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ON TOPOLOGICAL PROPERTIES OF OPERATORS

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In this paper we study some topological properties of centroid, convexoid and spectraloid operators.

Recently Prasanna (1981) introduced two new classes of operators namely centroid and quasicentroid operators on a complex Hilbert space H. For a bounded linear operator T let d_T be the smallest disc containing the spectrum $\sigma(T)$ of T and let Z_T, C_T and R_T be the centre, the boundary and the radius of d_T respectively. T is said to be centroid if $T-Z_T$ is normaloid i.e. T is centroid if and only if $\| T-Z_T \| = R_T$. Stampfli (1970) observed that for every operator T there exists a unique complex number m_T called the centre of mass of T such that $\| T-m_T \|^2 + |Z|^2 \leqslant \| T-m_T + Z \|^2$ for all complex Z and showed that $Z_T = m_T$ if T is normal or hyponormal. T is said to be quasicentroid if $Z_T = m_T$. The author has shown (Prasanna 1981) that the class of centroid operators is a proper subset of the class of quasicentroid operators and that these two classes of operators are translation invariants.

Luecke (1972) studied the topological properties of G_1 operators and convexoid operators and showed that if dim $H \geqslant 2$ then the class $\mathscr{L}$ of convexoid operators is nowhere dense. In this paper using Luecke's technique and the faithful * representation of Berberian (1962) we prove some topological properties of centroid operators. This modification enables us to give a simpler proof of Luecke's theorem on convexoid operators and to improve a result of Patel (1974) on a topological property of spectraloid operators.

Throughout this paper we assume that $\beta(H)$ has the norm topology and invoke the Hausdorff metric topology for the compact subsets of the complex plane.

Theorem 1—Let T be a quasicentroid operator. If T can be approximated in the norm by centroid operators then T is centroid.

PROOF : Let T be a quasicentroid operator and let $\{T_n\}$ be a sequence of centroid operators such that $\| T_n - T \| \to 0$. We shall show that $\| T-Z_T \| = R_T$.

Since each T_n is centroid, $Z_{T_n} = m_{T_n}$. Also the mapping $T \to m_T$ is continuous with respect to the norm topology (Stampfli 1970) Hence $Z_{T_n} = m_{T_n} \to m_T = Z_T$ or $\{T_n - Z_{T_n}\} \to T - Z_T$ in the norm. Suppose $\| T-Z_T \| \neq R_T$. Set $\epsilon = d/4$ where $d = \| T-Z_T \| - R_T$. Since the spectrum is upper semicontinuous (Halmos 1967), there exists a positive integer N_1 such that for all $n \geqslant N_1$,

$$\sigma(T_n - Z_{T_n}) \subset \sigma(T - Z_T) + \epsilon \subset R_T D + \epsilon$$

where D is the unit disc. Now $d_{T_n} - Z_{T_n} = R_{T_n} D$. Hence $R_{T_n} D \subset R_T D + d/4$ for all $n \geqslant N_1$

or $\quad R_{T_n} = \| T_n - Z_{T_n} \| \leqslant R_T + d/4$ for all $n \geqslant N_1$. ...(I)

Since $\| T_n - Z_{T_n} \| \to \| T - Z_T \|$, there exists an integer N_2 such that for all $n \geqslant N_2$

$$| \, \| T_n - Z_{T_n} \| - \| T - Z_T \| \, | \leqslant \epsilon. \qquad \text{...(II)}$$

If $n_o > \max \{N_1, N_2\}$ then

$\| T_{n_0} - Z_{T_{n_0}} \| \leqslant R_T + d/4$ from (I) and

$R_T + 3\,d/4 \leqslant \| T_{n_0} - Z_{T_{n_0}} \| \leqslant R_T + 5d/4$ from (II), which is absurd. Hence $\| T - Z_T \| = R_T$ or T is centroid.

Theorem 2—Let $\mathscr{Z}$ be the class of centroid operators on H. If dim $H \geqslant Z$ then $\mathscr{Z}$ has empty interior.

PROOF : We shall first show that if $T \in \text{Int}(\mathscr{Z})$ then $\sigma(T)$ must have atleast two points.

Suppose $\sigma(T) = \{a\}$. Since T is centroid $T = aI$ (Fujii and Prasanna 1981). Write $H = M \oplus M^1$ where M is a two dimensional subspace of H. Given $\epsilon > 0$, define an operator A on H as follows:

$$A = \begin{bmatrix} 0 & \epsilon \\ 0 & 0 \end{bmatrix} \text{ on } M \text{ and } A = 0 \text{ on } M^1.$$

Then $\sigma(T + A) = \{a\}$. Suppose $T + A$ is centroid. Then $T + A - aI = 0$ or A is the zero operator which is a contradiction. Hence $T + A$ is not centroid. However, $\| T + A - T \| = \| A \| = \epsilon$. Since ϵ is arbitrary, it follows that $T \notin \text{Int}(\mathscr{Z})$.

