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DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS AND AN EXTENSION OF LYAPUNOV'S METHOD

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Most of the research on the theory of perturbed systems is on stability under continuous perturbations. In this paper, Lyapunov's second method is employed to investigate sufficient conditions for eventual uniform asymptotic stability of differential systems with impulsive perturbations.

§ 1. The second method of Lyapunov plays a vital role in studying the qualitative behaviour of solutions of ordinary and functional differential equations. This method depends on estimating the derivative of a scalar function along the solutions of the given differential equation. But this type of technique is not directly applicable in the study of the behaviour of solutions of measure differential equations and differential equations with impulsive perturbations. Since the solutions of such differential equations are discontinuous, most differential inequalities are not applicable.

The object of this paper is to extend Lyapunov's second method and investigate sufficient conditions for eventual uniform asymptotic stability of differential equations with impulsive perturbations. The stability of systems with respect to impulsive perturbations has been considered by Barbashin (1966) and Zavalishchin (1966). Our results improve and include some of the results of Barbashin (1966), Das and Sharma (1972), Raghavendra and Rao (1974) and Zavalishchin (1966), simple example is constructed to illustrate the results.

§ 2. We shall use the following notation throughout this paper :

R^n = space of n-vectors

$$| x | = \sum_{i=1}^{n} | x_i | , \quad x \in R^n,$$

$J = [0, \infty)$,

R^+ = nonnegative real line.

$S_r = x \in R^n : x < r, \quad r > 0$,

$\mathscr{K}$ = the class of functions $b(u)$, defined and continuous on $[0, r)$, $b(0) = 0$ and monotonic increasing in u,

C_0 = the class of functions having uniform Lipschitz constants on $J \times S_r$.

We shall consider the measure differential equation

$$Dx = f(t, x) + g(t, x)\, Du \qquad \text{...(2.1)}$$

where $x \in R^n$, Du denotes the distributional derivative of the function u. f, g $J \times R^n \to R^n$ and $u : J \to R$ is a right continuous function of bounded variation on every compact subinterval of J. Here Du can be identified with a Stieltjes measure and has the effect of suddenly changing the state of the system at the points of discontinuity of u.

Equation (2.1) may be regarded as a perturbed system of the ordinary differential equation

$$x' = f(t, x) \qquad \text{...(2.2)}$$

where the perturbation $g(t, x)\, Du$ is impulsive. A question arises under what conditions the stability properties of (2.2) are shared by the solutions of (2.1). A satisfactory answer to this question is rather difficult, possibly, because of the crucial role of differential and integral inequalities in stability theory. Since solutions of (2.1) are discontinuous, most differential inequalities are inapplicable, furthermore, integral inequalities are not available for Stieltjes integral. However, here we investigate some stability properties of ordinary differential systems with respect to impulsive perturbations under somewhat restrictive conditions. Such conditions are natural to expect since the impulsive perturbations may cause considerable changes in the state variables to make the system unstable.

We require the following conditions in our subsequent discussion.

(H_1) : There exists $r > 0$ such that for every b, $0 < b < r$, there exist $\tau_b > 0$ and a measurable function $\gamma_b(t)$ defined on $[\tau_b, \infty]$ such that $|g(t, x)| \leqslant \gamma_b(t)$ for all $b \leqslant |x| < r$ and $t \geqslant \tau_b$, where

$$G_b(t) = \int_t^{t+1} \gamma_b(s)\, ds \to 0 \text{ as } t \to \infty.$$

Define

$$Q_b(t) = \sup\{G_b(s) : t-1 \leqslant s < \infty\}.$$

Obviously $Q_b(t) \downarrow 0$ as $t \to \infty$.

(H_2) : u is a right continuous real valued function defined on J and is a function of bounded variation on every compact subinterval of J. The discontinuities $t_1 < t_2 < t_3 < \ldots < t_k < \ldots$ of u tend to ∞ as k tends to ∞

(H_3) : The series $\sum_{n=1}^{\infty} \gamma_b(t_n) h_n$ and $\sum_{n=1}^{\infty} Q_b(t_n)$ are convergent, where h_n represents the jump at t_n of the function u.

