$, $\Phi'_{0,\beta,C}(1-u)$ satisfies the requirements of $K(u)$ is Lemma 3.1.1.

Let $C_1 > C_0$ be given and let $C > 1$.
From (3.1.2), we have

$$A_{0,\beta,C_1}(w) = \int_0^1 \Phi'_{0,\beta,C_1}(1-u)\, A(wu)\, du$$

and $$A_{0,\beta,C}(w) = \int_0^1 \Phi'_{0,\beta,C}(1-u)\; A\,(wu)\, du,$$

where $A(x) = \sum_{n<x} a_n$.

For $u \neq 1$, writing $K(u) = \Phi'_{0,\beta,C_1}(1-u)$ and

$$k\,(u) = \frac{\Phi'_{0,\beta,C}(1-u)}{\Phi'_{0,\beta,C_1}(1-u)}\,,$$

we get $$A_{0,\beta,C_1}(w) = \int_0^1 K(u)\; A(wu)\; du$$

and $$A_{0,\beta,C}(w) = \int_0^1 k(u)\, K(u)\, A(wu)\, du.$$

By routine calculation it is seen that $k(u)$ and $K(u)$ satisfy the requirements of Lemma 3.1.1 and hence by this lemma we have

$$A_{0,\beta,C_1}(w) \in BV(0, \infty) \Rightarrow A_{0,\beta,C}(w) \in BV(0, \infty).$$

It follows from the theorem that in the definition of $|\,0, \beta\,|$-summability it is immaterial whether we say for an arbitrary large C, for every sufficiently large C or for every $C > 1$. This means in our definition of $|\,0, \beta\,|$-summability. 'C' is a dummy parameter.

§ 4. Absolute regularity of (α, β)-summability method is an immediate consequence of the following consistency theorem of $|\,\alpha, \beta\,|$-summability due to Boyer and Holder (1963).

Theorem 4.1 (Boyer and Holder 1963)—If Σa_n is summable $|\,\alpha, \beta\,|$, then it is summable $|\,\alpha', \beta'\,|$, for $\alpha' > \alpha$, or $\alpha' = \alpha$, $\beta' > \beta$.

However for the completeness of the work we establish the absolute regularity of (α, β)-summability method by providing an alternative general proof*.

Let $h(u)$ be any function defined for $u \geqq 0$. Let us consider the summability method given by the transformation

$$T(w) = \sum_{n=0}^{\infty} h\left(\frac{n}{w}\right) a_n. \qquad \ldots(4.1.1)$$

It is trivial that in order that $\sum_{n=0}^{\infty} |a_n| < \infty$ should imply that $T(w) \in BV$ $(0, \infty)$, it is sufficient to show that $h(u) \in BV(0, \infty)$. For, we then have

$$\int_0^{\infty} |d\,T(w)| \leqq \sum_{n=0}^{\infty} |a_n| \int_0^{\infty} |d\,h(nw^{-1})|. \qquad \ldots(4.1.2)$$

When $n = 0$ the integral on the right of (4.1.2) is 0. If $n \geqq 1$ then, as w increase from 0 to ∞, we see nw^{-1} decreases from ∞ to 0, so that this integral is equal to

$$\int_0^{\infty} |d\,h(u)| = I, \text{ say, where } I \text{ is a constant,}$$

Thus the expression on the right of (4.1.2) is equal to

$$I \sum_{n=0}^{\infty} |a_n| < \infty.$$

It is easily verified that the series to function transformation (4.1.1) is absolutely regular (that is to say, the above holds and whenever $\sum_{n=0}^{\infty} |a_n| < \infty$ the limit $\lim_{w\to\infty} T(w)$ agrees with the sum $\sum_{n=0}^{\infty} a_n$) if, in addition to $h(u) \in BV(0, \infty)$, we have $h(0) = h(0+) = 1$. These conditions are satisfied, in particular, when we take

$$h(u) = \begin{cases} B(1-u)^{\alpha} \log^{-\beta} \dfrac{C}{1-u}, & (0 \leqq u < 1) \\ 0 & (u \geqq 1) \end{cases}$$

for $\alpha > 0$ (any real β) or $\alpha = 0$, $\beta > 0$.

