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STATIC CHARGED DUST CYLINDERS IN GENERAL RELATIVITY

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The statie cylindrically symmetric interior solutions of Einstein Maxwell's equations for dust have been obtained. The solutions are singularity free, physically reasonable and can be interpreted as sources of Bonnor's radial solution.

1. INTRODUCTION

Krori and Barua (1974) have obtained two static internal solutions of charged dust cylinder of finite radius. One of them has singularity at $r = 0$. We have generalized their solutions. The solutions are physically possible, singularity free and of finite radius. Bonnor's (1953) solution is found to be an appropriate external solution. It is matched to the internal solutions obtained.

2. THE FIELD EQUATIONS AND THE METRIC

The field equations are

$$R^i_j - \frac{1}{2} R g^i_j = -8\pi T^i_j = -8\pi \left(M^i_j + E^i_j \right) \qquad \ldots(1)$$

where M^i_j, the material energy-momentum tensor, is given by

$$M^i_j = \rho u^i u_j \qquad \ldots(2)$$

and E^i_j, the electromagnetic energy-momentum tensor, is given by

$$4\pi E^i_j = - F_{jk} F^{ik} + \frac{1}{4} g^i_j F_{kl} F^{kl}. \qquad \ldots(3)$$

The electromagnetic field tensor F_{ij} satisfies the Maxwell's equations

$$F_{(ij;k)} = 0, \; F^{ij}{}_{;j} = 4\pi J^i. \qquad \ldots(4)$$

Since field is static F_{14} is the only non-vanishing component of F_{ij} and the four current vector

$$J^i = (0,0,0,\sigma u^4). \qquad \ldots(5)$$

From eqns. (1) we have

$$R^3_3 + R^4_4 = 0$$

and therefore the line element can be taken in Weyl's cononical form (Synge 1960)

$$ds^2 = - e^{2(\nu-\lambda)} (dr^2 + dz^2) - r^2 e^{-2\lambda} d\varphi^2 + e^{2\lambda} dt^2 \qquad \ldots(6)$$

where ν and λ are functions of r alone.

The field equations (1) for metric (6) are

$$e^{-2(\nu-\lambda)}\left(\frac{\nu_{,1}}{r} - \lambda_{,1}^{2}\right) = F_{14}\,F^{14} = -\,E^2 \qquad \ldots(7)$$

$$e^{-2(\nu-\lambda)}\left(\nu_{,11} + \lambda_{,1}^{2}\right) = -\,F_{14}\,F^{14} = E^2 \qquad \ldots(8)$$

$$e^{-2(\nu-\lambda)}\left(\nu_{,11} + \lambda_{,1}^{2} - 2\lambda_{,11} - \frac{2\lambda_{,1}}{r}\right) = -\,8\pi\rho + F_{14}\,F^{14} = -\,8\pi\rho - E^2 \qquad \ldots(9)$$

where the suffix 1 after comma denotes differentiation with respect to r and from eqns. (4), we get

$$\frac{d}{dr}\left(re^{2(\nu-\lambda)}\,F^{41}\right) = 4\pi\sigma re^{2\nu-3\lambda}. \qquad \ldots(10)$$

We also have $T^{ij}_{;j} = 0$ which lead to the equation

$$\sigma F_{14} = \rho\lambda_{,1}\,e^{\lambda}. \qquad \ldots(11)$$

Krori and Barua have shown that ν must be a constant for the solution to be regular at $r = 0$ and then that $\rho = \pm\,\sigma$. From eqns. (7) and (11) it follows that if $\rho = \pm\,\sigma$, ν is a constant which can be taken as zero without loss of generality. Therefore we consider the metric

$$ds^2 = -\,e^{-2\lambda}\,(dr^2 + r^2\,d\varphi^2 + dz^2) + e^{2\lambda}\,dt^2 \qquad \ldots(12)$$

which belongs to Papapetrou (1974), Majumdar (1947) class. The field equations (7)-(10) now reduce to the following equations

$$E^2 = -\,F_{14}\,F^{14} = \lambda_{,1}^{2}e^{2\lambda} \qquad \ldots(13)$$

$$4\pi\rho = \pm\,4\pi\sigma = e^{2\lambda}\left(\lambda_{11,} - \lambda_{,1}^{2} + \frac{\lambda_{,1}}{r}\right). \qquad \ldots(14)$$

3. Internal Solutions

We have two equations (13) and (14) in three variables λ, $F_{14}\,F^{14}$ and $\rho\,(=\pm\sigma)$, so one assumption is necessary to obtain the solution. Therefore we must either assume some condition on physical ground or choose λ as function of r which gives physically possible solution. In the following the latter one is considered.