(H_4) : The derivative of u on $[t_k, t_{k+1}]$, $k = 1, 2, 3, \ldots$ exists and is bounded by a constant $M > 0$, where the derivative at t_k, $k = 1, 2, 3, \ldots$ is to be understood as the right hand derivative.

We state the following existence and uniqueness theorem whose proof is quite similar to that of Theorem 1 of Das and Sharma (1972).

Theorem — Suppose that f and g satisfy the following conditions on the set

$E = \{ (t, x) : t \in [t_0, T], \ x \in S_r \}$,

(i) f is continuous in x for each fixed t and measurable in t for fixed x and satisfies a local Lipschitz condition in x,

(ii) there exists a Lebesgue integrable function m, such that

$|f(t, x)| \leqslant m(t)$;

(iii) $g(t, x)$ is du-integrable for each $x(.) \in BV([t_0, T], S_r)$;

(iv) $g(t, x)$ is continuous in x for each t;

(v) there exists a dv_u-integrable function w such that

$| g(t, x) | \leqslant w(t)$

where v_u denotes the total variation function of u. Then there exists a unique solution $x(.)$ of (2.1) on some interval $[t_0, t_0 + a]$ satisfying the initial condition $x(t_0) = x_0 \in S_r$.

§ 3. In this section, we shall give sufficient conditions for eventual uniform asymptotic stability (cf. Lakshmikantham and Leela 1969, p. 222) of the trivial solution of (2.1).

Theorem 3.1— Assume that the conditions (H_1) to (H_4) hold and $f \in C_0$, $f(t,0) \equiv 0$ for $t \geqslant 0$. Let the trivial solution of (2.2) be eventually uniformly asymptotically stable. Then there exist $T_0 \geqslant 0$ and $\delta_0 > 0$ such that if $t_0 \geqslant T_0$ and $|x_0| < \delta_0$, then the solution $y(t) = y(t, t_0, x_0)$ of (2.1) satisfies $| y(t) | \to 0$ as $t \to \infty$. In particular, if $g(t,0) \equiv 0$, then the trivial solution of (2.1) is eventually uniformly asymptotically stable.

PROOF : Since the solutions of (2.2) are unique and continuous with respect to initial values, in view of Remark 3.14.1 in Lakshmikantham and Leela (1969), the trivial solution of (2.2) is uniformly asymptotically stable. Thus, by a theorem of Massera (1956) there exists a Lyapunov funcion $V(t, x)$ on $J \times S_r$ satisfying

(i) $a(|x|) \leqslant V(t, x) \leqslant c(|x|)$, $a, c \in \mathscr{K}$;

(ii) $\dot{V}(t, x) \dfrac{\partial V(t, x)}{\partial t} + V(t, x) \cdot f(t, x) \leqslant -\sigma(|x|)$,

where $V = \left(\dfrac{\partial V}{\partial x_1}, \dots, \dfrac{\partial V}{\partial x_n} \right)$ and $\sigma \in \mathscr{K}$;

(iii) $| V(t, x) | \leqslant K_1$, where K_1 is a positive constant.

Let $0 < \epsilon < r$ be given. Choose $\delta = \delta(\epsilon)$, $0 < \delta < \epsilon$ so that

$$3c(\delta) < a(\epsilon). \qquad \text{...(3.1)}$$

Let $\alpha > 0$ be any number such that $\alpha < \delta$. Choose $N_1 = N_1(\epsilon)$ so that $t_{N_1} \geqslant \tau_\delta + 1$

and

$$\sum_{n=N_1}^{\infty} [K_1 \gamma_\alpha (t_{n+1}) h_{n+1} + MK_1 Q_\alpha (t_{n+1})] < a(\epsilon)/3 \quad \ldots(3.2)$$

Choose $T_1 = T_1(\epsilon) \geqslant t_{N_1}$ so large that

$$3MK_1 Q_\alpha (T_1) < \min [\sigma(\alpha), a(\epsilon)]. \quad \ldots(3.3)$$

Let $|x_0| < \delta$ and $t_0 \geqslant T_1$. Then, we claim that

$$|y(t, t_0, x_0)| < \epsilon \text{ for } t \in [t_0, \infty). \quad \ldots(3.4)$$

Suppose not. Let T_3 be the first point larger then t_0 such that $|y(T_3)| \geqslant \epsilon$; its existence is guaranteed by the right continuity of $y(t)$.

