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NODE ELIMINATION IN THREE-MACHINE FLOWSHOP SCHEDULING PROBLEM

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This paper discusses an elimination rule which avoids the creation of many nodes when n-job, 3-machine flowshop problem with minimum total elapsed time is solved by branch and bound technique.

1. INTRODUCTION

Consider three-machine flowshop scheduling problem in which n jobs are processed over three machines M_1, M_2 and M_3 in this fixed order. Given the processing time of job i on each machine M_k, $k = 1, 2, 3$, the problem is to find the sequence of jobs to be processed on the machines so that the total elapsed time is minimized. Efficient solutions exist only for special cases of three-machine flowshop (Achuthan 1977; Burns and Rooker 1976, 1978; Szwarc 1974).

The general three-machine flowshop problem has been solved by using branch and bound technique (Bestwick and Hastings 1976, Ignall and Schrage 1965, Lomnicki 1965, McMohon and Burton 1967, Nabeshima 1967) which consists of branching and obtaining the lower bound on the total elapsed time associated with any completion of partial sequence σ of the jobs attached to a node. In this procedure it is observed that the computational time varies with the number of nodes created, i. e., more the nodes more will be the computational time.

In this paper, an elimination rule is developed by which many nodes are eliminated when using branch and bound technique. Two numerical problems have been given in the end to illustrate the use of node elimination criterion.

2. NOTATIONS

p_{ik} = processing time of job i on machine M_k

J_r = a scheduled partial sequence containing r jobs

J'_r = complement of J_r in the set of n jobs

S = any permutation of n jobs

S_{pq} = Johnson's optimal sequence (Johnson 1954) for two-machine flow-shop with machine pair (M_p, M_q), $1 \leqslant p < q \leqslant 3$

$T(\sigma, k)$ = the total elapsed time of subsequenoe σ of S on machine M_k, $1 \leqslant k \leqslant 3$.

3. Node Elimination Criterion

The total elapsed time of subsequence σi on machine M_k, when job i is adjoined to σ is given by

$$T(\sigma i, k) = \max\,[T(\sigma i, k-1), T(\sigma, k)] + p_{ik} \qquad \ldots(1)$$

where

$$T(\varphi, k) = 0 = T(\sigma, 0).$$

Hence $T(S, 3)$ is the total elapsed time for permutation S on machine M_3.

The following theorem proves an adjacent ordering property of jobs i and j.

Theorem — If a job i precedes another job j in each of the sequences S_{12}, S_{23} and S_{13}, then in the optimal sequence job i precedes job j for any adjacent position.

Proof : We prove that sequence $S' = \sigma i j \pi$ is preferred to sequence $S'' = \sigma j i \pi$, where $\sigma\pi$ is any permutation of $n-2$ jobs other than jobs i and j. Now

$$T(\sigma i j\, \pi, 3) \leqslant T(\sigma j i\, \pi, 3) \qquad \ldots(2)$$

if $\quad T(\sigma i j, k) \leqslant T(\sigma j i, k)$ for all $1 \leqslant k \leqslant 3$. $\qquad \ldots(3)$

We prove relation (3) for all possible values of k.

(i) When $k = 1$, then

$$\begin{aligned} T(\sigma i j, 1) &= T(\sigma, 1) + p_{i1} + p_{j1} \\ &= T(\sigma, 1) + p_{j1} + p_{i1} \\ &= T(\sigma j i, 1) \end{aligned}$$

(ii) When $k = 2$, then relation (1) gives

$$T(\sigma i j, 2) = \max \begin{pmatrix} T(\sigma i j, 1) + p_{j2}, \\ T(\sigma i, 1) + p_{i2} + p_{j2}, \\ T(\sigma, 2) + p_{i2} + p_{j2} \end{pmatrix}$$

$$T(\sigma j i, 2) = \max \begin{bmatrix} T(\sigma j i, 1) + p_{i2}, \\ T(\sigma j, 1) + p_{j2} + p_{i2}, \\ T(\sigma, 2) + p_{j2} + p_{i2} \end{bmatrix}.$$

Hence

$$T(\sigma i j, 2) \leqslant T(\sigma j i, 2) \qquad \ldots(4)$$

if

$$\max \begin{pmatrix} T(\sigma i j, 1) + p_{j2}, \\ T(\sigma i, 1) + p_{i2} + p_{j2}, \\ T(\sigma, 2) + p_{i2} + p_{j2} \end{pmatrix} \leqslant \max \begin{pmatrix} T(\sigma j i, 1) + p_{i2}, \\ T(\sigma j, 1) + p_{j2} + p_{i2} \\ T(\sigma, 2) + p_{j2} + p_{i2} \end{pmatrix}$$

i. e., if

$$\max\begin{bmatrix} T(\sigma ij, 1) + p_{j2}, \\ T(\sigma i, 1) + P_{i2}+p_{j2} \end{bmatrix} \leqslant \max\begin{bmatrix} T(\sigma ji, 1) + p_{i2}, \\ T(\sigma j, 1) + p_{j2}+p_{i2} \end{bmatrix}. \qquad \ldots(5)$$

