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ON GENERALIZED LOTOTSKY SUMMABILITY

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In this paper we study the summability properties of the generalized Lototsky method. This method include as special cases the generalizations of the Lototsky methods studied by Gaston Smith (1965) and Amnon Jakimovski (1959).

1. INTRODUCTION

Let $\{f_\nu(z)\}$ be a sequence of entire functions and $\{h_\nu\}$ a sequence of values. Define a matrix (a_{nk}) by the relations

$$\left.\begin{array}{l} a_{00}=1 \ ; \ a_{0k} = 0 \ (k>0) \\ \prod\limits_{\nu=1}^{n} (f_\nu(z)h_\nu+1-h_\nu) = \sum\limits_{k=0}^{\infty} a_{nk}\, z^k \end{array}\right\} \qquad ...(1.1)$$

Denote by $L(f_\nu,h_\nu)$ the summability transform associated with the matrix (a_{nk}). When $h_\nu = 1/1+d_\nu$ is a bounded sequence of scalars and $f_\nu(z) = f(z)$ for each ν and $f_\nu(1) = 1$, then (a_{nk}) represents the (f, d_n) matrix (Smith 1965). With the further assumption that $f(z) = z$, we have $[F, d_n]$ matrix (Jakimovski 1959). Both (f, d_n) and $[F, d_n]$ matrices are generalizations of the Lototsky matrix (Angew 1957). In this paper we study the summability properties of the $L(f_\nu, h_\nu)$ transform.

In section 2 we discuss the regularity of the $L(f_\nu, h_\nu)$ transform. Sections 3 and 4 are devoted to the study of the $L(f_\nu, h_\nu)$ summability of the geometric series as well as the Legendre series. We determine in section 5 sufficient conditions on x which ensure that each sequence summable by the Euler summability method $E(x)$ is also summable by the $L(f_\nu, h_\nu)$ transform to the same value. The theorem of this section generalizes a corresponding theorem proved by Agnew (1957) for the Lototsky method. A similar inclusion relation between the Euler summability method and the generalized Euler transform was discussed by Wood (1968).

2. REGULARITY

We prove the following theorem concerning the regularity of the $L(f_\nu, h_\nu)$ transform.

Theorem 2.1—Let $\{f_\nu(z)\}$ be a sequence of entire functions and let $\{h_\nu\}$ a sequence of values such that $0 \leqslant h_\nu \leqslant 1$.
Further let

(i) $f_\nu^{(k)}(0) \geqslant 0 \ (\nu \geqslant 1,\ k \geqslant 0)$

(ii) $f_\nu\ (1) = 1\ (\nu \geqslant 1)$

and

(iii) $\lim_\nu \sup f_\nu(0) < 1$.

Then the $L(f_\nu, h_\nu)$ transform is regular if and only if $\sum_\nu h_\nu = \infty$.

PROOF : Write

$$P_n(z) = \prod_{\nu=1}^{n} (f_\nu(z)\, h_\nu + 1 - h_\nu) = \sum_{k=0}^{\infty} a_{nk}\, z^k . \qquad \ldots(2.1)$$

First assume that the $L(f_\nu, h_\nu)$ transform is regular. Let $\{h_{\nu k}\}$ denote the subsequence of $\{h_\nu\}$ consisting of all those $h_\nu \neq 0$. Such a subsequence exists. For if $h_\nu = 0$ for all ν sufficiently large, then for each fixed k, $a_{nk} = a_{n+1,k}$ for all n sufficiently large, in which case, the $L(f_\nu, h_\nu)$ transformation is not regular. It can be easily seen that the $L(f_\nu, h_\nu)$ transform is equivalent to the $L(f_{\nu k}, h_{\nu k})$ transform. Hence we may assume that $h_\nu \neq 0$ for any ν. Then $0 < h_\nu \leqslant 1$. If $h_\nu = 1$ for infinitely many values of ν, then

$$\sum_\nu h_\nu = \infty.$$

Henee assume that $h_\nu < 1$ for all except a finite number of values of ν, say for all $\nu > N$. Since $f_\nu(1) = 1$ for each ν, there exists integers $k_\nu \geqslant 0$ such that $f_\nu^{(k_\nu)}(0) > 0$. Let $m = k_1 + \ldots + k_N$. Calculation of $a_{n,m}$ from eqn. (2.1) shows that for $n > N$.