Since the solution $y(t)$ of (2.1) is unique to the right of t_0, there exists a non-negative number δ_1 such that

$$\inf_{t \in [t_0, T_3]} |y(t)| = \delta_1.$$

If $\delta_1 = 0$ for some $t^* \in [t_0, T_3]$ then due to the uniqueness of solution $y(t)$ of (2.1) to the right of t_0, we have $y(t) \equiv 0$ for all $t \geqslant t^*$ and therefore the proof is trivial in this case. Thus we consider the case when $\delta_1 > 0$. Then $\delta_1 \leqslant |y(t)| < r$ between t_0 and T_3.

Since u is differentiable on $[t_k, t_{k+1})$, y is also differentiable on $[t_k, t_k + 1)$, $k = 1, 2, 3, \ldots$ For as long as a solution $y(t)$ of (2.1) exists, and is differentiable for $t \in [t_k, t_{k+1})$, we have

$$\frac{d}{dt} V(t, y(t)) = \frac{\partial}{dt} V(t, y(t)) + \nabla V(t, y(t)) \cdot [f(t, y(t)) + g(t, y(t)) \dot{u}(t)].$$

By integrating from t_k to t and using (ii) and (iii),
we get

$$V(t, y(t)) \leqslant V(t_k, y(t_k)) - \int_{t_k}^{t} \sigma(|y(s)|)\, ds + MK_1 \int_{t_k}^{t} |g(s, y(s))|\, ds.$$

Thus, if $0 < \delta_1 \leqslant |y(s)| < r$ between t_k and t, where $t \in [t_k, t_{k+1})$, then Lemma 3.4 of Strauss and Yorke (1967) yields

$$V(t, y(t)) \leqslant V(t_k, y(t_k)) - \sigma(\delta_1)(t - t_k) + MK_1 Q_{\delta_1}(t_k)(t - t_k + 1). \quad \ldots(3.5)$$

Since $V(t, x)$ is continuous in t for each fixed x, we have

$$\begin{aligned} V(t_{k+1}, y(t_{k+1})) &\leqslant |V(t_{k+1}, y(t_{k+1})) - V(t_{k+1}^-, y(t_{k+1}^-))| + V(t_{k+1}^-, y(t_{k+1}^-)) \\ &\leqslant |V(t_{k+1}, y(t_{k+1})) - V(t_{k+1}, y(t_{k+1}^-))| \\ &\quad + \lim_{h \to 0} + V(t_{k+1} - h, y(t_{k+1} - h)). \end{aligned} \quad \ldots(3.6)$$

Further, we know (Das and Sharma 1972) that $y(\cdot)$ is a solution of (2.1) through (t_0, x_0) on J if and only if it satisfies the integral equation

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds + \int_{t_0}^{t} g(s, y(s))\, du(s)$$

for $t > t_0 \in J$. Hence,

$$| y(t_{k+1}) - y(t_{k+1}^-) | = \lim_{h \to 0^+} | y(t_{k+1}) - y(t_{k+1} - h) |$$

$$= \lim_{h \to 0^+} \left| \int_{t_{k+1}-h}^{t_{k+1}} f(s, y(s))\, ds + \int_{t_{k+1}-h}^{t_{k+1}} g(s, y(s))\, du(s) \right|$$

$$\leqslant | g(t_{k+1}, y(t_{k+1})) \| u(t_{k+1}) - u(t_{k+1}^-) | .$$

Thus, if $0 < \delta_1 \leqslant | y(t_{k+1}) | < r$, we have

$$| y(t_{k+1}) - y(t_{k+1}^-) | \leqslant \gamma_{\delta_1}(t_{k+1})\, h_{k+1}. \quad \ldots(3.7)$$

