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A TWO-OBJECTIVE CAPACITY EXPANSION AND WATER ALLOCATION PROGRAMME FOR RESERVOIRS WITH RESTRICTIONS ON AUGMENTATION OF THEIR CAPACITIES TO MEET DRINKING WATER REQUIREMENTS

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The problem of determination of an optimal capacity expansion and water allocation programme with objectives to minimize the total water supply cost and the total capital expenditure to be incurred in augmenting capacities of the reservoirs taking restrictions on augmentation of their capacities into considerations to meet drinking water requirements of a number of regions has been considered. The two objectives have been assigned ordinal ranking of priorities so that the low priority objective is considered only after the high priority objective has been realized to the fullest extent possible. An algorithm has been developed to obtain the solution of the problem.

1. INTRODUCTION

The applications of the transportation method of linear programming abound in many areas of interest which have nothing to do with transportation. The applications of this method have been reported by many workers—Dantzig and Fulkerson (1954), Flood (1954), Bowman (1956), Hadley (1962), Aguilar (1973), Prakash (1978). This list of workers is not exhaustive and the applications of this method have been steadily growing with time. Though the method is an offshoot of the simplex method, it is computationally far more efficient than the simplex method itself. This accounts for the wide popularity of the method. Recently Prakash (1979) has applied the transportation method to a machine-assignment problem with two objectives. In the present paper, the method has been applied to the problem of determination of an optimal capacity expansion and water allocation programme with two objectives—one high priority and another low priority—for a number of reservoirs of existing given capacities with restrictions on augmentation of their capacities to meet present enhanced drinking water requirements of a number of regions. The high priority objective is to minimize the total water supply cost and the low priority objective is to minimize the total capital expenditure to be incurred in augmenting existing capacities of the reservoirs. The problem is of current interest as the present enhanced demands of drinking water cannot be met fully by the reservoirs built in the past without augmenting thier existing capacities. The problem has been formulated as a goal programming-type problem which ultimately assumes the form of a transportation-type problem.

An algorithm has been developed to obtain the solution of the problem. The method has been illustrated through a numerical example toward the end.

2. Formulation of the Problem

Consider m reservoirs built in the past which are now to fulfil enhanced drinking water requirements of n regions. Let a_i $(i=1,\ldots,m)$ be the units of existing capacity of reservoir i, b_{n+i} $(i=1,\ldots,m)$ the units beyond which the capacity of reservoir i cannot be augmented, b_j $(j=1,\ldots,n)$ the units of requirement of drinking water in region j, c'_{ij} $(i=1,\ldots,m;\ j=1,\ldots,n)$ the units of cost inclusive of pumping and treatment of supplying one unit of drinking water from reservoir i to region j, $c'_{i(n+i)}$ $(i=1,\ldots,m)$ the units of capital expenditure to be incurred in augmenting one unit of capacity of reservoir i, $x_{i(n+i)}$ $(i=1,\ldots,m)$ the number of units by which capacity of reservoir i should be augmented to fulfil present enhanced drinking water requirements and x_{ij} $(i=1,\ldots,m;\ j=1,\ldots,n)$ the number of units of water to be supplied from reservoir i to region j after capacities of the reservoirs have been augmented. It is required to determine the number of units by which capacities of the reservoirs should be augmented and also the number of units of water to be supplied from each of the reservoirs to each of the regions to meet present enhanced water requirements of the regions completely fulfilling two objectives—one high priority and another low priority. The high priority objective is to minimize the total water supply cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}\, x_{ij}$ and the low priority objective is to minimize the total capital expenditure $\sum_{i=1}^{m} c'_{i(n+i)}\, x_{i(n+i)}$ to be incurred in augmenting existing capacities of the reservoirs without violating the restrictions on augmentation of their capacities so as to meet water requirements of the regions fully.