$$a_{nm} \geqslant \frac{f_1^{(k_1)}(0)}{k_1!} \cdots \frac{f_N^{(k_N)}(0)}{k_N!} h_1 \ldots h_N \prod_{\nu=N+1}^{n} (f_\nu(0) h_\nu + 1 - h_\nu).$$

Also by the regularity of the $L(f_\nu, h_\nu)$ transform,

$$\lim_{n\to\infty} a_{nm} = 0$$

Further

$$\frac{f_1^{(k_1)}(0)}{k_1!} \cdots \frac{f_N^{(k_N)}(0)}{k_N!} h_1 \ldots h_N \neq 0.$$

Hence

$$\prod_{\nu=N+1}^{\infty} (f_\nu(0)\, h_\nu + 1 - h_\nu) = \prod_{\nu=N+1}^{\infty} (1 - (1 - f_\nu(0) h_\nu)) = 0. \qquad \ldots(2.2)$$

By assumption $h_\nu < 1$ for $\nu > N$ and so for such ν, $0 \leqslant h_\nu (1 - f_\nu(0)) \leqslant h_\nu < 1$. This together with (2.2) implies that

$$\sum_{\nu=N+1}^{\infty} h_\nu (1 - f_\nu(0)) = \infty.$$

Consequently

$$\sum_\nu h_\nu = \infty.$$

Conversely assume that $\sum_\nu h_\nu = \infty$. By (i), $f_\nu(z)$ has non negative Taylor coefficients. Also $0 \leqslant h_\nu \leqslant 1$. Hence from (2.1)

$$a_{nk} \geqslant 0 \ (n \geqslant 1\ ;\ k \geqslant 0). \qquad \ldots(2.3)$$

Since $\limsup_\nu f_\nu(0) < 1$, we can choose $r_0 > 0$ such that

$$\limsup_\nu f_\nu(0) < r_0 < 1.$$

By Cauchy's formula,

$$a_{nk} = \frac{1}{2\pi i}\int_{|z|=r_1} z^{-k-1} P_n(z)\,dz,$$

where $r_1 > 0$ is so chosen that $r_1 < 1 - r_0$.
Hence

$$|\,a_{nk}\,| \leqslant \frac{r_1^{-k}}{2\pi}\int_0^{2\pi} |\,P_n(r_1 e^{i\theta})\,|\,d\theta.$$

By Schwarz lemma,

$$|\,f_\nu(z) - f_\nu(0)\,| \leqslant |\,z\,|,\ (\,|\,z\,| \leqslant 1).$$

In particular,

$$|\,f_\nu(r_1 e^{i\theta})\,| \leqslant f_\nu(0) + r_1.$$

Hence

$$|\,P_n(r_1 e^{i\theta})\,| \leqslant \prod_{\nu=1}^{n} [(f_\nu(0)+r_1)\,h_\nu + 1 - h_\nu]$$

$$= \prod_{\nu=1}^{n} [1 - h_\nu\,(1 - f_\nu(0) - r_1)].$$

Therefore

$$|\,a_{nk}\,| \leqslant r_1^{-k} \prod_{\nu=1}^{n} \ [1 - h_\nu\,(1 - f_\nu(0) - r_1)]. \qquad \ldots(2.4)$$