§5. We now prove the following theorems

Theorem 5.1—If $\phi(t) \in BV(0, \pi)$ then the series (1.1) is summable $|0, 1+\delta|$, $\delta > 0$.

Theorem 5.2—If (i) $\psi(t) \in BV(0, \pi)$, (ii) $\int_0^{\pi} \frac{|\psi(t)|}{t} dt < \infty$ then the series (1.2) is summable $|0. 1+\delta|$, $\delta > 0$.

* The author is indebted to the referee for suggesting the present proof.

In conclusion of Theorems 5.1 and 5.2, δ cannot be replaced by -1. Since the absolute convergence of Fourier Series and its allied series cannot be ensured by the existing conditions on the generating function (see examples given in Mohanty and Ray 1967). Now it is natural to enquire what happens to the validity of the theorems when δ satisfies the inequality $-1 < \delta \leqq 0$. As we have not been able to answer this question, it still remains as an open problem.

From theorem 4.1 it follows that summability $| C, \alpha |, \alpha > 0$ includes summability. $| 0, 1+\delta |, \delta > 0$ and hence our theorem 5.1 and theorem 5.2 include the following results on absolute Cesàro summability of Fourier Series and Allied series respectively.

Theorem B (Bosanquet 1936)—If $\phi(t)$ is BV in the interval $(0, \pi)$ then the Fourier Series of $f(t)$ is summable $| C, \delta |$ at the point $t = x$ for every $\delta > 0$.

Theorem BH (Bosanquet and Hyslop 1937—If

(i) $\psi(t) \in BV(0, \pi)$

and (ii) $\dfrac{\psi(t)}{t} \in L\ (0, \pi)$

then the allied series of the Fourier series of $f(t)$ at $t = x$ is summable $| C,\delta |, \delta>0$.

We write throughout the paper $F(u) = (1-u)^{-1} \log^{-2-\delta} C/1-u$ for $0 \leqq u < 1$. Without loss of generality we assume throughout the paper that $C \geqq e^{2+\delta}, \delta > 0$. This ensures that $F(u)$ is nondecreasing in $[0, 1)$.

In order to prove the theorems we need a number of lemmas.

Lemma 1—Uniformly in $0 < t < \pi$, $wt \geqq 1$, we have

$$\sum_{n<w-1} F\left(\frac{n}{w}\right) \begin{matrix}\sin nt\\ \cos nt\end{matrix} = 0 \{ w \log^{-1-\delta} C\, wt \}.$$

Proof of Lemma 1 : $\displaystyle\sum_{n<w-1} F\left(\frac{n}{w}\right) \begin{matrix}\sin nt\\ \cos nt\end{matrix} = \sum_{0\leqq n\leqq w-t^{-1}} + \sum_{\substack{w-t^{-1}<n\\ <w-1}} = \sum_1 + \sum_2$, say.

$$\left|\sum_1\right| = \left|\sum_{0\leqq n \leq w-t^{-1}} F\left(\frac{n}{w}\right) \begin{matrix}\sin nt\\ \cos nt\end{matrix}\right|$$

$$\leqq F\left(\frac{w-t^{-1}}{w}\right) \max_{0\leqq M,M'\leqq w-t^{-1}} \left| \sum_{M}^{M'} \begin{matrix}\sin nt\\ \cos nt\end{matrix} \right|$$

$$= O(wt \log^{-2-\delta} C\, wt)\, t^{-1}$$

$$= O(w \log^{-2-\delta} G\, wt) \qquad \ldots(5.1)$$

$$\left|\sum_2\right| = \left|\sum_{w-t^{-1}n<w<-1} F\left(\frac{n}{w}\right) \begin{matrix}\sin nt\\ \cos nt\end{matrix}\right|$$

$$< \int_{w-t^{-1}}^{w} F\left(\frac{x}{w}\right) dx = O(w \log^{-1-\delta} C\, wt). \qquad \ldots(5.2)$$

combining (5.1) and (5.2) we get lemma 1.

Lemma 2—$\sum_{n<w-1} F(nw^{-1}) \sin nt = O(w^2 t)$.

Proof of Lemma 2 : Since $|\sin nt| \leqq nt < wt$, we have,

$$\left|\sum_{n<w-1} F\left(\frac{n}{w}\right) \sin nt\right| < wt \sum_{n<w-1} F\left(\frac{n}{w}\right)$$

$$< wt \int_0^w F\left(\frac{x}{w}\right) dx = O(w^2 t).$$

This completes the proof of Lemma 2.