By using (3.5), (3.6), (3.7) and the uniform Lipschitz property of V [this property is guaranteed because of (iii)],
we obtain

$$V(t_{k+1}, y(t_{k+1})) \leqslant V(t_k, y(t_k)) + \lim_{h \to 0^+} [\sigma(\delta_1)(t_{k+1} - h - t_k)$$

$$+ MK_1\, Q_{\delta_1}(t_k)(t_{k+1} - h - t_k + 1)] + K_1\, \gamma_{\delta_1}(t_{k+1})\, h_{k+1}.$$

That is,

$$(t_{k+1}, y(t_{k+1})) \leqslant V(t_k, y(t_k)) - \sigma(\delta_1)(t_{k+1} - t_k) + MK_1\, Q_{\delta_1}(t_{k+1} - t_k + 1)$$

$$+ K_1\, \gamma_{\delta_1}(t_{k+1})\, h_{n+1}. \quad \ldots(3.8)$$

Thus, if $0 < \delta_1 \leqslant | y(s) | < r$ between t_k and t_{k+1}, then for any $t \in [t_k, t_{k+1}]$, from (3.5) and (3.8), we have

$$V(t, y(t)) \leqslant V(t_k, y(t_k)) - \sigma(\delta_1)(t - t_k)$$

$$+ MK_1\, Q_{\delta_1}(t_k)(t - t_k + 1) + K_1\, \gamma_{\delta_1}(t_{k+1})\, h_{k+1}. \quad \ldots(3.9)$$

Similarly, if $\delta_1 \leqslant | y(s) | < r$ between t_{k+1} and t_{k+2},
then for $t \in [t_{k+1}, t_{k+2}]$, we have

$$V(t, y(t)) \leqslant V(t_{k+1}, y(t_{k+1})) - \sigma(\delta_1)(t - t_{k+1})$$

$$+ MK_1\, Q_{\delta_1}(t_{k+1})\, t - t_{k+1} + 1) + K_1\, \gamma_{\delta_1}(t_{k+2})\, h_{k+2}. \quad \ldots(3.10)$$

Thus, if $\delta_1 \leqslant | y(s) | < r$ between t_k and t_{k+2}, by (3.9), (3.10) and the decreasing property of $Q_{\delta_1}(t)$, we have for $t \in [t_k, t_{k+2}]$

$$V(t, y(t)) \leqslant V(t_k, y(t_k)) - \sigma(\delta_1)(t - t_k) + MK_1\, Q_{\delta_1}(t_k)(t - t_{k+1})$$

$$+ MK_1 Q_{\delta_1}(t_{k+1}) + K_1\, \gamma_{\delta_1}(t_{k+1})\, h_{k+1} + K_1\, \gamma_{\delta_1}(t_{k+2})\, h_{k+2}.$$

By repeating the above process, in general, if $\delta_1 \leqslant | y(s) | < r$ between t_k and t, then we have

$$V(t, y(t)) \leqslant V(t_k, y(t_k)) - \sigma(\delta_1)(t-t_k) + MK_1 \; Q_{\delta_1}(t_k)(t-t_k+1)$$

$$+ \sum_{i=N_1}^{\infty} [K_1 \gamma_{\delta_1} \; (t_{i+1}) h_{i+1} + MK_1 Q_{\delta_1} \; (t_{i+1}). \qquad ...(3.11)$$

Note that one may replace t_k in (3.11) by any number in (t_k, t). Thus for $t \in [t_0, T_3]$

$$V(t, y(t)) \leqslant V(t_0, y(t_0)) - \sigma(\delta_1) \; (t-t_0) + MK_1 \; Q_{\delta_1}(t_0)(t-t_0+1)$$

$$+ \sum_{i=N_1}^{\infty} \; [K_1 \gamma_{\delta_1} \; (t_{i+1}) h_{i+1} + MK_1 Q_{\delta_1}(t_{i+1})]. \qquad ...(3.12)$$

