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MHD DUSTY VISCOUS FLOW THROUGH A CIRCULAR PIPE

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In this paper an exact solution of the problem of the motion of a dusty viscous fluid through a circular pipe with an insulating wall under the influence of a transverse magnetic field is presented. From numerical computations it is seen that the fluid particles reach maximum velocity earlier than the dust particles and that the fluid particles, because of less inertia (compared with the dust particles) attain the state of rest earlier than the dust particles. The increase of the Hartmann number delays the fluid or dust particles in reaching the maximum or zero velocity.

INTRODUCTION

Recently, several research workers have investigated the flow of a dusty viscous fluid through a circular pipe. Newal Kishore and Pandey (1977) gave an analytical solution of the flow of a dusty viscous fluid through a circular pipe when the flow is created by a pressure gradient yarying harmonically with time. Verma and Mathur (1973) studied the problem in the case when a constant pressure gradient is impulsively applied. Singh and Dube (1975) considered the problem in the cases of linearly and exponentially varying pressure gradient. Sambasiva Rao (1969) investigated the same when the pressure gradient decays exponentialy with time. In this paper an attempt is made to generalize the work of Pathak (1974), who has obtained an exact solution of the unsteady hydromagnetic pipe flow of a clean fluid under a transverse magnetic field, to the case of a dusty viscous fluid.

MATHEMATICAL FORMULATION OF THE PROBLEM

We now consider the motion of an incompressible, electrically conducting, dusty viscous fluid through a circular pipe with an insulating wall. The fluid is initially at rest and the motion is created by an exponentially-decaying pressure gradient applied along the axis of the pipe. The direction of application of the pressure gradient is chosen as the positive direction of the x-axis. The only non-zero components of the velocities of the fluid and dust particles are the axial ones and let u and v respectively denote these components. Based on Saffman's (1962) model of a dusty gas, the equations govesning the motion of the fluid, when the number density N of the dust particles is taken as a constant, are

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \mu \nabla^2 u + KN(v-u) - \frac{B_0}{r}\sin\theta \frac{\partial H_z}{\partial \theta} + B_0 \cos\theta \frac{\partial H_z}{\partial r} \quad ...(1)$$

$$m \frac{\partial v}{\partial t} = K(u - v) \quad \ldots(2)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\sigma\mu_e} \nabla^2 H_z + \frac{B_0}{\mu_e}\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \frac{\partial u}{\partial \theta}\right) \quad \ldots(3)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

K is the Stokes coefficient of resistance of the dust particles, m the mass of each dust particle, p, ρ, μ, σ and μ_e are the pressure, density, coefficient of viscosity, conductivity and magnetic permeability of the fluid and H_z is the component of the intensity of the magnetic field along the z-direction. u, v and H are functions of r, θ and t only.

Initial conditions : $u = 0 = v = H_z$ at $t = 0$ ...(4)

Boundary conditions : $u(1, \theta, t) = 0 = H_z(1, \theta, t)$ for all t. ...(5)

Let us assume that $\frac{\partial p}{\partial z} = l f(t)$ ...(6)

$$\text{where } f(t) = \begin{cases} 0, & t \leqslant 0 \\ \exp(-\lambda'(t), & t > 0 \end{cases} \quad \ldots(7)$$

and l is a (dimensional) constant.

We define the following dimensionless parameters:

$$u^* = u/V_0,\ v^* = v/V_0,\ H^* = H_z/H_0,\ r^* = r/r_0,\ t^* = tV_0/r_0 \quad \ldots(8)$$

where V_0 and H_0 are the characteristic velocity and intensity of the magnetic field respectively and r_0 is the radius of the pipe. Let $\nabla^* = r_0 \nabla$.