Now we shall formulate this two-objective problem as a goal programming-type problem. To do this, we follow the method discussed by Hughes and Grawiog (1973) after modification and assign the priority factors M and L to the total supply cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}\, x_{ij}$ and the total capital expenditure $\sum_{i=1}^{m} c'_{i(n+i)}\, x_{i(n+i)}$ respectively. The priority factors M and L are positive and have the relationship

$$M \gg p\, L, \qquad \ldots(1)$$

which implies that M is so much larger than L that no number p, however large it might be, can make $p\,L$ equal to or greater than M. The mathematical formulation of this problem is as follows. Find $x_{i(n+i)}$ and x_{ij} $(i=1,\ldots,m;\ j=1,\ldots,n)$ which minimize

$$z = M\sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}\, x_{ij} + L\sum_{i=1}^{m} c'_{i(n+i)}\, x_{i(n+i)} \qquad \ldots(2)$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} - x_{i(n+i)} = a_i \ (i=1,\ldots,m) \qquad \ldots(3)$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ (j = 1,\ldots,n) \qquad \ldots(4)$$

$$x_{i(n+i)} \leqslant b_{n+i} \ (i = 1,\ldots,m). \qquad \ldots(5)$$

In view of relationship (1), the problem will seek to minimize the total water supply cost $\sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij} x_{ij}$ and the total capital expenditure $\sum_{i=1}^{m} c'_{i(n+i)} x_{i(n+i)}$ with priorities being accorded to them in the order mentioned above.

3. Solution Procedure

The mathematical problem formulated above is a transportation-type problem consisting of the objective function given by (2), the m row constraints given by (3), the $(n+m)$ column constraints given by (4) and (5), the relationship given by (1). Now we shall describe a procedure to obtain its solution by modifying the procedure for solving the generalized transportation problem given by Hadley (1962). The notation of Hadley (1962) is followed.

We introduce slack variables $x_{(m+1)(n+i)} \geqslant 0$ $(i=1,\ldots,m)$ into constraints (5) to transform them from inequalities to equations so as to assume the form

$$x_{(n+i)} + x_{(m+1)\,(n+i)} = b_{n+i} \ (i=1,\ldots,m). \qquad \ldots(6)$$

To overcome the difficulty in obtaining an initial basic feasible solution of the problem, we introduce artificial variables $x_{(m+2)\,(n+i)} \geqslant 0$ $(i=1,\ldots,m)$ into eqns. (6) with a plus or minus sign appended to them. The question of selecting appropriate signs will be decided at the time of obtaining an initial basic feasible solution. Equations (6) then assume the form

$$x_{i(n+i)} + x_{(m+1)\,(n+i)} \pm x_{(m+2)\,(n+i)} = b_{n+i} \ (i=1,\ldots,m). \qquad \ldots(7)$$

We associate a cost zero with each of the slack variables and a cost N with each of the artificial variables where N is a positive number subjected to the relationship

$$N \gg p\ M. \qquad \ldots(8)$$

After doing all this, the original problem is reduced to the following equivalent problem which requires finding $x_{i(n+i)}$, $x_{(m+1)\,(n+i)}$, $x_{(m+2)\,(n+i)}$ and x_{ij} $(i=1,\ldots,m;$ $j=1,\ldots,n)$ which minimize

$$z = N \sum_{i=1}^{m} x_{(m+2)\,(n+i)} + M \sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij}\, x_{ij} + L \sum_{i=1}^{m} c'_{i(m+i)}\, x_{i(n+i)} \qquad \ldots(9)$$

subject to the constraint eqns. (3), (4) and (7), the relationships (1) and (8).

The tableau representation of this equivalent problem is shown in Table I. In this table, row i of the tableau is denoted by R_i and its column j is denoted by P_j. The ith cell in P_{n+i} $(i=1,\ldots,m)$ corresponds to the variable $x_{i(n+i)}$ representing the number of units of capacity of reservoir i to be augmented, the $(m+1)$ th cell in it corresponds to the slack variable $x_{(m+1)\,(n+i)}$ and the $(m+2)$th cell in it corresponds to the artificial variable $x_{(m+2)\,(n+i)}$; while the remaining cells in it are blank. All