Therefore for $t = T_3$, (3.12) yields that

$$a(\epsilon) \leqslant a(|y(T_3)|) \leqslant V(|T_3, y(T_3)) \leqslant V(t_0, y(t_0)) - \sigma(\delta_1)(T_3 - t_0)$$

$$+ MK_1 Q_{\delta_1}(t_0)(T_3 - t_0+1)$$

$$+ \sum_{i=N_1}^{\infty} \; [K_1 \gamma_{\delta_1} \; (t_{i+1}) h_{i+1} + MK_1 Q_{\delta_1} \; (t_{i+1})]$$

$$\leqslant c(|y(t_0)|) + MK_1 Q_{\delta_1}(T_1)$$

$$+ \sum_{i=N_1}^{\infty} \; [K_1 \gamma_{\delta_1} \; (t_{i+1}) h_{i+1} + MK_1 Q_{\delta_1} \; (t_{i+1})]$$

$$+ \; [MK_1 \; Q_{\delta_1}(T_1) - \sigma(\delta_1)](T_3 - t_0).$$

Thus in view of (3.1) and (3.2), (3.3) with $\alpha = \delta_1$,
we have

$$a(\epsilon) < c(\delta) + \frac{a(\epsilon)}{3} + \frac{a(\epsilon)}{3}$$

$$< \frac{a(\epsilon)}{3} + \frac{a(\epsilon)}{3} + \frac{a(\epsilon)}{3} = a(\epsilon)$$

a contradiction, proving (3.4). This proves that the trivial solution of (2.1) is eventually uniformly stable for the case $g(t, 0) \equiv 0$. For the rest of the proof fix $r_1 < r$ and choose $\delta_0 = \delta(r_1)$ and $T_0 = T_1(r_1)$. Fix $t_0 \geqslant T_0$ and $|x_0| < \delta_0$. Then (3.4) implies that $|y(t, t_0, x_0)| < r_1$ on $[t_0, \infty)$.

Let $0 < \eta < r_1$. Choose $\delta = \delta(\eta)$, $\quad 0 < \delta < \eta$

so that

$$3c(\delta) < a(\eta). \qquad ...(3.13)$$

and $N_2 = N_2(\eta)$ so large, $t_{N_2} \geqslant \tau_\delta + 1$ so that

$$\sum_{n=N_2}^{\infty} \; [K_1 \gamma_\delta (t_{n+1}) h_{n+1} + MK_1 Q_\delta (t_{n+1})] < a(\epsilon)/3 \qquad ...(3.14)$$

Then, choose $T_1 = T_1(\eta) \geqslant t_{N_2}$ such that

$$3 MK_1 Q_\delta (T_1) < \min[\sigma(\delta), a(\eta)]. \qquad ...(3.15)$$

Select

$$T = \left[\frac{3\,c\,(r_1) + 3MK_1\,Q_\delta\,(1) + a\,(\eta) + 2\,T_1\,(\eta)\,\sigma\,(\delta)}{2\,\sigma\,(\delta)}\right] > T_1\,(\eta)$$

and it is clear that T depends only on η, not on t_0 or x_0.

We now claim that

$$|\, y(t_1, t_0, x_0) \,| < \delta \qquad \text{...(3.16)}$$

for some $t_1 \in [t_0 + T_1, t_0 + T]$.

Suppose not. Then $\delta \leqslant |\, y\ (t, t_0, x) \,| < r_1$ on $[t_0 + T_1, t_0 + T]$. Let $y_0 = y\,(t_0 + T_1, t_0, x_0)$. Then by (3.11), (3.13) to (3.15), we have

$$\begin{aligned} 0 < a\,(\delta) &\leqslant a\,(\,|\, y\,(t_0 + T, t_0 + T_1, y_0)\,|\\ &\leqslant V\,(t_0 + T, y\,(t_0 + T))\\ &\leqslant c\,(\,|\, y_0 \,|\,) + [MK_1\,Q_\delta\,(t_0 + T_1) - \sigma\,(\delta)\,]\,(T - T_1)\\ &\quad + MK_1\,Q_\delta\,(t_0 + T_1) + \sum_{i=N_2}^{\infty}\ [K_1\,\gamma_\delta\,(t_{i+1})\,h_{i+1} + MK_1\,Q_\delta\,(t_{i+1})\,]\\ &< c\,(r_1) - \tfrac{2}{3}\,\sigma\,(\delta)\,(T - T_1) + MK_1\,Q_\delta\,(1) + \tfrac{1}{3}\,a\,(\eta) = 0, \end{aligned}$$

a contradiction, proving (3.16). Thus by (3.4)

$$|\, y\,(t, t_1, y\,(t_1, t_0, x_0))\,| < \eta \text{ on } [t_1, \infty)$$

because $t_1 \geqslant t_0 + T_1 \geqslant T_1$ and $|\, y\,(t_1, t_0, x_0)\,| < \delta$.

