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ON AN ERROR TERM OF LANDAU

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Let ϕ denote the Euler totient function. Landau proved that $\sum_{1 \leqslant n \leqslant x} 1/\phi(n) = A(\log x + B) + E(x)$ where $A > 0$ and B are constants and $E(x) = O(\log x/x)$. Using a theorem of Walfisz based on Weyl's inequality for exponential sums, we prove that $E(x) = O((\log x)^{2/3}/x)$.

1. INTRODUCTION

Let $\phi(n)$ denote the Euler totient function defined to be the number of positive integers $\leqslant n$ and prime to n. Landau (1900, p. 184), proved that

$$\sum_{1 \leqslant n \leqslant x} 1/\phi(n) = \frac{315\,\zeta(3)}{2\pi^4}\left\{\log x + \gamma - \sum_{p} \frac{\log p}{p^2 - p + 1}\right\} + O(\log x/x) \qquad \ldots(1.1)$$

where ζ denotes the Riemann Zeta function, γ denotes the Euler-Mascheroni constant and the sum on the right extends over all primes p. In this note we prove that the error term can be improved to $O(\log x)^{2/3}/x)$. Our improvement appears to be new and depends on estimates for the sum

$$S(x) = \sum_{n \leqslant x} \frac{1}{n} P(x/n) \qquad \ldots(1.2)$$

where $P(x) = \{x\} - 1/2$, $\{x\}$ denoting the fractional part of x. Walfisz (1927) proved that $S(x) = O(\log x/\log\log x)$ while in his recent work (cf. Walfisz 1963, p. 98), using Vinogradov's improvements on Weyl's inequality, he proved that

$$S(x) = O((\log x)^{2/3}). \qquad \ldots(1.3)$$

Throughout the paper, the constants implied by the symbols O and $\ll$ will be absolute and refer to the passage as $x \to \infty$.

2. PRELIMINARIES

Let μ denote the Möbius function, σ denote the sum of all the positive divisors function and ψ be the Dedekind's ψ-function defined by $\psi(n) = \sum_{d\delta = n} \mu^2(d)\,\delta$.

Lemma 2.1— $\sum_{n \leqslant x} 1/n = \log x + \gamma - P(x)/x + O(1/x^2)$.

PROOF : This is an easy consequence of the Euler-Maclaurin's summation formula (cf. Rademacher 1973, p. 15) and the well-known fact that

$$\sum_{n\leqslant x} 1/n = \log x + \gamma + O(1/x).$$

Lemma 2.2— $\displaystyle\sum_{n\leqslant x} \frac{\mu^2(n)}{\phi(n)} P(x/n) = O((\log 2x)^{2/3})$.

PROOF : By (1.2) and (1.3), we have for positive integral k

$$S_k(x) = \sum_{n\leqslant x,\,(n,k)=1} \frac{1}{n} P(x/n) = \sum_{n\leqslant x} \frac{1}{n} P(x/n) \sum_{d|(n,k)} \mu(d)$$

$$= \sum_{d\delta\leqslant x,\, d|k} \frac{\mu(d)}{d\delta} P(x/d\delta) = \sum_{d|k} \frac{\mu(d)}{d} S(x/d)$$

$$\ll \sum_{d|k} \frac{\mu^2(d)}{d} (\log(2x/d))^{2/3} \ll \frac{\psi(k)}{k} (\log 2x)^{2/3}. \qquad \ldots(2.1)$$

Also since $\mu^2(n) = \sum_{d^2\delta=n} \mu(d)$, we have by (2.1)

$$\sum_{n\leqslant x,(n,k)=1} \frac{\mu^2(n)}{n} P(x/n) = \sum_{d^2\delta\leqslant x,(d\delta,k)=1} (\mu(d)/(d^2\delta))\, P(x/d^2\delta)$$

$$= \sum_{d\leqslant x^{1/2},(d,k)=1} \frac{\mu(d)}{d^2} S_k(x/d^2)$$

$$\ll \sum_{d\leqslant\sqrt{x}} \frac{1}{d^2} \frac{\psi(k)}{k} (\log(2x/d^2))^{2/3}$$

$$\ll \frac{\psi(k)}{k} (\log 2x)^{2/3}. \qquad \ldots(2.2)$$

Finally since

$$n/\phi(n) = \prod_{p|n}\left(\frac{p}{p-1}\right) = \prod_{p|n} (1+1(p-1))^{-1} = \sum_{d\delta=n} \mu^2(d)/\phi(d) \qquad \ldots(2.3)$$

we have by (2.2)

$$\sum_{n\leqslant x} \frac{\mu^2(n)}{\phi(n)} P(x/n) = \sum_{n\leqslant x} \frac{\mu^2(n)}{n} P(x/n) \sum_{d\delta=n} \frac{\mu^2(d)}{\phi(d)}$$

$$= \sum_{d\delta\leqslant x,(d,\delta)=1} \frac{\mu^2(d)}{d\phi(d)} \frac{\mu^2(\delta)}{\delta} P(x/d\delta)$$

$$= \sum_{n\leqslant x} \frac{\mu^2(n)}{n\phi(n)} \sum_{m\leqslant x/n,(m,n)=1} \frac{\mu^2(m)}{m} P\left(\frac{x/n}{m}\right)$$

$$\ll \sum_{n\leqslant x} \frac{\psi(n)}{n^2\phi(n)} \left(\log\frac{2x}{n}\right)^{2/3} \ll \left(\log 2x\right)^{2/3} \sum_{n\leqslant x} \frac{\psi(n)}{n^2\phi(n)}.$$