TABLE I

	P_1	P_2		P_n	P_{n+1}		P_{n+m}	a_i	u_i
R_1	Mc'_{11} / 1 1	Mc'_{12} / 1 1	$\cdots$	Mc'_{1n} / 1 1	$Lc'_{1(n+1)}$ / -1 1	$\cdots$		a_1	u_1
R_2	Mc'_{21} / 1 1	Mc'_{22} / 1 1	$\cdots$	Mc'_{2n} / 1 1		$\cdots$		a_2	u_1
	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
R_m	Mc'_{m1} / 1 1	Mc'_{m2} / 1 1	$\cdots$	Mc'_{mn} / 1 1		$\cdots$	$Lc'_{m(n+n)}$ / -1 1	a_m	u_m
R_{m+1}			$\cdots$		0 / 1	$\cdots$	0 / 1		
R_{m+2}					N / ± 1	$\cdots$	N / ± 1		
b_j	b_1	b_2	$\cdots$	b_n	b_{n+1}	$\cdots$	b_{n+m}		
v_j	v_1	v_2	$\cdots$	v_n	v_{n+1}	$\cdots$	v_{n+m}		

other cells except the blank ones correspond to the variables x_{ij} $(i=1,\ldots,m; j=1,\ldots,n)$ representing the number of units of water to be supplied from each of the reservoirs to each of the regions. The first n cells in R_{m+1} and R_{m+2} are blank whereas there is no entry in the left bottom corners of the remaining cells in them. The ith, the $(m+1)$th and the $(m+2)$th cells in P_{n+i} have entries of 1, 1, and $\pm$ 1 (out of these only that one which enables us to obtain an initial basic feasible solution is retained) respectively in their right bottom corners. Also the ith cell in P_{n+i} has an entry of -1 in its left bottom corner. The left top corner of the ith cell in P_{n+i} contains capital expenditure to be incurred in augmenting one unit of capacity of reservoir i multiplied by L, while the left top corners of the $(m+1)$th and the $(m+2)$th cells in it contain costs associated with the slack variable and the artificial variable respectively. And the left top corners of all other cells (i, j) except the blank ones contain supply cost of one unit of water from reservoir i to region j multiplied by M. To obtain a_i in R_i $(i=1,\ldots,m)$, sum the products obtained multiplying x_{ij} by the entry in the left bottom corner of the associated cell (i, j) across the row skipping the blank cells. To obtain b_j in P_j $(j=1,\ldots,n+m)$ sum the products obtained multiplying x_{ij} by the entry in the right bottom corner of the associated cell (i, j) across the column skipping the blank cells. And z is obtained by summing the products obtained multiplying x_{ij}

by the entry in the left top corner of the associated cell (i, j) all over the tableau skipping the blank cells.

Now an initial basic feasible solution for the equivalent problem can be found almost in the same way as for the standard transportation problem. Several methods of obtaining an initial basic feasible solution for the standard transportation problem have been discussed by Hadley (1962). Any of these methods after some modification can be used to obtain an initial basic feasible solution for the present problem. All these methods for determining an initial basic feasible solution assign a positive value to one variable and, at the same time, satisfy either a resource constraint or a requirement constraint at each step. In the case of the standard transportation problem, x_{ij} is the amount used of resource a_i; it is also the amount satisfied of requirement b_j. But in the case of the present problem, x_{ij} multiplied by the entry in the left bottom corner of the associated cell (i, j) is the amount used of resource a_i and x_{ij} multiplied by the entry in the right bottom corner of the associated cell (i, j) is the amount satisfied of requirement b_j. After a basic feasible solution has been obtained, the values of the basic variables are entered inside { } in the right top corners of the appropriate cells so that the basic ceus are distinguished from other cells. It may be pointed out that the above referred process of determining an initial basic feasible solution also resolves the question of appending appropriate signs to the artificial variables so that only one out of $\pm$ 1 is retained in the right bottom corners of the cells $(m+2, n+i)$.