Since $r_0 > \limsup_\nu f_\nu(0)$, $f_\nu(0) < r_0$ for large values of ν.
Hence for such ν,

$$1 - f_\nu(0) - r_1 \geqslant 1 - r_0 - r_1, > 0.$$

Further $\sum_\nu h_\nu = \infty$. It follows that

$$\sum_\nu (1 - f_\nu(0) - r_1)\,h_\nu = \infty.$$

Therefore the infinite product

$$\prod_{\nu=1}^{\infty} \ [1 - h_\nu\,(1 - f_\nu(0) - r_1)] = 0.$$

From this and (2.4) it follows that

$$\lim_{n\to\infty} \ a_{nk} = 0. \qquad \ldots(2.5)$$

Clearly since $f_\nu(1) = 1$,

$$\sum_{k=0}^{\infty} a_{nk} = P_n(1) = 1. \qquad \ldots(2.6)$$

From (2.3), (2.5) and (2.6) it follows that (a_{nk}) satisfies the well known Silverman—Toeplitz conditions for regularity. Hence the $L(f_\nu, h_\nu)$ transform is regular.

We have as an easy consequence the following corollary.

Corollary 2.2—Let $\{f_\nu, (z)\}$, $\{h_\nu\}$ be as in Theorem 2.1. Further let h_ν converge to $h \neq 0$. Then the $L(f_\nu, h_\nu)$ transform is regular.

3. Summability of the Geometric Series

Theorem 3.1—Let$\{f_\nu(z)\}$, $\{h_\nu\}$ be as in Theorem 2.1. Suppose further that $f_\nu(z)$ converges to a function $f(z)$ uniformly on compact subsets of the plane and that h_ν converges to h. If $h=0$ assume further that $\Sigma\, h_\nu = \infty$. Let the region Δ be defined by

$$\Delta = \begin{cases} \{z: |\, f(z)\, h+1-h \,| < 1\}, & \text{if } h \neq 0 \\ \{z: \operatorname{Re} f(z) < 1\}, & \text{if } h = 0. \end{cases}$$

Then $L(f_\nu, h_\nu)$ continues the geometric series into Δ. For $z = \in \overline{\Delta}^{\sim}$, the complement of Δ closure, $L(f_\nu, h_\nu)$ continues the geometric series if and only if $f_\nu(z)\, h_\nu + 1 - h_\nu = 0$ for some ν. Further there exist at most a countable number of points in $\overline{\Delta}^{\sim}$, where $L(f_\nu, h_\nu)$ continues the geometric series.

Proof: Let $s_k(z)$ be the kth partial sum of the geometric series $\sum_{n=0}^{\infty} z^n$ and let $\{t_n(z)\}$ be the $L(f_\nu, h_\nu)$ transform of $s_k(z)$. Then

$$t_n(z) = \sum_{k=0}^{\infty} a_{nk}\, s_k(z)$$

$$= \sum_{k=0}^{\infty} a_{nk} \frac{1-z^{k+1}}{1-z}$$

$$= \frac{1}{1-z} - \frac{z}{1-z} P_n(z).$$

Thus for $z \neq 1$, $t_n(z) \to 1/1-z$ if and only if $P_n(z) \to 0$.

First assume that $h=0$ so that $h_\nu \to 0$ as $\nu \to \infty$.

Taking z as fixed, write

$$1 - f(z) = \alpha + i\beta.$$

Then

$$|\, 1 - h_\nu (1 - f_\nu(z)) \,|^2 = |\, 1 - h_\nu (\alpha + i\beta) + 0(h_\nu) \,|^2$$
$$= 1 - 2h_\nu \alpha + 0(h_\nu).$$

It therefore follows from the divergence of $\sum_\nu h_\nu$ that

$$\prod_{\nu=1}^{\infty} |\, 1 - h_\nu (1 - f_\nu(z)) \,|^2 = \begin{cases} 0, & \text{if } \alpha > 0 \text{ or } 1 - h_\nu + f_\nu(z) h_\nu = 0 \text{ for some } \nu. \\ \infty, & \text{if } \alpha < 0 \text{ and } 1 - h_\nu + f_\nu(z) h_\nu \neq 0 \text{ for any } \nu \end{cases}$$

Hence if $\alpha > 0$, then $|\, P_n(z) \,|^2 \to 0$ as $n \to \infty$ and if $\alpha < 0$, $|\, P_n(z) \,|^2 \to \infty$ as $n \to \infty$ provided $f_\nu(z) h_\nu + 1 - h_\nu \neq 0$ for any ν. This gives the conclusion.

