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A NOTE ON FINITELY DIVISIBLE MODULES

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In this note the author introduces the notion of finitely divisible modules and study some of their properties. He gives a necessary and sufficient condition for a module to be finitely divisible and obtains a structure theorem for these modules. Characterizations for two important torsion theories are obtained in terms of finitely divisible modules and the possibility of characterizations of some other torsion theories in terms of them is left open.

1. PRELIMINARIES

Throughout this note $(\mathscr{T}, \mathscr{F})$ will denote a hereditary torsion theory of R-mod, the category of all left R-modules over a ring R with identity. By T and $\mathscr{G}$ we shall denote the torsion functor and the Gabriel filter of left ideals associated with $(\mathscr{T}, \mathscr{F})$ respectively. For various properties of torsion theories and all other relevant notions we refer to Stenstrom (1971).

2. FINITELY DIVISIBLE MODULES

Definition 2.1— Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory of R-mod. An R-module M is called finitely divisible relative to $(\mathscr{T}, \mathscr{F})$ if given $O \to L \to N$ with L finitely generated dense in N and $f: L \to M$ there exists $g: N \to M$ such that $gi = f$.

Remark 2.2 : If $(\mathscr{T}, \mathscr{F})$ is the trivial torsion theory (R-mod, $\{0\}$) then the concept of finitely divisible modules coincides with the concept of finitely injective modules defined by Rangaswamy and Ramamurthi (1973).

Remark 2.3 : Clearly a divisible module is finitely divisible and a finitely injective module is finitely divisible with respect to any torsion theory.

The following properties of finitely divisible modules are obvious.

Proposition 2.4 — (a) Direct sums and direct products of finitely divisible modules are finitely divisible.

(b) A direct summand of a finitely divisible module is finitely divisible.

We now give a characterization of finitely divisible modules.

Proposition 2.5 — An R-module M is finitely divisible if and only if it contains a copy of divisible hull of each of its finitely generated submodule.

PROOF: We only prove the necessary part. Let M be a finitely divisible module and N be a finitely generated submodule of M. Let $D(N)$ be the divisible hull of N. Let $i: N \to D(N)$ and $j: N \to M$ be the inclusion maps. Since N is dense in $D(N)$ and

M is finitely divisible, there exists $g : D(N) \to M$ such that $gi = j$. Since N is essential in $D(N)$, $g : D(N) \to M$ is a monomorphism.

Corollary 2.6— Every finitely generated finitely divisible module is divisible.

Definition 2.7 (Golan 1975)— A module M is called T-finitely generated relative to a torsion theory $(\mathcal{T}, \mathcal{F})$ if there exists a finitely generated submodule N of M such that $M/N \in \mathcal{T}$.

Proposition 2.8— Let M be a countably generated module which is also T-finitely generated. Then M is finitely divisible if and only if M is a direct sum of countable number of divisible modules.

PROOF : Let $N \subseteq M$ be a finitely generated submodule of M such that M/N is torsion. Let $\{x_i\}_{i=1}^{\infty}$ be a set of generators of M. Since N is finitely generated there exists an integer n such that $N \subseteq \sum_{i=1}^{n} Rx_i$. We then have a countable increasing chain $\{M_i\}_{i=1}^{\infty}$ of finitely generated submodules of M such that $M_1 \supseteq N$ and $M = \bigcup_{i=1}^{\infty} M_i$. Now M finitely divisible implies M contains a divisible hull $D(M_i)$ of M_i for all i. We can take $\{D(M_i)\}_{i=1}^{\infty}$ to be an increasing chain of submodules of M. Now, since $N \subseteq M_i$ for all i, $D(M_i)$ is a direct summand of $D(M_{i+1})$ for all i. Let $L_1 = D(M_1)$ and for $k > 1$, let L_k be such that $L_k \oplus D(M_k) = D(M_{k+1})$. Clearly $\{L_k\}_{k=1}^{\infty}$ is a direct family of submodules of M and $M = \sum_{k=1}^{\infty} \oplus L_k$.

Corollary 2.9— A countably generated torsion module is finitely divisible if and only if it is a countable direct sum of divisible modules.

Corollary 2.10— Let R satisfies ascending chain condition on left ideals of $\mathcal{T}$ and M be a T-finitely generated countably generated module. Then M is divisible if M is finitely divisible. In particular every countably generated finitely divisible torsion module is divisible.

PROOF : Follows from Proposition 2.8 and Lemma 2 Golan and Teply (1973).

The next proposition gives the structure of finitely divisible modules.

Proposition 2.11— An R-module M is finitely divisible if and only if $M = \varinjlim \{M_i, \phi_{ij}\}$ where M_i's are divisibles and ϕ_{ij}'s are monomorphisms.