To determine whether the basic feasible solution is optimal, the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells (i, j) are computed and their values are entered in the right top corners of these cells. To compute the $(z_{ij}-c_{ij})$, we proceed as follows. Denote by $c^B_{\alpha\beta}$ the costs corresponding to the basic variables associated with the basic cells (α, β). Then we compute the m u_i $(i=1,\ldots,m)$ and the $(n+m)$ v_j $(j=1,\ldots,n+m)$ from the following $(2m+n)$ equations of four different forms

$$\left.\begin{array}{lll} u_\alpha + v_\beta = M\, c'_{\alpha\beta} & & \text{(for the basic cells in the first } n \text{ columns of the tableau)} \\ -u_\alpha + v_\beta = L\, c'_{\alpha\beta} & & \text{(for the basic cells in the last } m \text{ columns and the first } m \text{ rows of the tableau)} \\ v_\beta = 0 & & \text{(for the basic cells in the last } m \text{ columns and the } (m+1)\text{th row of the tableau)} \\ \pm\, v_\beta = N & & \text{(for the basic cells in the last } m \text{ columns and the } (m+2)\text{th row of the tableau).} \end{array}\right\} \quad \ldots(10)$$

Once the m u_i and the $(n+1)$ v_j are known, we can compute the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells from the following formulae

$$\left.\begin{array}{lll} z_{ij} - c_{ij} = u_i + v_j - M c'_{ij} & & \text{(for the nonbasic cells in the first } n \text{ columns of the tableau)} \\ z_{ij} - c_{ij} = - u_i + v_j - L c'_{ij} & & \text{(for the nonbasic cells in the last } m \text{ columns and the first } m \text{ rows of the tableau)} \\ z_{ij} - c_{ij} = v_j & & \text{(for the nonbasic cells in the last } m \text{ columns and the } (m+1)\text{th row of the tableau)} \\ z_{ij} - c_{ij} = \pm v_j - N & & \text{(for the nonbasic cells in the last } m \text{ columns and the } (m+2)\text{th row of the tableau).} \end{array}\right\} \quad \ldots(11)$$

After the values of the $(z_{ij}-c_{ij})$ have been calculated, they are entered in the right top corners of the nonbasic cells (i, j). If all the $(z_{ij}-c_{ij})$ are nonpositive, then the basic feasible solution is optimal.

On the other hand, if one or more of the $(z_{ij}-c_{ij})$ is or are positive, the basic feasible solution is not optimal. Then the vector $\bar{p}_{st}$ corresponding to the cell (s, t) to enter the basis is determined from $(z_{st}-c_{st}) = \max (z_{ij}-c_{ij})$ among all the positive $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells. And to determine the vector to leave the basis, we proceed as follows. Denote by $\bar{p}^B_{\alpha\beta}$ and $x^B_{\alpha\beta}$ respectively the $(2m + n)$ basis vectors and basic variables corresponding to the basic cells. Then the $y^{\alpha\beta}_{st}$ are determined from the vector equation

$$\bar{p}_{st} = \sum_{\alpha\beta} y^{\alpha\beta}_{st} \, \bar{p}^B_{\alpha\beta} , \quad \ldots(12)$$

where $\sum_{\alpha\beta}$ means the summation over the basis vectors. After having calculated the $y^{\alpha\beta}_{st}$, the criterion used in the simplex method is employed to determine the vector $\bar{p}^B_{qr}$ to leave the basis which in the present case reduces to

$$x^B_{qr} = \theta = \min \left\{ \frac{x^B_{\alpha\beta}}{y^{\alpha\beta}_{st}} \,\middle|\, y_{st} > 0. \right\} \quad \ldots(13)$$

Then the new basic feasible solution is obtained by setting $\hat{x}_{st} = \theta$, $\hat{x}_{qr} = 0$ and making the appropriate adjustments in the tableau to obtain the values of the remaining basic variables.

For this new basic feasible solution, a new tableau is constructed and the whole previous procedure is repeated. This repetition is continued till we reach the state where all the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells are nonpositive. And when this happens we have actually obtained the desired solution.

