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APPLICATION OF THE CONFLUENT HYPERGEOMETRIC FUNCTION TO NON-CENTRAL WISHART DISTRIBUTIONS

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In the present paper addition theorems of the confluent hypergeometric function ($_1F_1$) have been applied for the investigation of relations for moments of the "generalised variance" of non-central Wishart distributions. Multiplication theorems have been used for obtaining the probability density function of non-central Wishart distributions.

1. INTRODUCTION

James (1960, 1961) has derived the distribution of the characteristic roots of the co-variance matrix and the non-central Wishart distributions. He expressed the density function of these distributions as series of zonal polynomials. The distribution derived by Constantine (1963) has been also expressed as series of zonal polynomials

Many of the distributions arising in the univariate normal sampling theory, such as non-central x^2, non-central F, etc. have been expressed in terms of the generalized hypergeometric function $_pF_q$ given in Erdélyi *et al.* (1953). This work was done by Herz (1955), who has also showed that the non-central Wishart distribution involves the function $_pF_q$.

Here an attempt is being made to enlarge the theory non-central Wishart distributions. Five new results are being obtained for moments of generalized variance of non-central Wishart distributions with the help of addition theorems for $_1F_1$ given in Slater (1960). Multiplication theorems of $_1F_1$ have been utilized in the investigation of six new results for probability density function of non-central Wishart distributions.

2. GENERAL FORMULAE

The density function of the matrix $S = X X'$, when X has the distribution of the form

$$(\det 2\,\pi\,\Sigma)^{-\frac{1}{2}n} \exp\left(tr - \tfrac{1}{2}\,\Sigma^{-1}\,(X-M)\,(X-M')\right) \qquad \ldots(2.1)$$

is given by

$$\left\{ \Gamma_m\left(\frac{1}{2}\;n\right)^{-1} \left(\det 2\,\Sigma\right)^{-\frac{1}{2}n} \exp\left(t\,r - \Omega\right) \exp\left(t\,r - \tfrac{1}{2}\,\Sigma^{-1}\,s\right) \left(\det s\right)^{\frac{1}{2}(n-m-1)}\;{}_0F_1\left(\tfrac{1}{2}\,n;\,\tfrac{1}{2}\,\Sigma^{-1}\,\Omega\,s\right)\right\} \qquad \ldots(2.2)$$

From above relation Constantine (1963) obtained following important relation

$$\mathscr{E}\left[(\det s)^t\right] = \left\{\frac{\Gamma_m(t+\frac{1}{2}n)}{\Gamma_m(\frac{1}{2}n)} (\det 2\Sigma)^t \exp(tr-\Omega)\, {}_1F_1(\tfrac{1}{2}n+t; \tfrac{1}{2}n; \Omega)\right\} \quad ...(2.3)$$

where $\mathscr{E}$ denotes the expectation.

Above relation is for the moments of the generalized variance of non-central Wishart distributions. For moments we will obtain five new results by the application of following relation given in (Slater 1960)

$${}_1F_1(a; b; x+y) = \left\{\left(\frac{x}{x+y}\right)^a \sum_{n=0}^{\infty} \frac{(a)_n\, y^n}{n!\,(x+y)^n}\, {}_1F_1\, a+n; b+n; x)\right\}. \quad ...(2.4)$$

Now for finding the probability density function of the matrix $R = A(A+B)^{-1}$, Constantine (1963) has given following relation

$$\left\{\frac{\exp(tr-\Omega)\,\Gamma_m(\frac{1}{2}t)}{\Gamma_m(\frac{1}{2}s)} \exp(tr-A-B)\,(\det A)^{\frac{1}{2}(s-m-1)} (\det B)^{\frac{1}{2}(t-m-1)}\, {}_0F_1(\tfrac{1}{2}s; \Omega A)\right\}. \quad ...(2.5)$$

With the help of following multiplication theorem of ${}_1F_1$

$${}_1F_1(a; b; xy) = \sum_{n=0}^{\infty} \frac{(a)_n\, x^n\,(y-1)^n}{(b)_n\, n!}\, {}_1F_1(a+n; b+n; x) \quad ...(2.6)$$

we will try to investigate probability density function of non-central Wishart distribution.