Hence, by uniqueness of solutions of (2.1), we have

$$|\, y\,(t, t_0, x_0)\,| < \eta \text{ for } t \geqslant t_0 + T.$$

Since η is arbitrary, $|\, y\,(t, t_0, x_0)\,| \to 0$ as $t \to \infty$. Since T depends only on η and δ depends only on ϵ, the trivial solution of (2.1) is eventually uniformly asymptotically stable if $g\,(t, 0) \equiv 0$. This completes the proof.

Remark 3.2 : Das and Sharma (1972) have proved a theorem on quasi-equi-asymptotic stability of the null solution of (2.1), assuming that the null solution of (2.2) is exponentially asymptotically stable. Thus, although they have assumed a stronger type of stability for (2.2), they could only obtain a weaker type of stability for (2.1). Obviously our results improve and include the results of Das and Sharma (1972). Further, they assumed that the jumps of u at t_k ($k = 1, 2,\ldots$) are bounded by $\alpha\, e^{-c(t_k - t_0)}$, α, $c > 0$, $t_k \geqslant t_0$ (see Das and Sharma 1972, p. 151) so that the jumps ultimately die down to zero. This is clearly a stronger assumption than is made in our present study. In other words, by assuming a weaker type of condition on the jumps of u at t_k, we obtain sufficient conditions for eventual uniform asymptotic stability of the null solution of (2.1), assuming that the null solution of (2.2) is eventually uniformly asymptotically stable.

Remark 3.3 : If the perturbations in eqn. (2.1) are not impulsive so that the state of the system changes continuously with respect to time, then our results reduce to some of the results of Lakshmikantham and Leela (1960) and Strauss and Yorke (1967).

Example 3.4 : Define a scalar function $g(t, x)$ for $0 \leqslant t < \infty, 0 \leqslant |x| < r$ as follows :

$$g(t, x) = \begin{cases} 1 & \text{for } t = n, \quad n \text{ is a positive integer} \\ \dfrac{1}{tx+1} & \text{for other values of } t. \end{cases}$$

Let $\tau_{\delta_1} = \delta_1^{-1} + 1$, $\gamma_{\delta_1}(n) = 1$, and $\gamma_{\delta_1}(t) = \dfrac{1}{t\delta_1 - 1}$ for t not an integer.

Then $\gamma_{\delta_1}(t) \to 0$ as $t \to \infty$ and $\int_{\delta_1}^{\infty} \gamma_{\delta_1}(t)\, dt = \infty$.

However,

$$G_{\delta_1}(t) = \int_t^{t+1} \gamma_{\delta_1}(s)\, ds \to 0 \text{ as } t \to \infty.$$

In this case $Q_{\delta_1}(t) = \dfrac{1}{\delta_1} \log\left(1 + \dfrac{\delta_1}{t\delta_1 - \delta_1 - 1}\right)$ and $Q_{\delta_1}(t) \to 0$ as $t \to \infty$.

Choose $u(t) = \sum_{k=1}^{\infty} u_k(t)$

where $u_n(t) = 0$ for $t < n^2 + \frac{1}{2}$

$= 1$ for $t \geqslant n^2 + \frac{1}{2}$.

Then $h_n = 1$ and $t_n = n^2 + \frac{1}{2}$ for $n = 1, 2 \ldots$

Clearly the discontinuities of u tend to ∞. Further, the series $\sum_{n=1}^{\infty} \gamma_{\delta_1}(t_n) h_n$ and $\sum_{n=1}^{\infty} Q_{\delta_1}(t_n)$ are convergent. Thus the conditions (H_1) to (H_4) are satisfied.

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