Now suppose $h \neq 0$. Then

$$| 1-h_\nu(1-f_\nu(z)) | \to | 1-h(1-f(z)) | .$$

Hence if $| 1-h(1-f(z)) | <1$, then $| P_n(z) | \to 0$ as $n \to \infty$ and if $| 1-h(1-f(z)) | >1$, then $| P_n(z) | \to \infty$ as $n \to \infty$, provided $f_\nu(z)h_\nu+1-h_\nu \neq 0$ for any ν. This gives the conclusion in this case.

On the boundary of Δ nothing explicitly could be said about the continuation of the geometric series, for there exist $L(f_\nu, h_\nu)$ transforms that continue the geometric series to uncountably in finite number of points on the boundary and also $L(f_\nu, h_\nu)$ transforms that do not continue the geometric series to any point on the boundary. For example, consider

$$f_\nu(z) = z \text{ for all } \nu \text{ and } h_\nu = h\left(1-\frac{1}{\nu}\right),$$

where h is a constant such that $0<h\leqslant 1$. Then for any z on the boundary of Δ, $(1-f(z))h=1-e^{i\theta}$ for some real θ, so that

$$\begin{aligned} |f_\nu(z)h_\nu+1-h_\nu|^2 &= \left| 1-\left(1-\frac{1}{\nu}\right)\left(1-e^{i\theta}\right)\right|^2 \\ &= \frac{1}{\nu^2}+\left(1+\frac{1}{\nu^2}-\frac{2}{\nu}\right)+\frac{2}{\nu}\left(1-\frac{1}{\nu}\right)\cos\theta \\ &= 1-\frac{2(1-\cos\theta)}{\nu}+0\left(\frac{1}{\nu^2}\right). \end{aligned}$$

Thus we now have $P_n(z)\to 0$ as $n\to\infty$ for all z on the boundary of Δ except for the point corresponding to $\cos\theta=1$, that is to say $z = 1$.

On the other hand, suppose $f_\nu(z) = f(z)$ and $h_\nu=h$ for all ν, where h is a constant such that $0<h\leqslant 1$. Then for all z on the boundary of Δ, $| P_n(z) | =1$ and so does not tend to zero as $n\to\infty$. Hence the $L(f_\nu, h_\nu)$ transform does not continue the geometric series to any point on the boundary of Δ.

4. Summability of a Series of Legendre Polynomials

For $n=0,1,2,\ldots$,let $P_n(z)$ and $Q_n(t)$ denote respectively the Legendre functions of the first and second kind of the nth degree. Then for each n, $P_n(z)$is a polynomial of degree n. The Laplace integral representations of $P_n(z)$ for any z and $Q_n(t)$ for t not in $[-1,1]$ are given by (Whiittaker and Watson 1952, Chapter XV)

$$\left.\begin{aligned} P_n(z) &= \frac{1}{\pi}\int_0^\pi \zeta^n\, d\phi \\ \text{and} & \\ Q_n(t) &= \int_0^\infty \tau^{-n-1}\, ds \\ \text{where} & \\ \zeta &= \zeta(\phi) = z+(z^2-1)^{\frac{1}{2}}\cos\phi \\ \text{and} & \\ \tau &= \tau(s) = t+(t^2-1)^{\frac{1}{2}}\cosh s \end{aligned}\right\} \qquad \ldots(4.1)$$

In the case of $P_n(z)$ we may choose any branch of $(z^2-1)^{1/2}$. But in the case of $Q_n(t)$ we choose that analytic branch of $(t^2-1)^{1/2}$ for which $|\, t+(t^2-1)^{1/2} \,| > 1$.