(I) Suppose

$$e^{2\lambda} = (1 - br^n)^2\,e^{2ar^n} \qquad \ldots(15)$$

where a and b are constants. Then eqns. (13) and (14) give

$$E^2 = -\,F_{14}\,F^{14} = (a-b)\,n^2r^{2n-2}\,e^{2_{a}r^n} \qquad \ldots(16)$$

and

$$4\pi\rho = \pm\,4\pi\sigma = n^2\,r^{n-2}\,e^{2ar^n}\Big[\,a - b - \left(a^2 + b^2\right)r^n + ab\,(2a-b)\,r^{2n} - a^2b^2r^{3n}\Big] \qquad \ldots(17)$$

respectively. We observe the following:

(i) When $b = 0$ and $a > 0$, we get Krori and Barua's solution.

(ii) From eqns. (17) it follows that for ρ to be finite at $r = 0$ we must have $n \geqslant 2$. If $0 < n < 2$, the solution (15) is non-singular but ρ and $F_{14}F^{14}$ become infinite at $r = 0$.

(iii) Equation (16) shows that in order to have $F_{14}\, F^{14} < 0$ a must be greater than b.

(iv) $a \neq b$, otherwise $F_{14}\, F^{14} = 0$ and ρ will be negative as follows from

$$4\pi\rho = -n^2 a^2 r^{2n-2} e^{2ar^n}\left[\left(ar^n - \frac{1}{2}\right)^2 + \frac{7}{4}\right]. \qquad ...(18)$$

(v) If $a = 0$,

$$4\pi\rho = -bn^2 r^{n-2}(1 + br^n). \qquad ...(19)$$

This shows that b must be less than zero and the boundary $r = r_0$ of the solution must satisfy the relation $r_0^n < (-1/b)$.

Now we take $a \neq 0$, $b \neq 0$ and consider the following two cases where $r = r_0$ is the boundary.

(a) $b > 0$. For ρ to be non-negative, a and b $(a > b)$ must satisfy the inequality

$$\frac{1 + k_0 - \sqrt{1 - 2k_0 - 3k_0^2}}{2k_0(1 - k_0)} \leqslant \frac{a}{b} \leqslant \frac{1 + k_0 + \sqrt{1 - 2k_0 - 3k_0^2}}{2k_0(1 - k_0)} \qquad ...(20)$$

where $0 < k_0 = br_0^n \leqslant \frac{1}{3}$. Clearly a must be positive.

(b) $b < 0$. In this case $\rho \geqslant 0$ provided $a > b$ and

$$\frac{1 - k_0 - \sqrt{1 + 2k_0 - 3k_0^2}}{2k_0(1 + k_0)} \leqslant \frac{a}{-b} \leqslant \frac{1 - k_0 + \sqrt{1 + 2k_0 + 3k_0^2}}{2k_0(1 + k_0)} \qquad ...(21)$$

where $0 < k_0 = -br_0^n \leqslant 1$. Constant a can be positive as well as negative.

(II) Suppose

$$e^{-2\lambda} = F^2(\log R)^{2p} \qquad ...(22)$$

where $R = a + br^n$, $p \neq 0$, $b \neq 0$ and $a > 0$. Then from (13) and (14)

$$-F_{14}\, F^{14} = \frac{p^2 n^2 b^2 r^{2n-2}}{F \cdot R^2 (\log R)^{2p+2}} \qquad ...(23)$$

$$4\pi\rho = \pm\, 4\pi\sigma = \frac{n^2 r^{n-2}}{F^2 R^2 (\log R)^{2p+2}}\left\{pb\left[br^n(1-p) - a\log R\right]\right\}. \qquad ...(24)$$

For ρ to be finite at $r = 0$, we must have $n \geqslant 2$. If $0 < n < 2$, the solution (22) is non-singular but ρ and $-F_{14}\, F^{14}$ become infinite at $r = 0$.

When $p = 1$, eqn. (24) shows that ρ is non-negative if either

$$\left.\begin{array}{l} b < 0,\ a > 1 \text{ and } r < \left(\dfrac{a-1}{-b}\right)^{1/n} \\ \text{or} \\ b > 0,\ a < 1 \text{ and } r < \left(\dfrac{1-a}{b}\right)^{1/n}. \end{array}\right\} \qquad ...(25)$$

If $p \neq 1$, we found that the solution (22) is physically plausible at least in the cases given in the following table which also contains the range of validity of the solution in each case.