The equations of motion in dimesionless form (on dropping the stars) are

$$R \frac{\partial u}{\partial t} = -LF(t) + \nabla^2 u + \lambda_1 (v - u) - \frac{M^2}{R_M}\left(\frac{\sin\theta}{r}\frac{\partial H}{\partial \theta} - \cos\theta \frac{\partial H}{\partial r}\right) \quad \ldots(9)$$

$$R_M \frac{\partial H}{\partial t} = \nabla^2 H - R_M \left(\frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} - \cos\theta \frac{\partial u}{\partial r}\right) \quad \ldots(10)$$

and $$\frac{\partial v}{\partial t} = \lambda_2 (u - v) \quad \ldots(11)$$

where $R = \rho V_0 r_0/\mu$ = Reynolds number, $M = \sqrt{\frac{\sigma}{\mu}}\, r_0 \mu_e H_0$ = Hartmann number,

$R_M = \sigma\mu_e r_0 V_0$ = magnetic Reynolds number

$$L = \frac{l r_0^2}{\mu_0 V_0},\ \lambda = \frac{\lambda' r_0}{V_0}$$

$$\text{and } F(t) = \begin{cases} 0 & , t \leqslant 0 \\ \exp(-\lambda t) & , t > 0. \end{cases}$$

$\lambda_1 = \frac{KNr_0^2}{\mu}$ and $\lambda_2 = \frac{Kr_0}{mV_0}$ are the dimensionless mass concentration and relaxation time of the dust particles.

Solution of the Problem

Taking Laplace transforms of equations (9)-(11),

$$Rs\bar{u} = -L\bar{F}(s) + \nabla^2\bar{u} + \lambda_1(\bar{v} - \bar{u}) - \frac{M^2}{R_M}\left(\frac{\sin\theta}{r}\frac{\partial\bar{H}}{\partial\theta} - \cos\theta\frac{\partial\bar{H}}{\partial r}\right) \quad ...(12)$$

$$R_M s\bar{H} = \nabla^2\bar{H} - R_M\left(\frac{\sin\theta}{r}\frac{\partial\bar{u}}{\partial\theta} - \cos\theta\frac{\partial\bar{u}}{\partial r}\right) \quad ...(13)$$

and $s\bar{v} = \lambda_2(\bar{u} - \bar{v})$ ...(14)

where $\bar{u} = \bar{u}(r,\theta,s) = \int_0^\infty \exp(-st)\, u(r,\theta,t)\, dt$ ...(15)

is the Laplace transform of $u(r,\theta,t)$, Likewise $\bar{v}$ and $\bar{H}$ are defined. Eliminating $\bar{v}$ between eqns. (12) and (14) we obtain

$$(\nabla^2 - g_1)\bar{u} = g_2 + \frac{M^2}{R_M}\left(\frac{\sin\theta}{r}\frac{\partial\bar{H}}{\partial\theta} - \cos\theta\frac{\partial\bar{H}}{\partial r}\right) \quad ...(16)$$

where $g_1 = g_1(s) = \dfrac{s(\lambda_1 + \lambda_2 R + sR)}{s + \lambda_2}$ ...(17)

and $g_2 = g_2(s) = \dfrac{L}{s+\lambda}$ . ...(18)

The transformed boundary conditions are

$$\bar{u}(1,\theta) = 0 = \bar{H}(1,\theta) \quad ...(19)$$

Making the substitutions

$$\left.\begin{aligned}\phi(r,\theta) &= \bar{u} + \frac{g_2}{g_1} + \frac{M}{R_M}\bar{H}\\ \psi(r,\theta) &= \bar{u} + \frac{g_2}{g_1} - \frac{M}{R_M}\bar{H}\end{aligned}\right\} \quad ...(20)$$

to uncouple the equations (13) and (16) and simplifying, we obtain

$$(\nabla^2 - g_3^2)[\phi\exp(\alpha r\cos\theta)] = g_4\psi\exp(\alpha r\cos\theta) \quad ...(21)$$

$$(\nabla^2 - g_3^2)[\psi\exp(-\alpha r\cos\theta)] = g_4\phi\exp(-\alpha r\cos\theta) \quad ...(22)$$

where $g_3^2 = \alpha^2 + \frac{1}{2}(g_1 + s\,R_M)$, ...(23)

$g_4 = \frac{1}{2}(g_1 - s\,R_M)$ ...(24)

and $\alpha = M/2$. ...(25)

Substituting $\phi(r,\theta) = \sum_{m=1}^{\infty} F_{1m}(r)\cos m\theta$ ...(26)

$$\psi(r,\theta) = \sum_{m=1}^{\infty} G_{1m}(r)\cos m\theta \quad ...(27)$$

in eqns. (21) and (22) and taking Fourier transforms of the resulting equations, we obtain