Lemma 3— $\sum_{n<w-1} n F\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u} du = O(w^2)$.

Proof of Lemma 3: $\left|\sum_{n<w-1} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u} du\right| = \left|\sum_{n<w-1} n F\left(\frac{n}{w}\right) \int_{nt}^{n\pi} \frac{\sin u}{u} du\right|$

$$< K \sum_{n<w-1} nF\left(\frac{n}{w}\right) = O(w^2),$$

proceeding in the similar way as in case of lemma 2 and K is an absolute constant.

This terminates the proof of lemma 3.

Lemma 4—For $wt \geqq 1$,

$$\sum_{n<w-1} F\left(\frac{n}{w}\right) n \int_t^\pi \frac{\sin nu}{u} du = O\{t^{-1} w \log^{-1-\delta} Cwt\}.$$

Proof of Lemma 4: $\sum_{n<w-1} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u} du = \int_t^\pi \frac{1}{u}\left\{\sum_{n<w-1} nF\left(\frac{n}{w}\right) \sin nu\right\} du$

$$= t^{-1} \int_t^\xi \sum_{n<w-1} nF\left(\frac{n}{w}\right) \sin nu \, du, \quad t < \xi < \pi$$

$$= t^{-1} \sum_{n<w-1} nF\left(\frac{n}{w}\right) \left[-\frac{\cos nu}{n}\right]_t^\xi$$

$$= t^{-1} \left\{\sum_{n<w-1} F\left(\frac{n}{w}\right) \cos nt - \sum_{n<w-1} F\left(\frac{n}{w}\right) \cos n\xi\right\}$$

$$= O\{w t^{-1} \log^{-1-\delta} C wt\}, \text{ by Lemma 1.}$$

Hence the lemma.

Lemma 5 (Mohanty and Ray 1969)—Let $\rho > 0$ be given. In order that

(i) $\psi(t) \in BV(0, \pi)$ and (ii) $\dfrac{|\psi(t)|}{t} \in L(0, \pi)$ it is necessary and sufficient that

$$\int_0^\pi t^{-\rho} \, |dH(t)| < \infty$$

and $H(+0) = 0$, where $H(t) = t^\rho \psi(t)$.

§ 6. *Proof of Theorem* 5.1—For $n \geqq 1$, we obtain on integration by parts,

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \phi(t) \cos nt \, dt$$

$$= \frac{2}{\pi} \left[\phi(t) \frac{\sin nt}{n} \right]_0^\pi - \frac{2}{\pi} \int_0^\pi \frac{\sin nt}{n} \, d\phi(t)$$

$$= -\frac{2}{\pi} \int_0^\pi \frac{\sin nt}{n} \, d\phi(t) \, .$$

Using the above formula, $\Sigma A_n(x)$ is summable $|0, 1+\delta|$ if,

$$I = \frac{2}{\pi} (1+\delta) \int_e^\infty \frac{dw}{w^2} \left| -\sum_{n<w} F\left(\frac{n}{w}\right) \int_0^\pi \sin nt \, d\phi(t) \right| < \infty.$$

Since $I \leqq \int_0^\pi |d\phi(t)| \int_e^\infty \dfrac{dw}{w^2} \left| \sum_{n<w} F\left(\dfrac{n}{w}\right) \sin nt \right|$, and by

hypothesis, $\int_0^\pi |d\phi(t)| < \infty$, it is sufficient to show that uniformly in $0<t<\pi$,

$$q = \int_e^\infty \frac{dw}{w^2} \left| \sum_{n<w} F\left(\frac{n}{w}\right) \sin nt \right| = O(1). \qquad \text{...(6.1)}$$

$$\text{Now } q \leqq \int_e^\infty \frac{dw}{w^2} \left| \sum_{n<w-1} F\left(\frac{n}{w}\right) \sin nt \right| + \int_e^\infty \left| \frac{d(w)}{w^2} \right| \left| \sum_{w-1 \leqq n<w} F(n/w) \sin nt \right|$$

$$= q_1 + q_2, \text{ say.} \qquad \text{...(6.2)}$$

Now let $N = N(w)$ be the integer such that

$N < w \leqq N+1$, then

$$q_2 = \int_e^\infty \frac{dw}{w^2} \left| \sum_{w-1 \leqq n<w} F\left(\frac{n}{w}\right) \sin nt \right| = \int_e^\infty \frac{dw}{w^2} \left| F\left(\frac{N}{w^2}\right) \sin Nt \right|$$