4. A NUMERICAL EXAMPLE

Now we shall apply the above procedure to obtain the solution of a numerical problem which is obtained by taking $m = 2$, $n = 5$ and assigning numerical values to all other quantities in the problem formulated above in section 2. The tableau representation of the equivalent problem to this numerical problem is shown in Table II. For this equivalent problem, the objective function which we seek to minimize is

$$z = N(x_{46}+x_{47})+M(2x_{11}+7x_{12}+3x_{13}+4x_{14}+5x_{15}+5x_{21}+x_{22}$$
$$+3x_{23}+2x_{24}+6x_{25})+L(2x_{16}+3x_{27}). \qquad \ldots(14)$$

Table II

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	P_{5+2}	a_i	u_i
R_1	$2M$ {300} 1 1	$7M$ $-6M$ 1 1	$3M$ {600} 1 1	$4M$ $-2M$ 1 1	$5M$ M 1 1	$2L$ $N+L$ -1 1		900	$-N-3L$
R_2	$5M$ $-3M$ 1 1	M {200} 1 1	$3M$ {100} 1 1	$2M$ {900} 1 1	$6M$ {100} 1 1		$3L$ {300} -1 1	1000	$-N-3L$
R_{2+1}						0 {300} 1	0 $-N$ 1		
R_{2+2}						N $-N$ 1	N {100} -1		
b_j	300	200	700	900	100	300	200		
v_j	$N+2M+3L$	$N+M+3L$	$N+3M+3L$	$N+2M+3L$	$N+6M+3L$	0	$-N$		

To find an initial basic feasible solution for this equivalent problem, we apply the column-minima method after some modification. Following the procedure, an initial basic feasible solution is obtained deciding that the entries in the right bottom corners of the cells (4,6) and (4,7) corresponding to the artificial variables x_{46} and x_{47} should be selected as 1 and -1 respectively out of ± 1. The values of the basic variables of this initial solution are entered inside { } in the tableau of Table II. To determine whether the basic feasible solution is optimal, we require the values of the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells. For this purpose, we first calculate the values of the u_i and v_j, and enter their values in the last column and the last row of the tableau of Table II. Then the values of the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells are calculated and are entered in the right top corners of the associated nonbasic cells in the tableau of Table II. We now observe that $(z_{15}-c_{15})$ and $(z_{16}-c_{16})$ are positive and that the second one is greatest among them; therefore the basic feasible solution is not optimal and the vector $\bar{p}_{16}$ should enter the basis.

To determine the vector to leave the basis, the $y_{16}^{\alpha\beta}$ must be computed. They are determined from the vector equation

$$p_{16} = y_{16}^{11}\bar{p}_{11} + y_{16}^{13}\bar{p}_{13} + y_{16}^{22}\bar{p}_{22} + y_{16}^{23}\bar{p}_{23} + y_{16}^{24}\bar{p}_{24}$$

$$+ y_{16}^{25}\bar{p}_{25} + y_{16}^{27}\bar{p}_{27} + y_{16}^{36}\bar{p}_{36} + y_{16}^{47}\bar{p}_{47} \qquad \ldots(15)$$

which at once yields the solution

$$\left.\begin{array}{l} y_{16}^{11} = 0, y_{16}^{13} = -1, \quad y_{16}^{22} = 0, y_{16}^{23} = 1, y_{16}^{24} = 0, \; y_{16}^{25} = 0, \\ y_{16}^{27} = 1, \; y_{16}^{36} = 1, y_{16}^{47} = 1 \end{array}\right\} \qquad \ldots(16)$$

only $y_{16}^{23}, y_{16}^{27}, y_{16}^{36}, y_{16}^{47}$ are positive and

$$\min\left\{\frac{x_{23}}{y_{16}^{23}}, \frac{x_{27}}{y_{16}^{27}}, \frac{x_{36}}{y_{16}^{36}}, \frac{x_{47}}{y_{16}^{47}}\right\} = \min\left\{100, 300, 300, 100\right\} = 100.$$

Therefore the vector to leave the basis has to be chosen out of $\bar{p}_{23}$ and $\bar{p}_{47}$; but as $\bar{p}_{47}$ is the vector corresponding to the artificial variable, we allow $\bar{p}_{47}$ to leave the basis. The new basic feasible solution is obtained by setting $\hat{x}_{16} = 100$, $\hat{x}_{47} = 0$ and making the appropriate adjustments in the tableau of Table II to obtain the values of the remaining basic variables. The values of the new basic variables are entered inside { } in the tableau of Table III. We then complete Table III, proceeding in the same way as we did to complete Table II. After this, we construct Tables IV and V similarly.