Fix t so that t does not belong to $[-1,1]$. Write

$$s_k = \sum_{n=0}^{k} (2n+1)\, P_n(z) Q_n(t) \qquad \ldots(4.2)$$

and

$$d_n = P_{n+1}(t)\, Q_n(z) - P_n(z)\, Q_{n+1}(t). \qquad \ldots(4.3)$$

We have by Cristoffel formula (Whittaker and Watson 1952),

$$\frac{1}{t-z} = s_n + (n+1) \frac{1}{t-z}\, d_n. \qquad \ldots(4.4)$$

By Heine's theorem (Whittaker and Watson 1952, p. 321), if z lies in the interior of ellipse with foci $\pm$ 1 and passing through t, then $\{s_n\}$ converges to $1/t-z$. The following theorem asserts that, under suitable conditions, the sequence $\{s_k\}$ is summable by $L\,(f_\nu, h_\nu)$ method to $1/t-z$ in a wider region.

Theorem 4.1—Let $\{f_\nu\}$, $\{h_\nu\}$, f, h be as in Theorem 3.1. If $h \neq 0$, suppose that

$$\left|\, h f\left(\frac{\zeta(\phi)}{\tau(s)}\right) + 1 - h \,\right| < 1, \quad \text{for } 0 \leqslant \phi \leqslant \pi;\ s \geqslant 0.$$

If $h=0$, suppose that for the same range

$$\operatorname{Re} f\left(\frac{\zeta(\phi)}{\tau(s)}\right) < 1.$$

Then the sequence $\{s_k\}$ of partial sums (4.2) is $L\,(f_\nu, h_\nu)$ summable to $(t-z)^{-1}$.

PROOF: Let

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk}\, s_k.$$

Since $\sum_{k=0}^{\infty} a_{nk} = 1$, we have by (4.4),

$$\sigma_n = \frac{1}{t-z} - \frac{1}{t-z} \sum_{k=0}^{\infty} (k+1) d_k\, a_{nk}.$$

Therefore

$$\lim_{n\to\infty} \sigma_n = (t-z)^{-1}$$

if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} (k+1) d_k\, a_{nk} = 0.$$

By (4.1) and (4.3),

$$\sum_{k=0}^{\infty} (k+1)\, d_k\, a_{nk} = \sum_{k=0}^{\infty} \frac{1}{\pi} \int_0^{\infty} \int_0^{\pi} (k+1)\, a_{nk} \left(\frac{\zeta}{\tau}\right)^{k} \left(\frac{\zeta}{\tau} - \frac{1}{\tau^2}\right) d\phi\, ds. \qquad \ldots(4.5)$$

Now

$$\sum_{k=0}^{\infty}(k+1)\,a_{nk}\left(\frac{\zeta}{\tau}\right)^k=\sum_{k=0}^{\infty}a_{nk}\left(\frac{\zeta}{\tau}\right)^k+\frac{\zeta}{\tau}\sum_{k=0}^{\infty}k\,a_{nk}\left(\frac{\zeta}{\tau}\right)^{k-1}$$

$$=P_n\left(\frac{\zeta}{\tau}\right)+\frac{\zeta}{\tau}P_n'\left(\frac{\zeta}{\tau}\right), \qquad ...(4.6)$$

where $P_n'(z)$ stands for $d/dz\,P_n(z)$. Since each f_ν is entire $P_n(z)$ and $P_n'(z)$ are entire functions. Hence the series $\sum_{k=0}^{\infty}(k+1)a_{nk}\,z^k$ is uniformly convergent on every bounded subset of the complex plane. Also the set $\{\zeta/\tau : 0\leqslant\phi\leqslant\pi;\ s\geqslant 0\}$ is bounded. Hence the series $\sum_{k=0}^{\infty}(k+1)\,a_{nk}\,z^k$ is uniformly convergent in $0\leqslant\phi\leqslant\pi;\ s\geqslant 0$. We may therefore interchange the order of summation and integration in the integral in (4.5). Thus