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{n^2}{r^2} + g_3\right)\right] F_{1m}(r)\left[I_{m+n}(\alpha r) + I_{m-n}(\alpha r)\right]$$

$$= g_4\left[I_{m+n}(\alpha r) + I_{m-n}(\alpha r)\right] G_{1m}(r) \quad ...(28)$$

and

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\left(\frac{n^2}{r^2}+g_3^2\right)\right]G_{1m}(r)\left[I_{m+n}(\alpha r)+I_{m-n}(\alpha r)\right]$$
$$=g_4\left[I_{m+n}(\alpha r)+I_{m-n}(\alpha r)\right]F_{1m}(r) \qquad ..(29)$$

where $I_n(x)$ is the modified Bessel function of the first kind. On solving for $F_{1m}(r)$, by eliminating $G_{1m}(r)$ between the equations (28) and (29), we get

$$F_{1m}(r)\left[I_{m+n}(\alpha r)+I_{m-n}(\alpha r)\right]=C_nI_n(\beta r)+D_nI_n(\gamma r) \qquad ...(30)$$

where $\beta^2=\alpha^2+g_1$ ..(31)

and $\gamma^2=\alpha^2+sR_M$, ...(32)

C_n and D_n are arbitrary constants and as the second kind of modified Bessel functions have a singularity at $r=0$, they have been omitted.

Inverting eqn. (30), and using eqn. (20), we obtain

$$\phi\exp(\alpha r\cos\theta)=\sum_{n=0}^{\infty}[C_nI_n(\beta r)+D_nI_n(\gamma r)]\cos n\theta. \qquad ...(33)$$

Likewise solving for $G_{1m}(r)$ we get

$$\psi\exp(-\alpha r\cos\theta)=\sum_{n=0}^{\infty}(-1)^n[C_nI_n(\beta r)-D_nI_n(\gamma r)]\cos n\theta. \qquad ..(34)$$

Using the boundary conditions (19) in eqns. (33) and (34),

we obain $C_n=\epsilon_n\dfrac{g_2}{g_1}\dfrac{I_n(\alpha)}{I_n(\beta)}$ ...(35)

and $D_n=0$

where $\epsilon_n=\begin{cases}1, & n=0\\ 2, & n>0.\end{cases}$ ...(36)

Substituting for ϕ and ψ, from eqns. (21), in eqns. (33) and (34) and inverting the resulting equations, we get

$$u+\frac{L}{2\pi i}\int_{\Gamma-i\infty}^{\Gamma+i\infty}\frac{(s+\lambda_2)}{s(s+\lambda)(\lambda_1+\lambda_2R+sR)}\exp(st)\,ds+\frac{M}{R_M}H$$
$$=\frac{L}{2\pi i}\sum_{n=0}^{\infty}\epsilon_nI_n(\alpha)\exp(-\alpha r\cos\theta)\cos n\theta\left[\int_{\Gamma-i\infty}^{\Gamma+i\infty}\frac{(s+\lambda_2)}{s(s+\lambda)(\lambda_1+\lambda_2R+sR)}\times\frac{I_n(\beta r)}{I_n(\beta)}\exp(st)\,ds\right] \qquad ...(37)$$

and

$$u+\frac{L}{2\pi i}\int_{\Gamma-i\infty}^{\Gamma+i\infty}\frac{(s+\lambda_2)}{s(s+\lambda)(\lambda_1+\lambda_2R+sR)}\exp(st)\,ds-\frac{M}{R_M}H$$

(*equation continued on p.* 815)

$$= \frac{L}{2\pi i} \sum_{n=0}^{\infty} (-1)^n \epsilon_n I_n(\alpha) \exp(\alpha r \cos\theta) \cos n\theta$$

$$\times \left[\int_{\Gamma - i\infty}^{\Gamma + i\infty} \frac{(s+\lambda_2)}{s(s+\lambda)(\lambda_1 + \lambda_2 R + sR)} \frac{I_n(\beta r)}{I_n(\beta)} \exp(st)\, ds \right]. \qquad \ldots(38)$$