Subtract $T(\sigma, 1)+ p_{i1} + p_{j1} + p_{i2} + p_{j2}$ from both sides of relation (5). Then inequality (4) is true if

$$\max(-p_{i2}, -p_{j1}) \leqslant (-p_{j2}, -p_{i1})$$

i.e.,

$$\min(p_{i1}, p_{j2}) \leqslant \min(p_{j1}, p_{i2}).$$

But this is a fact, because job i precedes job j in S_{12}. Therefore relation (4) is true.

(iii) When $k = 3$, then relation (1) gives

$$T(\sigma ij, 3) = \max(T(\sigma ij, 2), T(\sigma i, 3)) + p_{j3}$$

$$T(\sigma ji, 3) = \max(T(\sigma ji, 2), T(\sigma j, 3)) + p_{i3}.$$

Again using the relation (1) successively, we obtain

$$T(\sigma ij, 3) = \max\begin{bmatrix} \max\begin{bmatrix} T(\sigma ij, 1) + p_{j2}, \\ T(\sigma i, 2) + p_{j2} \end{bmatrix} \\ \max\begin{bmatrix} T(\sigma i, 2) + p_{i3} \\ T(\sigma, 3 + p_{i3} \end{bmatrix} \end{bmatrix} + p_{j3}$$

$$= \max\begin{bmatrix} T(\sigma i_j, 1) + p_{j2}, \\ T(\sigma i, 2) + p_{j2}, \\ T(\sigma i, 2) + p_{i3}, \\ T(\sigma, 3) + p_{i3} \end{bmatrix} + p_{j3}$$

$$= \max\begin{bmatrix} T(\sigma i, 1)+p_{j1}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{i2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{i2}+p_{j2}+p_{j3}, \\ T(\sigma, 3)+p_{i3}+p_{j3} \end{bmatrix}$$

Similarly,

$$T(\sigma ji, 3) = \max\begin{bmatrix} T(\sigma j, 1)+p_{i1}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{j3}+p_{i3}, \\ T(\sigma, 2)+p_{j2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{j2}+p_{i2}+p_{i3}, \\ T(\sigma, 3)+p_{i3}+p_{j3} \end{bmatrix}.$$

Hence

$$T(\sigma ij, 3) \leqslant T(\sigma ji, 3) \qquad ..(6)$$

if

$$\max \begin{bmatrix} T(\sigma i, 1)+p_{j1}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{i2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{i2}+p_{j2}+p_{j3}, \\ T(\sigma, 3)+p_{j3}+p_{i3} \end{bmatrix}$$

$$\leqslant \max \begin{bmatrix} T(\sigma j, 1)+p_{i1}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{j3}+p_{i3} \\ T(\sigma, 2)+p_{j2}+p_{i3}+p_{j3} \\ T(\sigma, 2)+p_{j2}+p_{i2}+p_{i3} \\ T(\sigma, 3)+p_{i3}+p_{j3} \end{bmatrix}$$

i.e. if,

$$\max \begin{bmatrix} T(\sigma i, 1)+p_{j1}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{j2}+p_{j3}, \\ T(\sigma i, 1)+p_{i2}+p_{i3}+p_{j3} \end{bmatrix}$$

$$\leqslant \max \begin{bmatrix} T(\sigma j, 1)+p_{i1}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{i2}+p_{i3}, \\ T(\sigma j, 1)+p_{j2}+p_{j3}+p_{i3} \end{bmatrix} \qquad ...(7)$$

and

$$\max \begin{bmatrix} T(\sigma, 2)+p_{i2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{i2}+p_{j2}+p_{j3} \end{bmatrix}$$

$$\leqslant \max \begin{bmatrix} T(\sigma, 2)+p_{j2}+p_{i3}+p_{j3}, \\ T(\sigma, 2)+p_{j2}+p_{i2}+p_{i3} \end{bmatrix}. \qquad ...(8)$$

Subtract $T(\sigma, 2) + p_{i2} + p_{i3} + p_{j2} + p_{j3}$ from both sides of relation (8). Then relation (8) reduces to proving

$$\max(-p_{j2}, -p_{i3}) \leqslant \max(-p_{i2}, -p_{j3})$$

i. e.,

$$\min(p_{i2}, p_{j3}) \leqslant \min(p_{j2}, p_{i3})$$

which is given because job i precedes job j in S_{23}.