Case No.	Value of b	Value of P	Condition on a	Value of n	Range of r for which ρ is non-negative and $R > 0$
(i)	$b<0$	$P>0$			
		(ia) $p=\frac{1}{2}$	$a>1$	$n\geqslant 2$	(1) $0\leqslant r^n \leqslant \frac{2\,a \log a}{-b\,(1+2\log a)}$ if $0<2\log a<a-1$. (2) $0\leqslant r^n < \frac{a-1}{-b}$ if $2\log a \geqslant a-1$.
		(ib) $0<p<\frac{1}{2}$	$a>1$	$n\geqslant 2$	(1) $0\leqslant r^n \leqslant r_1$ if $2p> \frac{2a\log a}{a-1}-a+1$. (2) $0\leqslant r^n<\frac{a-1}{-b}$ if $2p\leqslant \frac{2a\log a}{a-1}-a+1$.
		(ic) $p>\frac{1}{2}$	$a>1$	$n\geqslant 2$	(1) $0\leqslant r^n \leqslant r_1$ if p is an integer greater than 1. (2) $0\leqslant r^n \leqslant r_2$ where r_2 is the smaller of r_1 and $\frac{a-1}{-b}$, if p is not an integer.
(ii)	$b>0$	$p>0$ p is an integer greater than 1	$a<1$	$n\geqslant 2$	$0\leqslant r^n \leqslant r_1$.
(iii)	$b<0$	$p<0$	(iii a)		
		p is a negative integer	$p^2 \geqslant -2 \log a > 0$	$n\geqslant 2$	(1) $0\leqslant r^n < \frac{a}{-b}$ if $p<-1$ but $1+2p> 2\log a$. (2) $0\leqslant r^n \leqslant \frac{a}{b}\left[p+\sqrt{p^2+2\log a}\right]$ if $p<-1$ but $1+2p<2\log a$.
			(iii b)		
			$p^2+2 \log a<0$	$n\geqslant 2$	$0\leqslant r^n <\left(\frac{a}{-b}\right)$.
(iv)	$b>0$	$p<0$	$a>1$	$n\geqslant 2$	$0\leqslant r^n \leqslant \frac{a}{b}\left[p+\sqrt{p^2+2\log a}\right]$.

In the table $r_1 = \frac{a}{b\,(1-2p)}\left[\, p + \log a - \sqrt{(p-\log a)^2 + 2\log a}\,\right].$

4. External Solution and the Matching at the Boundary

The external solution must be a cylindrically symmetric static solution of Einstein-Maxwell's equations for empty space and have the form (12) and hence satisfies eqns. (13) and (14) with $\rho = \sigma = 0$. The solution is found to be

$$ds^2 = -(d + c' \log r)^2 (dr^2 + r^2 d\varphi^2 + dz^2) + \frac{dt^2}{(d + c' \log r)^2} \qquad \ldots(26)$$

which c' and d are arbitrary constants. This solution has also been obtained by Bonnor. From equation (13), we obtain

$$F^{41} = \pm \frac{c'}{r} e^{2\lambda} \qquad \ldots(27)$$

for the solution (26)

We assume the continuity of g_{ij} and their first derivatives across the boundary of the distribtion $r = r_0$. Therefore from eqn. (13) it follows that F_{14} is continuous across the boundary. Now define $q\ (r)$, the charge contained in the cylinder of radius r and of unit length, as

$$q\ (r) = 2\pi \int_0^r J^4 \sqrt{-g}\, dr. \qquad \ldots(28)$$

Equation (10) with $\nu = 0$ then gives

$$F^{41} = \frac{2\, q(r)}{r} \left(e^{2\lambda} \right) \text{int.}. \qquad \ldots(29)$$

From the continuity of F^{41} across $r = r_0$ and eqns. (27) and (29) it follows that

$$\pm c' = 2q(r_0) = 2q \text{ (say)}.$$

When $q \neq 0$, the solution (26) can be written as

$$ds^2 = -4q^2 (\log cr)^2 (dr^2 + r^2 d\varphi^2 + dz^2) + \frac{dt^2}{4q^2 (\log cr)^2}. \qquad \ldots(30)$$