$$\sum_{k=0}^{\infty}(k+1)\,d_k\,a_{nk}=\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\sum_{k=0}^{\infty}(k+1)\,a_{nk}\left(\frac{\zeta}{\tau}\right)^k\left[\frac{\zeta}{\tau}-\frac{1}{\tau^2}\right]d\phi ds$$

$$=\frac{1}{\pi}\int_0^{\infty}\int_0^{\pi}\left[P_n\left(\frac{\zeta}{\tau}\right)+\frac{\zeta}{\tau}P_n'\left(\frac{\zeta}{\tau}\right)\right]\left(\frac{\zeta}{\tau}-\frac{1}{\tau^2}\right)d\phi ds$$

using (4.6). Hence to establish the theorem it suffices to prove that $P_n(\zeta/\tau)$ and $P_n'(\zeta/\tau)$ converge to zero uniformly in $0\leqslant\phi\leqslant\pi;\ 0\leqslant s<\infty$.

Now let Δ be as in Theorem 3.1. The proof of Theorem 3.1. shows that for $z\in\Delta$, we have $P_n(z)\to 0$ as $n\to\infty$ uniformly on any compact subset of Δ. Hence the same holds for $P_n'(z)$. By hypothesis $\frac{\zeta(\phi)}{\tau(s)}\in\Delta$ for $0\leqslant\phi\leqslant\pi;\ 0\leqslant s<\infty$. Since $\frac{\zeta(\phi)}{\tau(s)}\to 0$ as $s\to\infty$ uniformly in ϕ and since $0\in\Delta$, it follows that for $0\leqslant\phi\leqslant\pi;\ 0\leqslant s<\infty$, $\frac{\zeta(\phi)}{\tau(s)}$ lies in a compact subset of Δ. Hence unifomly in ϕ, s,

$$P_n\left(\frac{\zeta(\phi)}{\tau(s)}\right)\to 0;\ P_n'\left(\frac{\zeta(\phi)}{\tau(s)}\right)\to 0 \text{ as } n\to\infty.$$

and the theorem follows.

5. Relation between the Summability Methods of Euler and $L(f_\nu, h_\nu)$

Suppose x is real and positive. Let $E(x)$ denote the Euler transform given by

$$t_n=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\ s_k.$$

It is well known that $E(x)$ is regular for $0<x\leqslant 1$ and that if $x\neq 0$, $E(1/x)$ is the inverse of $E(x)$. For any x, real and positive, the connection between $E(x)$ and the $L(f_\nu, h_\nu)$ transform is given by the following theorem.

Theorem 5.1 – Let $\{f_\nu(z)\}$, $\{h_\nu\}$ be as in Theorem 2.1. Assume further that $\sum\limits_{\nu} h_\nu = \infty$ and that each f_ν is linear, say $f_\nu(z) = a_\nu z + b_\nu$ $(\nu = 1, 2, \ldots)$. Then

$$E(x) \subset L(f_\nu, h_\nu)$$

if $$\sum_{\nu}^{1} \left(\frac{a_\nu h_\nu}{x} - 1 \right) < \infty,$$

where $\overset{1}{\underset{<}{\Sigma}}$ denotes the sum over those ν for which $\frac{a_\nu h_\nu}{x} > 1$.