3. Moments of Generalised Variance with the help of Addition Theorems

We will prove following result

$$\left\{\exp(tr-\Omega)\left(\frac{\lambda+\Omega}{\Omega}\right)^k \sum_{m=0}^{\infty} \frac{(\lambda+\Omega)^m\,(k)_m}{(\beta)_m}\, {}_1F_1(k+m; \beta+m; \Omega)\right\}$$

$$= \left\{\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)} (\det 2\Sigma)^{\beta-\alpha}\, \mathscr{E}\left[(\det s)^{(\alpha-\beta)}\right]\right\} \quad ...(3.1)$$

Proof : Result (2.3) can be written as

$$\exp(tr-\Omega)\, {}_1F_1(\alpha; \beta; \Omega) = \left\{\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)} (\det 2\Sigma)^{\beta-\alpha}\, \mathscr{E}\left[(\det s)^{\alpha-\beta}\right]\right\}. \quad ...(3.2)$$

Application of (2.4) to above relation gives

$$\left\{\exp(tr-\Omega)\left(\frac{\lambda+\Omega}{\Omega}\right)^k \sum_{m=0}^{\infty} \frac{m!\,(\Omega+\lambda)^m}{(k)_m\,(\lambda)^m}\, {}_1F_1(k; \beta; \Omega+\lambda)\right\}$$

$$= \left\{\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)} (\det 2\Sigma)^{\beta-\alpha}\, \mathscr{E}\left[(\det s)^{\alpha-\beta}\right]\right\} \quad ...(3.3)$$

where $k = \alpha - m$

The first result for addition theorems of ${}_1F_1$ given in Slater (1960) will give final result.

Applying same procedure we can obtain following four results by the applications of other results of addition theorems of ${}_1F_1$ given in Slater (1960) :

$$\left\{\exp tr-\Omega)\left(\frac{\lambda+\Omega}{\Omega}\right)^{k-\beta+1}\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda+\Omega)^m(1-\beta)_m}{\Omega^m}\,{}_1F_1(k;\beta-m;\Omega)\right\}$$

$$=\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)}(\det 2\Sigma)^{-\alpha+\beta}\mathscr{C}\left[(\det s)^{\alpha-\beta}\right] \qquad \ldots(3.4)$$

$$\left\{\exp(tr-\Omega)\exp(\lambda)\left(\frac{\lambda+\Omega}{\lambda}\right)^{k}\sum_{m=0}^{\infty}\frac{(-1)^m(\lambda+\Omega)^m(\beta-k)_m}{(\beta)_m}\,{}_1F_1(k;\beta+m;\Omega)\right\}$$

$$=\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)}(\det 2\Sigma)^{\beta-\alpha}\mathscr{C}\left[(\det s)^{\alpha-\beta}\right] \qquad \ldots(3.5)$$

$$\left\{\exp(tr-\Omega)\exp(\lambda)\left(\frac{\Omega}{\Omega+\lambda}\right)^{\beta}\sum_{m=0}^{\infty}(\lambda+\Omega)^{2m}(\beta-k)_m\,{}_1F_1(k-m;\beta;\Omega)\right\}$$

$$=\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)}(\det 2\Sigma)^{\beta-\alpha}\mathscr{C}\left[(\det s)^{\alpha-\beta}\right] \qquad \ldots(3.6)$$

and

$$\left\{\exp(tr-\Omega)\exp(\lambda)\left(\frac{\lambda+\Omega}{\Omega}\right)^{-\beta+k+1}\sum_{m+0}^{\infty}\frac{(-1)^m(1-\beta)_m(\lambda+\Omega)^m}{(\lambda)^m}\right.$$

$$\left.{}_1F_1(k-m;\beta-m;\Omega)\right\}=\frac{\Gamma_m(\beta)}{\Gamma_m(\alpha)}(\det 2\Sigma)^{\beta-\alpha}\mathscr{C}\left[(\det s)^{\alpha-\beta}\right] \qquad \ldots(3.7)$$

4. Probability Density Function with the help of Multiplication Theorems

First we will obtain following result

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}(\det B)^{\frac{1}{2}(t-m-1)}\right.$$

$$\exp(-2(\Omega A)^{1/2})\sum_{n=0}^{\infty}\frac{(\frac{1}{2}s-\frac{1}{2})_n(2\,\Omega^{1/2})^n(2\,A^{1/2}-1)}{(s-1)_n\,n\,!}$$

$$\left.{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2}+n;s-1+n;2\,\Omega^{1/2})\right\} \qquad \ldots(4.1)$$