From the continuity of g_{ij} and their first derivatives across the boundary we obtain the constants c anb q appearing in the external solution (30) in terms of the constants appearing in the internal solution. It is found that

$$c = \frac{1}{r_0} \exp \frac{1 - br_0^n}{nr_0^n \left[b - a\left(1 - br_0^n \right) \right]} \qquad \ldots(31)$$

$$q^2 = \frac{n^2 r_0^{2n} \left[b - a\left(1 - br_0^n \right) \right]^2}{4 \left(1 - br_0^n \right)^4 e^{2ar_0^n}} \qquad \ldots(32)$$

in case of the internal solution (15) and

$$c = \frac{1}{r_0} \exp \frac{\left(a + br_0^n \right) \log \left(a + br_0^n \right)}{n\, p\, br_0^n} \qquad \ldots(33)$$

$$q^2 = \frac{n^2 p^2 b^2 F^2 r_0^{2n} \left[\log \left(a + br_0^n \right) \right]^{2p-2}}{4 \left(a + br_0^n \right)^2} \qquad \ldots(34)$$

in case of the internal solution (22).

The direct calculation of $q\ (r_0)$ from (28) for the solutions (15) and (22) lead to the expressions for q as given in (32) and (34) respectively.

5. Discussion

The solution (30) has singularities at $r = 0$, $r = 1/c$ and $r = \infty$. Hence the solution is valid either in the range $0<r<1/c$ or in the range $1/c\ <r<\infty$. Assuming that the range for the solution (30) is the latter one i.e. $1/c$ lies inside the source, Bonnor has shown that the solution (30) corresponds to the field of a massless line charge. In our case ρ is non-negative and hence we assume the validity of the solution in the range $0<r<1/c$. We have verified in all cases considered in section 3 that r_0 is less than $1/c$. Hence solution (30) is valid only between r_0 and $1/c$.

The Gauss' gravitational theorem in its relativistic form as given by Whittaker (1935) states that

$$-\iint\left\{g^1 \frac{\partial(z,\varphi)}{\partial(u,v)} + g^2 \frac{\partial(\varphi,r)}{\partial(u,v)} + g^2 \frac{\partial(r,z)}{\partial(u,v)}\right\}(-g)^{1/2}\ du\, dv$$

$$= 8\pi\iiint\left(T_4^4 - \frac{1}{2} T \right)(-g)^{1/2}\ dr\, dz\, d\varphi \qquad \text{...(35)}$$

where g^i is the three vector representing gravitational force measured by an observer at rest which for the line element (6) is given by

$$g^i = \frac{1}{2}\,\frac{g^{ii}}{g_{44}}\,\frac{\partial g_{44}}{\partial x_i},\ (i = 1,2,3,\ x_1 = r,\ x_2 = z,\ x_3 = \varphi) \qquad \text{...(36)}$$

The integration in (35) is taken over any simple closed surface S in the instantaneous space of the observer and u and v are any two parameters which specify the position of points on S. The right-hand side of (35) is proportional to the quantity, which in relativity plays the part of gravitational mass, in classical mechanics. By denoting the left-hand side of (35) by $4\ \pi M$ where M is the gravitational mass inside unit length of the cylinder of radius r_0, Bonnor has obtained

$$M = -\frac{1}{2}\left[\log\left(c\, r_0\right)\right]^{-1}. \qquad \text{...(37)}$$

The mass M, from the right-hand side of (35) is found to be

$$M = \frac{n r_0^n}{2\left(1 - b r_0^n\right)}\left[a\left(1 - b r_0^n\right) - b\right] \qquad \text{...(38)}$$

and

$$M = -\frac{n\, p\, b\, r_0^n}{2\left(a + b r_0^n\right)\log\left(a + b r_0^n\right)} \qquad \text{...(39)}$$

for the solutions (15) and (22) respectively. Substitution of values of c from (31) and (33) in (37) yield (38) and (39) respectively.

It follows from the equations (37) and continuity of g_{ij} at $r = r_0$ that

$$\frac{M^2}{q^2} = \frac{1}{4q^2 (\log cr_0)^2} = \left[e^{2\lambda}_{\text{int. or ex.}} \right]_{r=r_0} \qquad \ldots(40)$$

and so $M^2 \gtreqless q^2$ according as $\left[e^{2\lambda} \right]_{r=r_0} \gtreqless 1$. For instance $M^2 > q^2$ if $b = 0$ and $a > 0$ or if $a = 0$ and $b < 0$ for the solution (15). In case of solution (22), the ratio M^2/q^2) depends on the choice of F also.

It should be noted that every physically possible solution given in section 3 is singularity free and is of finite radius.

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