PROOF : Let $\sigma = \{\sigma_j\}$ be the Euler transform $E(x)$ of $S = \{S_k\}$. Since each f_ν is linear,

$$a_{nk} = 0 \text{ for } k > n.$$

Hence

$$L(f_\nu, h_\nu)\, S = L(f_\nu, h_\nu)\, E\left(\frac{1}{x}\right) \sigma$$

$$= \sum_{k=0}^{n} a_{nk} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1}{x}\right)^{j} \left(1 - \frac{1}{x}\right)^{k-j} \sigma_j$$

$$= \sum_{j=0}^{n} \left(\frac{1}{x}\right)^{j} \sigma_j \sum_{k=j}^{n} \binom{k}{j} a_{nk} \left(1 - \frac{1}{x}\right)^{k-j}$$

$$= \sum_{j=0}^{n} b_{nj}\, \sigma_j,$$

where

$$b_{nj} = \left(\frac{1}{x}\right)^{j} \sum_{k=j}^{n} \binom{k}{j} a_{nk} \left(1 - \frac{1}{x}\right)^{k-j}.$$

Hence to establish the theorem we need only show that (b_{nj}) is regular. Now

$$\sum_{j=0}^{n} b_{nj} z^{j} = \sum_{j=0}^{n} \left(\frac{z}{x}\right)^{j} \sum_{k=j}^{n} \binom{k}{j} \left(1 - \frac{1}{x}\right)^{k-j} a_{nk}$$

$$= \sum_{k=0}^{n} a_{nk} \left(1 - \frac{1}{x} + \frac{z}{x}\right)^{k}$$

$$= P_n\left(1 - \frac{1}{x} + \frac{z}{x}\right).$$

Hence b_{nj} is the coefficient of z^j in $P_n\left(1 - \frac{1}{x} + \frac{z}{x}\right)$. Choose r_0 such that $0 < r_0 < 1$.

Then

$$b_{nj} = \frac{1}{2\pi i} \int_{|z| = r_0} z^{-j-1} P_n\left(1 - \frac{1}{x} + \frac{z}{x}\right) dz.$$

Therefore

$$|b_{nj}| \leqslant \frac{r_0^{-j}}{2\pi}\int_0^{2\pi}\left|P_n\left(1-\frac{1}{x}+r_0\frac{e^{i\theta}}{x}\right)\right| d\theta. \qquad \ldots(5.1)$$

By hypothesis, for each ν, $f_\nu(z) = a_\nu z+b_\nu$, where $a_\nu, b_\nu \geqslant 0$, $a_\nu+b_\nu=1$ and $\limsup_\nu b_\nu<1$. So

$$P_n\left(1-\frac{1}{x}+\frac{z}{x}\right)=\prod_{\nu=1}^{n}\left(1-\frac{a_\nu h_\nu}{x}+z\frac{a_\nu h_\nu}{x}\right)$$

$$=\prod_{\nu=1}^{n}\left(1-s_\nu+z s_\nu\right), \qquad \ldots(5.2)$$

where $s_\nu = \dfrac{a_\nu h_\nu}{x}$. From (5.1) and (5.2),

$$|b_{nj}| \leqslant r_0^{-j}\prod_{\nu=1}^{n}(|1-s_\nu|+r_0 s_\nu).$$

Now $\dfrac{|1-s_\nu|+r_0 s_\nu}{|1-s_\nu|+s_\nu}$ can be written in the form

$1-t_\nu+r_0 t_\nu$, where $t_\nu = \dfrac{s_\nu}{|1-s_\nu|+s_\nu}$. Then $0\leqslant t_\nu \leqslant 1$.

Hence

$$|b_{nj}| \leqslant r_0^{-j}\left[\prod_{\nu=1}^{n}(1-t_\nu+r_0 t_\nu)\right]\cdot\left[\prod_{\nu=1}^{n}(|1-s_\nu|+s_\nu)\right] \qquad \ldots(5.3)$$

Further, using the inequality $x \leqslant \exp(x-1)$ for any real x,

$$\prod_{\nu=1}^{\infty}(|1-s_\nu|+s_\nu) \leqslant \exp\left\{\sum_{\nu=1}^{\infty}(|1-s_\nu|+s_\nu-1)\right\}$$

$$= \exp\left\{\sum_{\substack{\nu=1\\ S_\nu>1}}^{\infty}(2s_\nu-2)\right\}$$

$$= \exp\left\{2\sum_\nu{}'(s_\nu-1)\right\} < \infty, \qquad \ldots(5.4)$$

by hypothesis. Also

$$\sum_\nu t_\nu = \sum_\nu \frac{s_\nu}{|1-s_\nu|+s_\nu}$$

$$= \sum_{0\leqslant s_\nu \leqslant 1} s_\nu + \sum_{s_\nu>1}\frac{s_\nu}{2s_\nu-1}.$$