The integrand on the L.H.S. of eqn. (37) or (38) has simple poles at $s = 0$, $s = -\lambda$ and $s = -(\lambda_1 R^{-1} + \lambda_2)$, whereas the integrand on R.H.S. of either of these equations has, in addition to these three simple poles, a doubly infinite number of simple poles at $s = \delta_{mn}$, δ^*_{mn}, where δ_{mn} and δ^*_{mn} are the roots of the quadratic equation

$$Rx^2 + (\lambda_1 + \lambda_2 R + \alpha^2 + \gamma^2_{mn})\, x + \lambda_2(\alpha^2 + \gamma^2_{mn}) = 0 \qquad \ldots(39)$$

and

$$\delta_{mn},\ \delta^*_{mn} = \frac{1}{2R}\left\{ -\left(\lambda_1 + \lambda_2 R + \alpha^2 + \gamma^2_{mn}\right) \pm \left[\left(\lambda_1 + \lambda_2 R + \alpha^2 + \gamma^2_{mn}\right)^2 - 4R\lambda_2\left(\alpha^2 + \gamma^2_{mn}\right)\right]^{1/2} \right\}. \qquad \ldots(40)$$

Both δ_{mn} and δ^*_{mn} are negative and γ_{mn} represents the mth zero of $J_n(x)$.

Equations (37) and (38), on completing the inversion and after simplification yield the following expressions for u and H.

$$u(r,\theta t) = L\left[\frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n I_n(\alpha)\cos n\theta \left\{\exp(\alpha r\cos\theta) + (-1)^n \exp(-\alpha r\cos\theta)\right\}\right.$$

$$\times \left[\left\{\frac{I_n(\alpha r)}{I_n(\alpha)} - 2\right]\left[A + B\exp\left\{-\left(\frac{\lambda_1}{R} + \lambda_2\right)t\right\}\right] + C\left\{\frac{I_n(\delta r)}{I_n(\delta)} - 2\right\}$$

$$\exp(-\lambda t) + \sum_{m=1}^{\infty} D_{mn}\exp(\delta_{mn} t) + \sum_{m=1}^{\infty} E_{mn}\exp(\delta^*_{mn} t) \qquad \ldots(41)$$

$$H(r,\theta,t) = \frac{LR_M}{2M}\sum_{n=0}^{\infty} \epsilon_n I_n(\alpha)\cos n\theta\left\{\exp(\alpha r\cos\theta) - (-1)^n\exp(\alpha r\cos\theta)\right\}$$

$$\times\left[\frac{I_n(\alpha r)}{I_n(\alpha)}\left[A + B\exp\left\{-\left(\frac{\lambda_1}{R} + \lambda_2\right)t\right\}\right] + C\frac{I_n(\delta r)}{I_n(\delta)}\exp(-\lambda t)\right.$$

$$\left. + \sum_{m=1}^{\infty} D_{mn}\exp(\delta_{mn} t) + \sum_{m=1}^{\infty} E_{mn}\exp(\delta^*_{mn} t)\right]\Bigg] \qquad \ldots(42)$$

where $A = \dfrac{\lambda_2}{\lambda(\lambda_1 + \lambda_2 R)}$, $B = \dfrac{\lambda}{(\lambda_1 + \lambda_2 R)\,[(\lambda - \lambda_2)R - \lambda_1]}$

$$C = \frac{\lambda - \lambda_2}{\lambda[\lambda_1 + (\lambda_2 - \lambda_1)R]},$$

$$D_{mn} \frac{-4\gamma_{mn}(\delta_{mn} + \lambda_2)^3 J_n(\gamma_{mn} r)}{[R(\delta_{mn} + \lambda_2)^2 + \lambda_1\lambda_2]\,\delta_{mn}(\delta_{mn} + \lambda)\,[\lambda_1 + \delta_{mn} + \lambda_n)R]\,[J_{n-1}(\gamma_{mn}) - J_{n+1}(\gamma_{mn})]}$$

and E_{mn} is D_{mn} with δ_{mn} replaced by δ^*_{mn} .