(equation continued on p. 792)

$$\leqq \sum_{N=2}^{\infty} N^{-1} \int_{N}^{N+1} F\left(\frac{N}{w}\right) \frac{N}{w^2}\, dw = \sum_{N=2}^{\infty} N^{-1} \left[\frac{1}{1+\delta} \log^{-1-\delta} \frac{C}{1-\frac{N}{w}} \right]_{N+0}^{N+1}$$

$$= \frac{1}{1+\delta} \sum_{N=2}^{\infty} N^{-1} \log^{-1-\delta} C\,(N+1) = O\,(1) . \qquad \ldots(6.3)$$

Now $q_1 = \int_{e}^{t^{-1}} + \int_{t^{-1}}^{\infty} = q_{1,1} + q_{1,2}$, say . ...(6.4)

By an appeal to lemma 2,

$$q_{1,2} = \int_{e}^{t^{-1}} \frac{dw}{w^2}\, O\,(w^2 t) = O\left\{ t\,[w]_{e}^{t^{-1}} \right\} = O\,(1), \qquad \ldots(6.5)$$

for $0 < t < \pi$.

Applying lemma 1, we have

$$q_{1,2} = \int_{t^{-1}}^{\infty} \frac{dw}{w^2}\, O\,(w \log^{-1-\delta} C\,wt) = O\left(\int_{t^{-1}}^{\infty} \frac{dw}{w \log^{1+\delta} C\,wt} \right)$$

$$= O\left(\int_{C}^{\infty} \frac{du}{u \log^{1+\delta} u} \right) = O\,(1) . \qquad \ldots(6.6)$$

Hence combining the results (6.2), (6.3), (6.4), (6.5), and (6.6) together we get (6.1).

This completes the proof of Theorem 5.1.

§ 7. Writing $t\,\psi\,(t) = H\,(t)$, by virtue of lemma 5, our theorem 5.2 takes the following equivalent form.

Theorem 5.2 *a*—If (i) $\int_{0}^{\pi} t^{-1}\,|\,d\,H\,(t)\,| < \infty$ and (ii) $H\,(+\,0) = 0$, then the series (1.2) is summable $|\,0,\ 1+\delta\,|$, $\delta > 0$.

Proof of Theorem 5.2 *a* :—On integration by parts,

$$B_n\,(x) = \frac{2}{\pi} \int_{0}^{\pi} \psi\,(t) \sin nt\, dt = \frac{2}{\pi} \int_{0}^{\pi} t\,\psi\,(t)\, \frac{\sin nt}{t}\, dt.$$

$$= \frac{2}{\pi} \left[-H(t) \int_{t}^{\pi} \frac{\sin nu}{u}\, du \right]_{0}^{\pi} + \frac{2}{\pi} \int_{0}^{\pi} d\,H\,(t) \int_{t}^{\pi} \frac{\sin nu}{u}\, du$$

$$= \frac{2}{\pi} \int_{0}^{\pi} d\,H\,(t) \int_{t}^{\pi} \frac{\sin nu}{u}\, du .$$

Thus it is enough to show that

$$J = \int_0^\pi |dH(t)| \int_1^\infty \frac{dw}{w^2} \left| \sum_{n<w} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u}\, du \right| < \infty .$$

Since by hypothesis $\int_0^\pi t^{-1}\, |dH(t)| < \infty$, it is sufficient to show that uniformly in $0 < t < \pi$,

$$g(t) = \int_1^\infty \frac{dw}{w^2} \left| \sum_{n<w} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u}\, du \right| = O(t^{-1}) . \qquad ...(7.1)$$

Now
$$g(t) \leqq \int_1^\infty \frac{dw}{w^2} \left| \sum_{n<w-1} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u}\, du \right|$$
$$+ \int_1^\infty \frac{dw}{w^2} \left| \sum_{w-1<1\leqq n<w} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u}\, du \right|$$
$$= g_1(t) + g_2(t) \text{, say .} \qquad ...(7.2)$$