Hence if $s_\nu>1$ for infinitely many values of ν, then

$$\sum_{s_\nu>1} \frac{s_\nu}{2s_\nu-1} > \sum_{s_\nu>1} \frac{1}{2} = \infty.$$

Therefore

$$\sum_{\nu} t_\nu = \infty.$$

Now suppose $0 \leqslant s_\nu \leqslant 1$ for all except a finite number of values of ν. In this case $\sum\limits_{0\leqslant s_\nu \leqslant 1} s_\nu = \infty$ if and only if $\sum\limits_{\nu=1}^{\infty} s_\nu = \infty$. But

$$\sum_{\nu=1}^{\infty} s_\nu = \frac{1}{x} \sum_{\nu=1}^{\infty} a_\nu\, h_\nu = \infty,$$

since $\sum\limits_{\nu=1}^{\infty} h_\nu = \infty$ and $\liminf\limits_{\nu} a_\nu = 1 - \limsup\limits_{\nu} b_\nu > 0$.

Therefore

$$\sum_{0\leqslant s_\nu<1} s_\nu = \infty.$$

Hence in this case also

$$\sum_{\nu} t_\nu = \infty.$$

It follows that

$$\prod_{\nu=1}^{\infty}\Big(1-t_\nu+r_0 t_\nu\Big) = \prod_{\nu=1}^{\infty}\Big\{1-(1-r_0)t_\nu\Big\} = 0. \qquad \text{...(5.5)}$$

From (5.3), (5.4) and (5.5), it follows that, for a fixed j

$$\lim_{n\to\infty} b_{nj} = 0. \qquad \text{...(5.6)}$$

Clearly

$$\sum_{j=0}^{n} b_{nj} = \left[P_n\left(1-\frac{1}{x}+\frac{z}{x}\right)\right]_{z=1} = 1. \qquad \text{...(5.7)}$$

Again since b_{nj} is the coefficient of z^j in $P_n\left(1-\frac{1}{x}+\frac{z}{x}\right)$, from (5.2) we get

$$b_{nj} = b_{n-1,j}\,(1-s_n) + b_{n-1,j-1}\, s_n,$$

so that

$$|\, b_{nj} \,| \leqslant |\, 1-s_n \,|\,|\, b_{n-1,j} \,| + s_n \,|\, b_{n-1,j-1} \,|.$$

Let $B_n = \sum_{j=0}^{n} \left| b_{nj} \right|$. Then

$$B_n \leqslant | 1 - s_n | B_{n-1} + s_n B_{n-1}$$
$$= (| 1 - s_n | + s_n) B_{n-1}.$$

Proceeding similarly we get

$$B_n \leqslant \left\{ \prod_{\nu=2}^{n} \left(\left| 1 - s_\nu \right| + s_\nu \right) \right\} B_1$$

$$= \prod_{\nu=1}^{n} \left(\left| 1 - s_\nu \right| + s_\nu \right)$$

$$= \prod_{\nu=1}^{\infty} \left(\left| 1 - s_\nu \right| + s_\nu \right) < \infty, \qquad \ldots(5.8)$$

by (5.4). From (5.6), (5.7) and (5.8), it follows that b_{nj} satisfies the Silverman-Toeplitz conditions for regularity. This completes the proof of the theorem.

Observe that if $x \geqslant a_\nu h_\nu$ for all except a finite number of values of ν, then $\sum_\nu' \left(\frac{a_\nu h_\nu}{x} - 1 \right)$ is trivially finite and hence by the above theorem, $E(x) \subset L(f_\nu, h_\nu)$.

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