Suppose $\sigma(T)$ has two or more points. Since the class of centroid operators is translation invariant we can assume that $Z_T = 0$. Since $Z_T \in$ conv. hull $[\sigma(T) \cap C_T]$ (Prasanna and Sheth 1980) there exist two or three distinct points in $\sigma(T) \cap C_T$ such that $Z_T = 0$ belongs to their convex hull. Let λ and μ be two such points. Since $|\lambda| = |\mu| = R_T$, λ and μ are normal approximate eigenvalue of T. Let T° be the image of T under the faithful * representation of Berberian. Since T is centroid if and only T° is centroid, it is sufficient to prove that T° is not in the interior of the class of centroid operators on H°, where H° is the extension of H in the Berberian representation. Hence without loss of generality we may assume that λ and μ are normal eigenvalues of T:

If x and y are the eigenvectors corresponding to λ and μ respectively then clearly $x \perp y$. Let M be the closed linear span of x and y. Define an operator S on H as follows:

$$Sx = \epsilon y, \; Sy = \theta \text{ and } Sz = \theta \text{ for all } z \in M^\perp.$$

Then $T + S = A \oplus B$ where

$$A = \begin{bmatrix} \lambda & \epsilon \\ 0 & \mu \end{bmatrix} \text{ on } M \text{ and } B = T/M^\perp \text{ on } M^\perp.$$

$\sigma(T + S) = \sigma(T)$. Hence $R_{T+S} = R_T$.

But $\| T + S \| = \sqrt{|\lambda|^2 + \epsilon^2} > |\lambda| = R_T = R_{T+S}$. Hence $T + S$ fails to be centroid. But $\|T+S-T\|=\|S\| = \epsilon$. Hence $T \notin \text{Int}(\mathscr{Z})$ and the proof is complete.

The technique of switching over to the Berberian representation helps us to give a simpler proof of Luecke's theorem.

Theorem 3 (Luecke 1972)—If dim $H \geqslant Z$ the class $\mathscr{L}$ of convexoid operators on H has empty interior.

Proof : Luecke showed that if $T \in \text{Int}(\mathscr{L})$ then $\sigma(T)$ must have atleast two points. We shall show that if $\sigma(T)$ has two or more points then $T \notin \text{Int}(\mathscr{L})$.

Since $\sigma(T)$has two or more points, $E\,(\overline{W(T)})$ contains atleast two points say λ and μ. By Krein-Milman theorem λ and μ are approximate eigenvectors of T and since $\lambda, \mu \in \partial W(T)$ they are normal approximate eigenvalues also. Since T is convexoid if and only if T° is convexoid, as in Theorem 2 we may assume that λ and μ are normal eigenvalues of T. Let S be as in Theorem 2. Then $\| T + S - T \| = \epsilon$ but $T + S$ is not convexoid (Luecke 1972). Hence $T \notin \text{Int}(\mathscr{L})$ or $\text{Int}(\mathscr{L})$ is empty.

Denoting the class of spectraloid operators by Sp, Patel (1974) showed that if $T \in Sp$ and there are two points a and b such that

(1) $a, b \in \partial W(T) \cap \pi_0(T)$ and

(2) $|a| = w(T)$, the numerical radius of T then $T \notin \text{Int}(Sp)$

We modify this result in the next theorem.

Definition—The peripheral spectrum $\sigma_{per}(T)$ is the set $\{\lambda : \lambda \in \sigma(T)$ and $|\lambda| = r(T)\}$ where $r(T)$ is the spectral radius of T.

Theorem—If $T \in Sp$ and $\sigma_{per}(T)$ has two or more points then $T \notin \text{Int}(Sp)$.

Proof : Let $\lambda, \mu \in \sigma_{per}(T)$. Since $|\lambda| = |\mu| = r(T) = w(T)$, λ and μ are normal approximate eigenvalues for T. Since T is spectraloid if and only if T° is spectraloid, as in the previous theorems we may assume that λ and μ are normal eigenvalues for T. Let S be as in Theorem 2. Then $\| T + S - T \| = \epsilon$ but $T + S$ is not spectraloid (Patel 1974). Hence $T \notin \text{Int}(Sp)$ and the proof is complete.

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References

Berberian, S.K. (1962). Approximate proper Vectors. *Proc. Am. Math. Soc.*, **13**, 111–14.

Fujii, M., and Prasanna, S. (1981). A lower bound for the Bjorck-Thomee constant of an operator. To appear in *Math. Japonica.*

Halmos, P.R. (1967). A Hilbert Space Problem Book. D. Van Nostrand, New York.

Luecke, G.R. (1972). Topological properties of paranormal operators on a Hilbert space. *Trans. Am. Math. Soc.*, **172**, 35–43.

Patel, S.M. (1974). Contributions to spectraloid operators. Ph.D. Thesis, Delhi University.

Prasanna, S. (1981). The norm of derivation and the Bjorck-Thomee-Istratescu theorem. To appear in *Math. Japonica.*

Prasanna, S., and Sheth, I.H. (1980). A remark on a paper of V. Istratescu. *Indian J. pure appl. Math.*, **11**, 212–14.

Stampfli, J.G. (1970). The norm of a derivation. *Pacific J. Math.*, **33**, 737–47.