Let $N = N(w)$ be the integer such that $N < w \leqq N+1$,

then
$$g_2(t) = \int_1^\infty \frac{dw}{w^2} \left| \sum_{w-1\leqq n<w} nF\left(\frac{n}{w}\right) \int_t^\pi \frac{\sin nu}{u}\, du \right|$$
$$= \int_1^\infty \frac{dw}{w^2} \left| NF\left(\frac{N}{w}\right) \int_t^\pi \frac{\sin Nu}{u}\, du \right|$$
$$= \int_1^\infty \frac{dw}{w^2} \left| t^{-1}\, NF\left(\frac{N}{w}\right) \int_t^\xi \sin Nu\, du \right| , \quad t < \xi < \pi$$
$$= \int_1^\infty \frac{dw}{w^2} \left| t^{-1} F\left(\frac{N}{w}\right) (\cos Nt - \cos N\xi) \right|$$
$$\leqq 2\, t^{-1} \int_1^\infty F\left(\frac{N}{w}\right) \frac{dw}{w^2}$$
$$= 2\, t^{-1} \sum_{N=1}^\infty N^{-1} \int_N^{N+1} F\left(\frac{N}{w}\right) \frac{N}{w^2}\, dw$$
$$= 2\, t^{-1} \sum_{N=1}^\infty N^{-1} \left[\frac{1}{1+\delta} \log^{-1-\delta} \frac{C}{1-\frac{N}{w}} \right]_{N+0}^{N+1}$$
$$= \frac{2\, t^{-1}}{1+\delta} \sum_{N=1}^\infty N^{-1} \log^{-1-\delta} C(N+1) = O(t^{-1}). \qquad ...(7.3)$$

Now $g_1(t) = \int_1^{t^{-1}} + \int_{1/t}^{\infty} = g_{1,1}(t) + g_{1,2}(t)$, say. ...(7.4)

By an appeal to Lemma 3,

$$g_{1,1}(t) = \int_1^{1/t} \frac{dw}{w^2}\, O(w^2) = O\left([w]_1^{t^{-1}}\right) = O(t^{-1}), \quad \text{...(7.5)}$$

and by an appeal to Lemma 4,

$$g_{1,2}(t) = \int_{t^{-1}}^{\infty} \frac{dw}{w^2}\, O(t^{-1} w \log^{-1-\delta} Cwt)$$

$$= O\left(t^{-1} \int_{t^{-1}}^{\infty} \frac{dw}{w \log^{1+\delta} C\, wt}\right) = O\left(t^{-1} \int_C^{\infty} \frac{du}{u \log^{1+\delta} u}\right)$$

$$= O(t^{-1})\,. \quad \text{...(7.6)}$$

Combining the results (7.2), (7.3), (7.4), 7.5) and (7.6) we get (7.1).
This establishes Theorem 5.2a completely.

ACKNOWLEDGEMENT

The author is indebted to Dr R. Mohanty for his kind encouragement during the preparation of this paper. He is also indebted to Dr B. K. Ray for his kind help for the improvement of this paper. He is thankful to the referee for his valuable suggestions for the completeness of this paper.

REFERENCES

Bosanquet, L.S. (1936). Notes on the absolute summability (*C*) of a Fourier series. *J. Lond. Math. Soc.*, 11–15.

Boanquet, L.S., and Hyslop, J. M. (1937). On the absolute summability of the allied series of a Fourier series. *Math. Z.*, **42**, 489–512.

Bosanquet, L.S., and Linfoot, E.H., 1931 a). On the zero order summability of a Fourier series. *J. Lond. Math. Soc.*, **6** No. 22. 117–26.

——— (1931b). Generalized means and the summability of Fourier series. *Quart. J. Math.*, **2**, 207–29.

Boyer, J., and Holder, L. I. (1963). A generalization of absolute Rieszian summability. *Proc. A. Math Soc.*, **14**, No. 3, 459–64.

Mohanty, R., and Ray, B. K. (1967). On the non-absolute summability of a Fourier series and conjugate of a Fourier series by a Nörlund method. *Proc. Camb. Phil. Soc.*, **63**, 407–11.

——— (1969). On the behaviour of a series associated with conjugate series of a Fourier series, *Canad. J. Math.*, **21**, 535–51.