Method for obtaining (4.1)

Result (2.5) can be written as

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$

$$\left.(\det B)^{\frac{1}{2}(t-m-1)}e^2(\Omega A)^{\frac{1}{2}}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2};s-1;4(\Omega A)^{\frac{1}{2}})\right\} \qquad \ldots(4.2)$$

In the above result ${}_0F_1$ has been converted into ${}_1F_1$ by the help of a result for Kummer's second theorem given in (Slater (1960).

Application of (2.6) to (4.2) will give finally result (4.1).

Applying the same method as adopted above we can establish following five new results with the help of multiplication theorems of ${}_1F_1$ given in (Slater (1960), which will be used in finding probability density function of non-central Wishart distributions.

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$
$$(\det B)^{(t-m-1)/2}\exp(-2(\Omega A)^{1/2})(4A)^{(2-s)/2}$$
$$\left.\sum_{n=0}^{\infty}\frac{(2-s)_n(1-2A)^{n/2}}{n!}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2};s-1-n;2\Omega^{1/2})\right\} \quad \ldots(4.3)$$

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$
$$(\det B)^{1/2(t-m-1)}\exp(-2(\Omega A)^{1/2}\left[(4A)^{\frac{1}{2}}\right]^{-1/2s-1}$$
$$\left.\sum_{n=0}^{\infty}\frac{(\frac{1}{2}s-\frac{1}{2})_n(s-2)^n}{n!(2A^{1/2})^n}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2}+n;s-1;2\Omega^{1/2})\right\} \quad \ldots(4.4)$$

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$
$$(\det B)^{1/2(t-m-1)}\exp(-2(\Omega A)^{1/2})\exp((s-2)2\Omega^{1/2})$$
$$\left.\sum_{n=0}^{\infty}\frac{(\frac{1}{2}s-\frac{1}{2})_n(1-2A^{1/2})^n(2\Omega^{1/2})^n}{(s-1)_n n!}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2};s-1+n;2\Omega^{1/2})\right\} \quad \ldots(4.5)$$

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$
$$(\det B)^{1/2(t-m-1)}\exp(-2(\Omega A)^{1/2})\exp((s-2)2\Omega^{1/2})(2A^{1/2})^{s-2}$$
$$\left.\sum_{n=0}^{\infty}\frac{(2-s)_n(1-2A^{1/2})^n}{n!}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2};s-1-n;2\Omega^{\frac{1}{2}})\right\} \quad \ldots(4.6)$$

and

$$\left\{\frac{\exp(tr-\Omega)}{\Gamma_m(\frac{1}{2}s)}\Gamma_m(\tfrac{1}{2}t)\exp(tr-A-B)(\det A)^{\frac{1}{2}(s-m-1)}\right.$$
$$(\det B)^{1/2(t-m-1)}\exp(-2(\Omega A)^{1/2})\exp((s-2)2\Omega^{1/2})(2A^{1/2})^{(s/2-\frac{1}{2})}$$
$$\left.\sum_{n=0}^{\infty}\frac{(\frac{1}{2}s-\frac{1}{2})_n(2A^{1/2}-1)^n}{n!(2A^{1/2})^n}\,{}_1F_1(\tfrac{1}{2}s-\tfrac{1}{2}-n;s-1;2\Omega^{\frac{1}{2}})\right\} \quad \ldots(4.7)$$

In all these results

(i) Ω, s, A, B and λ are matrices;

(ii) value of exp $(tr-\Omega)$ can be obtained from the following result

$$\Gamma_m(t,k) = \left\{ \int_{s>0} \exp(tr-\Omega) \det \Omega)^{t-\frac{1}{2}(m+1)} (\det \Omega\ 1)^{k1-k2} \right.$$

$$\left. (\det \Omega_2)^{k2-k3} \ \ldots\ldots (\det \Omega_{\ m})_{km}\ d\Omega \right\}$$

(iii) t denotes an integer;

(iv) tr denotes trace of a matrix.

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