PROOF : ($\Rightarrow$) First note that $M = \varinjlim \{M_i, \phi_{ij}\}$ where M_i $(i \in I)$ is the family of finitely generated submodules of M. Since by hypothesis M is finitely divisible, for each $i \in I$, M contains a divisible hull $D(M_i)$ of M_i. Now since $D(M_i)$'s are divisible we have $\phi'_{ij} : D(M_i) \to D(M_j)$ induced by $\phi_{ij} : M_i \to M_j$. It can be now easily seen that $\{D(M_i), \phi'_{ij}\}$ is a directed system with ϕ'_{ij} monomorphisms and $\varinjlim \{D(M_i), \phi'_{ij}\} = M$. ($\Leftarrow$) Easy to verify.

We now give characterizations of some special types of torsion theories in terms of finitely divisible modules.

Proposition 2.12— Following are equivalent for a hereditary torsion theory $(\mathscr{T}, \mathscr{F})$.

(1) Every left ideal in the Gabriel filter $\mathscr{G}$ is finitely generated.

(2) Every finitely divisible module is divisible.

Proof : (1) $\Rightarrow$ (2). By definition.

(2) $\Rightarrow$ (1). Follows from Proposition 2.11 and Theorem 2 of Golan and Teply (1973).

Proposition 2.13— Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory of R-mod. Then the following are equivalent.

(1) The Gabriel filter $\mathscr{G}$ of left ideals contains a cofinal subset of finitely generated left ideals.

(2) Every torsion-free finitely divisible module is divisible.

Proof : (1) $\Rightarrow$ (2). Let $I \in \mathscr{G}$ and M a finitely divisible torsion-free module. Let $f : I \to M$ be given. Since $\mathscr{G}$ contains a cofinal subset of finitely generated ideals, there exists $I' \subseteq I$ such that I' is finitely generated. Let $f' = f \mid I'$. By finite divisibility of M there exists $g : R \to M$ such that g extends f'. Using the torsion-freeness of M it can now be shown that g also extends f and hence M is divisible.

(2) $\Rightarrow$ (1). Follows from Proposition 2.11 and Proposition 12.2 of Stenstrom (1971).

Corollary 2.14 — Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory of R-mod. If every left ideal of R is T-finitely generated then every torsion-free finitely divisible module is divisible.

Proof : It can be easily seen that the Gabriel filter $\mathscr{T}$ in this case contains a cofinal family of finitely generated left ideals (Stenstrom 1971).

Remark 2.15 : Ramamurthi and Rangaswamy (1973) have shown that direct limits of finitely injective modules need not be finitely injective. Hence direct limits of finitely divisibles are not necessarily finitely divisible. It will be therefore interesting to characterize torsion theories for which direct limits of finitely divisibles are finitely divisible.

We now consider the property that homomorphic images of finitely divisibles are finitely divisible.

Proposition 2.16—Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory of R-mod. Then

(i) If homomorphic images of finitely divisibles are finitely divisible then every finitely generated left ideal in $\mathscr{T}$ is T-projective.

(ii) If homomorphic images of divisible are divisible then homomorphic images of finitely divisibles are finitely divisible.

Proof : (i) Let I be a finitely generated dense left ideal of R. Let $f : F \to F' \to O$ with F divisible and torsion-free and F' torsion-free be given. Let $g : I \to F'$ be any homomorphism. Since F is divisible, F' is finitely divisible by hypothesis. Hence there exists $h : R \to F'$ such that $hi = g$ where i is the inclusion of I in R.

Since R is projective there exists $k : R \to F$ such that $fk = h \Rightarrow fki = hi = g$. Hence by Beachy (1972), I is T-projective.

(ii) Easily follows from Proposition 2.5 and the hypothesis.

Proposition 2.17— If $\mathscr{T}$ contains a cofinal subset of finitely generated left ideals then every left ideal in $\mathscr{T}$ is T-projective if homomorphic images of finitely divisibles are finitely divisibles. Conversely, if every left ideal in $\mathscr{T}$ is projective then homomorphic images of finitely divisibles are divisible.

PROOF : ($\Leftarrow$) Let $I \in \mathscr{T}$ and $F \to F' \to O$ be given such that F is divisible and torsion-free and F' is torsion-free. Let $g : I \to F'$ be any homomorphism. Now by hypothesis F' is finitely divisible. Since F' is torsion-free by Proposition 2.13, F' is divisible. It is now easy to see that I is T-projective. ($\Rightarrow$) Follows from Proposition 2.16 (ii) and Proposition 4.6 of Golan (1975).

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