On account of $\psi(n) = O(n \sum_{d \leqslant n} 1/d) = O(n(1+\log n))$ and that there exists a $c > 0$ such that $\phi(n) > c\, n^{1/2}$ for all n, the series $\sum_{n=1}^{\infty} \psi(n)/n\,\phi(n)$ converges. This completes the proof of the lemma.

3. Main Result

For shortness, we write $A = 315\,\zeta(3)/2\pi^4$ and $B = \gamma - \sum_p (\log p)/(p^2-p+1)$.

Theorem— $\sum_{n \leqslant x} 1/\phi(n) = A(\log x + B) + ((\log 2x)^{2/3}/x)$.

Proof : By (2.3) and Lemma 2.1, we have

$$\sum_{n\leqslant x} 1/\phi(n) = \sum_{n\leqslant x} \frac{1}{n} \sum_{d\delta = n} \frac{\mu^2(d)}{\phi(d)} = \sum_{d\delta \leqslant x} \frac{\mu^2(d)}{d\phi(d)\delta}$$

$$= \sum_{d\leqslant x} \frac{\mu^2(d)}{d\phi(d)} \sum_{\delta \leqslant x/d} 1/\delta$$

$$= \sum_{d\leqslant x} \frac{\mu^2(d)}{d\phi(d)} \left\{ \log \frac{x}{d} + \gamma - \frac{P(x/d)}{x/d} + O\left(\frac{d^2}{x^2}\right)\right\}$$

$$= \sum_{n\leqslant x} \frac{\mu^2(n)}{n\phi(n)} \log \frac{x}{n} + \gamma \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n\phi(n)} + O\left(\sum_{n>x} \frac{1}{n\phi(n)}\right)$$

$$- \frac{1}{x} \sum_{n\leqslant x} \frac{\mu^2(n)}{\phi(n)} P(x/n) + O\left(\frac{1}{x^2} \sum_{n<x} n/\phi(n)\right). \quad \ldots(3.1)$$

It is well-known that $\sigma(n)\,\phi(n) > 6n^2\pi^{-2}$ for all n so that

$$\sum_{n>x} \frac{1}{n\phi(n)} \ll \sum_{n>x} \frac{\sigma(n)}{n^3} \ll 1/x \quad \ldots(3.2)$$

on account of $\sum_{n\leqslant x} \sigma(n) = O(x^2)$ and partial summation.

Also

$$\sum_{n\leqslant x} \frac{n}{\phi(n)} \ll \sum_{n\leqslant x} \sigma(n)/n = \sum_{d\delta\leqslant x} 1/\delta = \sum_{\delta\leqslant x} 1/\delta \sum_{d\leqslant x/\delta} 1$$

$$\leqslant \sum_{\delta \leqslant x} \frac{1}{\delta}\,\frac{x}{\delta} \ll x. \quad \ldots(3.3)$$

Also writing

$$\alpha = \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n\phi(n)} \quad \ldots(3.4)$$

we have by (3.2)

$$\sum_{n\leqslant x} \frac{\mu^2(n)}{n\phi(n)} \log (x/n) = \sum_{n\leqslant x} \frac{\mu^2(n)}{n\phi(n)} \int_n^x \frac{dt}{t} = \int_1^x (1/t) \Big(\sum_{n\leqslant t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt$$

$$= \int_1^x (1/t) \Big(\alpha - \sum_{n>t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt = \alpha \log x - \int_1^\infty (1/t) \Big(\sum_{n>t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt$$

$$+ O\Big(\int_x^\infty \frac{dt}{t^2} \Big) = \alpha \log x - \int_1^\infty (1/t) \Big(\sum_{n>t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt + O(1/x). \qquad \ldots(3.5)$$

Thus from (3.1) through (3.5) and Lemma 2.2, we have

$$\sum_{n\leqslant x} 1/\phi(n) = \alpha \log x + \alpha\gamma - \int_1^\infty (1/t) \Big(\sum_{n>t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt$$

$$+ O((\log 2x)^{2/3}/x).$$

On comparing this with (1.1), we find

$$\alpha = A \text{ and } \alpha\gamma - \int_1^\infty (1/t) \Big(\sum_{n>t} \frac{\mu^2(n)}{n\phi(n)} \Big) dt = AB.$$

This completes the proof of the theorem.

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