To prove (7), subtract $T(\sigma, 1) + \sum_{k=1}^{3} p_{ik} + \sum_{k=1}^{3} p_{jk}$ from both sides of relation (7).

Then

$$\max(-p_{i2}-p_{i3}, -p_{j1}-p_{j3}, -p_{j1}-p_{j2}) \leqslant \max(-p_{j2}-p_{j3}, -p_{i1}-p_{j3}, -p_{i1}-p_{i2})$$

or

$$\min(p_{i1}+p_{i2}, p_{j1}+p_{j3}, p_{j2}+p_{j3}) \leqslant \min(p_{j1}+p_{j2}, p_{j1}+p_{i3}, p_{i2}+p_{i3})$$

or

$$\min(p_{i1}+p_{i2}, p_{i1}+p_{j3}, p_{j2}+p_{j3}, p_{j2}+p_{i2}) \leqslant \min(p_{j1}+p_{j2}, p_{j1}+p_{i3}, p_{i2}+p_{i3}, p_{j2}+p_{i2})$$

i.e., if

$$\min(p_{i1}, p_{j2}) + \min(p_{i2}, p_{j3}) \leqslant \min(p_{j1}, p_{i2}) + \min(p_{j2}, p_{i3}). \quad \ldots(9)$$

Since

$$\min(p_{i1}, p_{j2}) \leqslant \min(p_{j1}, p_{i2}) \quad \ldots(10)$$

$$\min(p_{i2}, p_{j3}) \leqslant \min(p_{j2}, p_{i3}), \quad \ldots(11)$$

the truth of relation (9) follows by adding inequalities (10) and (11). Thus, relation (3) is true for all values of k, $1 \leqslant k \leqslant 3$. This completes the proof of the theorem.

Elimination criterion — If the first r positions in S_{12}, S_{13} and S_{23} contain the same jobs (may be in different order) denoted by the set J_r, then none of the remaining jobs J_r' can occupy any of the first r positions in optimal sequence.

PROOF : Start with any sequence S. It can be shown that through the following algorithm another sequence, having first r jobs belonging to set J_r, can always be obtained which has less total elapsed time.

Algorithm

Step 1 : Initialize $S = (i_1, i_2, \ldots, i_n)$ and $q = 1$.

Step 2 : Take the adjacent pair (i_q, i_{q+1}) of jobs in S. Interchange jobs i_q and i_{q+1} in S if and only if $i_q \in J_r'$ and $i_{q+1} \in J_r$, giving an improvement in the value of the total elapsed time. Go to step 1 if the interchange is possible, otherwise go to Step 3.

Step 3 : Set $q = q+1$ and proceed to Step 4.

Step 4 : If $q = n$, then S is the desired sequence, otherwise go to Step 2.

