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ON THE DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE CLASS LIP α

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The author defines $(N, p_n, q_n)_p$ method as

$$T_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k}\, s_{n-k,p} \to S \text{ as } n \to \infty$$

uniformly with respect to p, where $\{p_n\}$, $\{q_n\}$ are non-negative, non-increasing generating sequences,

$$s_{n-k,p} = \frac{1}{n-k+1} \sum_{\mu=p}^{n-k+p} s_\mu \text{ and } s_k \text{ is } k\text{-th partial sum of the Fourier series.}$$

We prove the theorem : If $f(x)$ is periodic function and belongs to the class Lip α for $0<\alpha\leqslant 1$ and if $\{p_n\}$, $\{q_n\}$ are defined for $(N. p_n\, q_n)_p$ method and $R(y)/y^\alpha$ is non-decreasing,
then

$$\left| f(x) - T_n^{p,q} \right| = O\left(\frac{1}{n^\alpha}\right).$$

§1. Let $f(x)$ be a function integrable in the sence of Lebesgue and periodic with period 2π. Let the Fourier series associated with $f(x)$ be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \qquad \text{...(1.1)}$$

A function $f \in$ Lip α if

$$f(x+h) - f(x) = O(|h|^\alpha) \text{ for } 0<\alpha\leqslant 1. \qquad \text{...(1.2)}$$

We say that $f(x) \in$ Lip (α, p) if

$$\left\{ \int_0^{2\pi} \left| f(x+t) - f(x) \right|^p dx \right\}^{1/p} = O(t^\alpha),\ 0<\alpha\leqslant 1\ . \qquad \text{...(1 3)}$$

Lorentz (1948) has defined:

Definition L_1—A sequence $\{s_n\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{(n+1)} \sum_{k=p}^{n+p} s_k = S, \qquad \text{...(1.4)}$$

(uniformly with respect to p.)

We defined (Qureshi 1981):

Definition L_2—A series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{s_n\}$ is said to be almost Nörlund summable to S, provided

$$t_{n,p} = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, s_{k,p} \to S \text{ as } n \to \infty \qquad \ldots(1.5)$$

uniformly with respect to p, where

$$s_{k,p} = \frac{1}{k+1} \sum_{\mu=p}^{k+p} s_\mu \qquad \ldots(1.6)$$

and $\{p_n\}$ is a sequence positive constants such that

$P_n = p_0 + p_1 + \ldots + p_n \to \infty$, as $n \to \infty$.

Following Khan (1974), we write:

Let $\{p_n\}$, $\{q_n\}$ be non-negative, non-increasing generating sequences for (N, p_n, q_n) method such that

$$P_n = p_o + p_1 + \ldots p_n \to \infty \text{ as } n \to \infty \qquad \ldots(1.7)$$

$$Q_n = q_o + q_1 + \ldots\ldots q_n \qquad \ldots(1.8)$$

$$R_n = p_o q_n + p_1 q_{n-1} + \ldots + p_n q_o \to \infty \text{ as } n \to \infty \qquad \ldots(1.9)$$

$$t_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} r_k s_{n-k} \qquad \ldots(1.10)$$

where s_k is the k-th partial sum of the series (1.1) and $r_k = p_k q_{n-k}$.

$P_n = P(n)$, $Q_n = Q(n)$, $r_n = r(n)$; $p(y) = p_{[y]}$ and $P(y) = P_{[y]}$, $r(y) = r_{[y]}$ and $R(y) = R_{[y]}$, where $[y]$ as usual denotes the greatest integer less than y.

For the first time we define:

Definition L_3—A series $\sum_{n=0}^{\infty} u_n$ with sequence of partial sums $\{s_n\}$ is said to be almost generalized Nörlund summable or summable $(N, p_n, q_n)_p$ to S, provided

$$T_n^{p,q} = \frac{1}{R_n} \sum_{k=0}^{n} r_k s_{n-k,p} \to S \text{ as } n \to \infty \qquad \ldots(1.11)$$

uniformly with respect to p, where

$$s_{n-k,p} = \frac{1}{n-k+1} \sum_{\mu=p}^{n-k+p} s_\mu . \qquad \ldots(1.12)$$

The following theorems are known:

Theorem A—If $f(x)$ is periodic and belongs to the class Lip (α,p), $0<\alpha\leqslant 1$ and let $\{p_n\}$ be defined as in (1.7) and

$$\left[\int_1^n \frac{(P(y))^q}{y^{q\alpha+1\cdot q}}\,dy\right]^{1/q} = O\left(\frac{P(n)}{n^{\alpha+(1/q)-1}}\right)$$

then

$$\min_{t_n}\left|\left|f - t_n\right|\right|_p = O\left(\frac{1}{n^{\alpha-(1/p)}}\right) \qquad \ldots(2.1)$$

where

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\, s_k$$

i.e. the (N,p_n) mean of the Fourier series (1.1).