Substituting for u from equation (41) in eqn. (11) and solving it for v, we obtain

$$v = L\left[\frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n I_n(\alpha)\cos n\theta \left\{\exp(-\alpha r\cos\theta) + (-1)^n \exp(\alpha r\cos\theta)\right\}\right.$$

$$\times\left[\left\{\frac{I_n(\alpha r)}{I_n(\alpha)} - 2\right\}\left[A - \frac{R\lambda_2}{\lambda_1} B\exp\left\{-\left(\frac{\lambda_1}{R} + \lambda_2\right)t\right\}\right]\right.$$

$$+ \frac{C\lambda_2}{\lambda_2 - \lambda}\left\{\frac{I_n(\alpha r)}{I_n(\alpha)} - 2\right\}\exp\left(-\lambda t\right) + \lambda_2\sum_{m=1}^{\infty}\frac{D_{mn}}{\delta_{mn}+\lambda_2}\exp\left(\delta_{mn}t\right)$$

$$\left.\left. + \lambda_2\sum_{m=n}^{\infty}\frac{E_{mn}}{\delta^*_{mn}+\lambda_2}\exp\left(\delta^*_{mn}t\right)\right]\right] \qquad \ldots(43)$$

Discussion

In Fig. 1, the mid-stream velocity, at different instants of time, of fluid and dus particles, when $\theta = 0$, has been plotted for $M = 1$ and 2. Calculations are made by

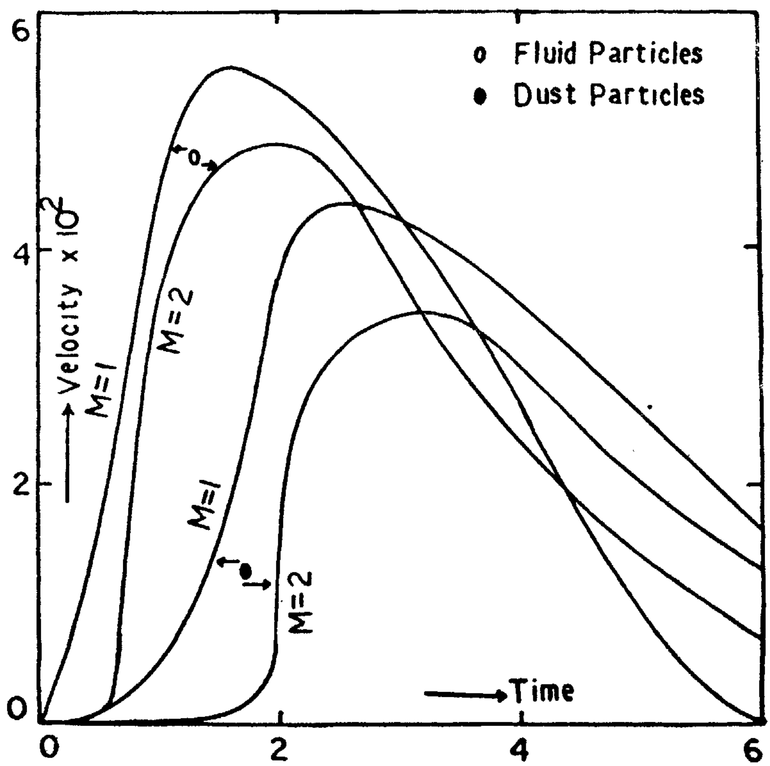

Fig. 1. Mid-Stream Velocity.

taking $\lambda_1 = 0.5$, $\lambda_2 = 1.25$, $R = 10$ and $L = -1$. When $M = 1$ the fluid particles attain their maximum velocity around $t = 1.5$ whereas the dust particles do so around $t = 2.5$. When $M = 2$ the maximum velocity of the fluid and dust particles are attained around $t = 2$ and $t = 3$ respectively. The maximum velocities of the fluid and dust particles at $M = 1$ are greater than the corresponding values at $M = 2$. The fluid particles attain zero velocity quicker than the dust particles. These observations are justifiable on physical grounds for the pressure gradient gives rise to the velocity of the fluid particles, which in turn is communicated to the dust particles. The effect of increasing the Hartmann number is to decrease the velocities of the fluid and dust particles and also to delay the time taken to attain maximum velocity and also state of rest.

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