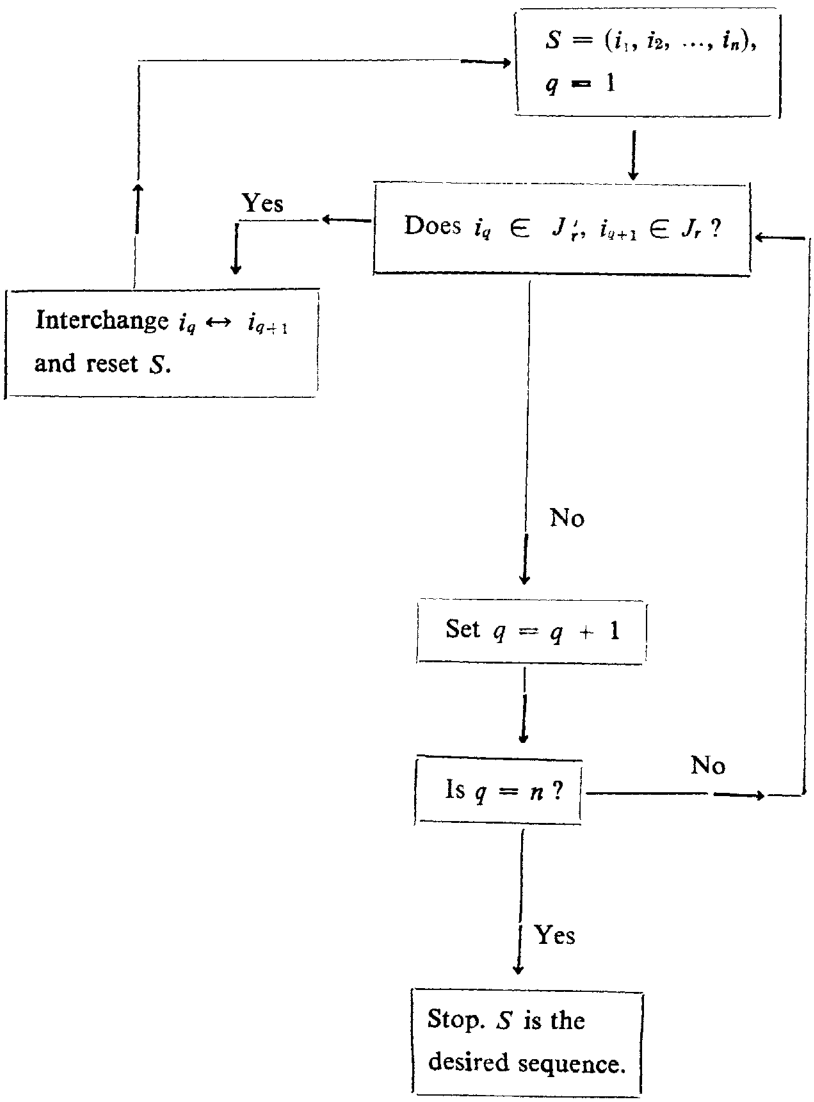
4. NUMERICAL EXAMPLES

Example 1

Consider the numerical problem whose data is given in Table I. The aim is to determine the sequence which minimizes the total elapsed time.

TABLE I
Data for a 5-job, 3-machine problem

Job (i)	Machines		
	M_1	M_2	M_3
1	123	300	76
2	57	156	200
3	198	201	211
4	154	162	122
5	92	99	211



Flow chart for the Algorithm.

We have

$$S_{12} = (25143)$$

$$S_{13} = (25341)$$

$$S_{23} = (52341).$$

We observe that two jobs namely, job 2 and job 5, occupy first two positions in each of the sequences S_{12}, S_{13} and S_{23}. Hence by the node elimination criterion jobs 1, 3 and 4 cannot occupy first two positions in optimal sequence. Using branch and bound solution procedure due to Nabeshima (1967) and the node elimination criterion, we obtain the complete calculations as shown in Fig. 1 (scheduling tree). Only 8 nodes are created to obtain the optimal sequence (25431) with minimum total elapsed time 1078; However, without using the node elimination criterion at least 14 nodes are created to solve this problem

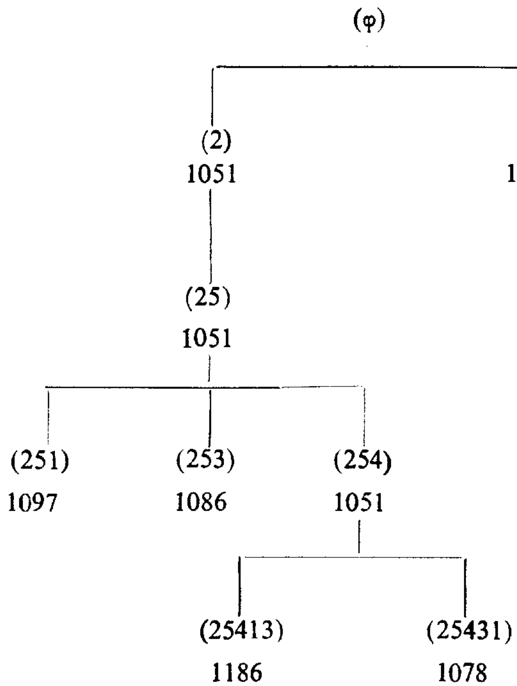


FIG. 1. Scheduling tree.

Example 2

Consider the problem whose data is given in Table II.

TABLE II
Data for 8 jobs, 3-machine problem

Job (i)	Machine M_1	M_2	M_3
1	2	3	5
2	8	6	9
3	7	6	8
4	8	5	4
5	7	4	3
6	6	4	2
7	8	3	1
8	9	2	1

In this case there are 4 Johnson's optimal sequences for machines (M_1, M_2), 4 Johnson's optimal sequences for machines (M_2,M_3) and 2 Johnson's, optimal sequences for machines (M_1,M_3). We select:

S_{12} = (13245678)
S_{23} = (13245678)
S_{13} = (13245678).

Hence using the node elimination criterion, we obtain S = (13245678) as an optimal sequence with minimum total elapsed time 58.

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