TABLE III

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	P_{5+2}	a_i	u_i
R_1	$2M$ {300} 1 1	$7M$ $-6M$ 1 1	$3M$ {700} 1 1	$4M$ $-2M$ 1 1	$5M$ M 1 1	$2L$ {100} -1 1		900	$-2L$
R_2	$5M$ $-3M$ 1 1	M {200} 1 1	$3M$ {0} 1 1	$2M$ {900} 1 1	$6M$ {100} 1 1		$3L$ {200} -1 1	1000	$-2L$
R_{2+1}						0 {200} 1	0 L 1		
R_{2+2}						N $-N$ 1	N $-N-L$ -1		
b_j	300	200	700	900	100	300	200		
v_j	$2M+2L$	$M+2L$	$3M+2L$	$2M+2L$	$6M+2L$	0	L		

TABLE IV

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	P_{5+2}	a_i	u_i
R_1	$2M$ {300} 1 1	$7M$ $-6M$ 1 1	$3M$ {600} 1 1	$4M$ $-2M$ 1 1	$5M$ {100} 1 1	$2L$ {100} -1 1		900	$-2L$
R_2	$5M$ $-3M$ 1 1	M {200} 1 1	$3M$ {100} 1 1	$2M$ {900} 1 1	$6M$ $-M$ 1 1		$3L$ {200} -1 1	1000	$-2L$
R_{2+1}						0 {200} 1	0 L 1		
R_{2+2}						N $-N$ 1	N $-N-L$ -1		
b_j	300	200	700	900	100	300	200		
v_j	$2M+2L$	$M+2L$	$3M+2L$	$2M+2L$	$5M+2L$	0	L		

TABLE V

	P_1	P_2	P_3	P_4	P_5	P_{5+1}	P_{5+2}	a_i	u_i
R_1	$2M$ {300} 1 1	$7M$ $-6M+L$ 1 1	$3M$ {700} 1 1	$4M$ $-2M+L$ 1 1	$5M$ {100} 1 1	$2L$ {200} -1 1		900	$-2L$
R_2	$5M$ $-3M-L$ 1 1	M {200} 1 1	$3M$ $-L$ 1 1	$2M$ {900} 1 1	$6M$ $-M-L$ 1 1		$3L$ {100} -1 1	1000	$-3L$
R_{2+1}						0 {100} 1	0 {100} 1		
R_{2+2}						N $-N$ 1	N $-N$ -1		
b_j	300	200	700	900	100	300	200		
v_j	$2M+2L$	$M+3L$	$3M+2L$	$2M+3L$	$5M+2L$	0	0		

Table V provides the desired solution of the numerical problem under consideration; because all the $(z_{ij}-c_{ij})$ corresponding to the nonbasic cells in this Table are positive indicating that the basic feasible solution is optimal. The minimum water supply cost and the minimum total capital expenditure to be incurred in augmenting capacities of the reservoirs to meet water requirements fully are $(2\times300+3\times700+5\times100+1\times200+2\times900)=5200$ units and $(2\times200+3\times100)=700$ units respectively.

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REFERENCES

Aguilar, R.J. (1973). Systems Analysis and Design. Prentice-Hall, Inc. Englewood Cliffs, New Jersey, pp. 202–209.

Bowman, E.H. (1956). Production scheduling by the transportation method of linear programming. *Op. Res.*, 4, 100–103.

Dantzig, G.B., and Fulkerson, D.R. (1954). Minimizing the number of tankers to meet a fixed schedule. *Nav. Res. Log. Quart.*, **1**, 217–22.

Flood, M.M. (1954). Application of transportation theory to scheduling a military tanker fleet. *Op. Res.*, **2**, 150–62.

Hadley, G. (1962). Linear Programming. Addison-Wesley Publishing Co., pp. 273–322, 437–47.

Hughes, A.J., and Grawiog, D.E. (1973). Linear Programming: An Emphasis on Decision Making. Addison-Wesley Publishing Co., pp. 300–312.

Prakash, S. (1978). On the machine-assignment problem. *Proc. Nat. Acad. Sci., India*, **48A**, 103–10.

————(1979). A machine-assignment problem with objectives to minimize duration of completing work and processing cost. *Proc. Nat. Acad. Sci., India*, 49 A, 229–38.