Theorem B—If $f(x)$ is a periodic function and belongs to the class Lip (α,p) for $0<\alpha\leqslant 1$ and if the sequences $\{p_n\}$, $\{q_n\}$ are defined as above,

$\frac{R(y)}{y^\alpha}$ is non-decreasing,

then

$$\min\left|\left|f - t_n^{p,q}\right|\right| = O\left(\frac{1}{n^{\alpha-(1/p)}}\right). \qquad \ldots(2.2)$$

Theorem A was proved by Sahney and Rao (1972) while Theorem B by Khan (1974).

We proved the following theorem (Qureshi 1981):

Theorem C—The degree of approximation of a periodic function f with period 2π and belonging to the class Lip α, $0<\alpha\leqslant 1$ by almost Nörlund means of its Fourier series is given by

$$\max_{0<t<2\pi}\left|f(t) - T_{n,p}(t)\right| = O\left(\frac{1}{n^\alpha}\right), \qquad \ldots(2.3)$$

where the sequence $\{p_n\}$ is non-negative and non-increasing such that

$$\sum_{k=0}^{n}\frac{p_{n-k}}{k+1} = O\left(\frac{P_n}{n}\right).$$

Our object of this paper is to prove the following theorem.

Theorem D—If $f(x)$ is a periodic function and belongs to the class Lip α for $0<\alpha\leqslant 1$ and if the sequences $\{p_n\}$, $\{q_n\}$ are defined as above,

$\frac{R(y)}{y^\alpha}$ is non-decreasing then

$$\left| f-T_n^{p,q} \right| = O\left(\frac{1}{n^\alpha}\right).$$

To prove the theorem we need the following lemma.

Lemma (Khan 1974)—If $\{p_n\}$ and $\{q_n\}$ are non-negative and non-increasing, then, for $0\leqslant a\leqslant b\leqslant\infty$, $0\leqslant t\leqslant\pi$ and any n, we have

$$\left| \sum_{k=a}^{b} p_k q_{n-k}\, e^{i(n-k)t} \right| \leqslant R\left(\frac{1}{t}\right), \text{ for any } a$$

PROOF OF THE THEOREM : Let $\{s_k\}$ be the k-th partial sum of the Fourier series (1.1), it is easy to show that

$$s_{n-k}(t)-f(t) = \frac{1}{\pi}\int_0^\pi \phi(t)\, \frac{\sin\left(n-k+\frac{1}{2}\right)t}{\sin t/2}\, dt,$$

where

$$\phi(t) = \tfrac{1}{2}\left[f(x+t)+f(x-t)-2f(x)\right]$$

and

$$s_{n-k,p}(x)-f(x) = \frac{1}{n-k+1}\sum_{\mu=p}^{n-k+p}\left\{ s_\mu(x)-f(x)\right\}$$

$$= \frac{1}{\pi(n-k+1)}\int_0^\pi \phi(t)\, \frac{(\cos pt - \cos(n-k+1)t)}{2\sin^2 t/2}\, dt.$$

Now, we have

$$T_n^{p,q}(x)-f(x) = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k}\left\{ s_{n-k,p}(x)-f(x)\right\}$$

$$= \frac{1}{\pi R_n}\int_0^\pi \phi(t)\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\,\frac{(\cos pt - \cos(n-k+p+1)t)}{2\sin^2 t/2}\, dt.$$

Therefore

$$\left|T_n^{p,q}(x)-f(x)\right| \leqslant \frac{1}{\pi R_n}\int_0^\pi \left|\phi(t)\right| \left|\sum_{k=0}^{n}\frac{p_k q_{n-k}}{n-k+1}\,\frac{(\cos pt - \cos(n-k+p+1)t)}{2\sin^2 t/2}\right| dt$$

$$= \frac{1}{\pi R_n}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] \left|\phi(t)\right|$$

(*equation continued on p.* 902)

$$\times \left| \sum_{k=0}^{n} \frac{p_k q_{n-k}}{(n-k+1)} \frac{(\cos pt - \cos (n-k+p+1)t)}{2\sin^2 t/2} \right| dt$$

$$= I_1 + I_2, \text{ say.}$$

Now

$$I_1 = O\left[\frac{1}{R_n}\int_0^{\pi/n} |\phi(t)| \left| \sum_{k=0}^{n} \frac{p_k q_{n-k}}{(n-k+1)} \frac{\sin \frac{1}{2}(n-k+2p+1)t \sin \frac{1}{2}(n-k+1)t}{\sin^2 t/2}\right| dt\right]$$

$$= O\left[\frac{1}{R_n}\int_0^{\pi/n} |\phi(t)| \sum_{k=0}^{n} \frac{p_k q_{n-k}}{(n-k+1)} \frac{(n-k+1)t}{t^2}\, dt\right]$$

$$= O\left[\frac{1}{R_n}\int_0^{\pi/n} t^{\alpha-1} \sum_{k=0}^{n} p_k q_{n-k}\, dt\right]$$

$$= O\left[\frac{1}{R_n}\int_0^{\pi/n} t^{\alpha-1}\, dt\right]$$

$$= O\left[\left(\frac{1}{n}\right)^{\alpha}\right].$$

Also

$$I_2 = O\left[\frac{1}{R_n}\int_{\pi/n}^{\pi} |\phi(t)| \left| \sum_{k=0}^{n} \frac{p_k q_{n-k}}{(n-k+1)} \frac{\sin \frac{1}{2}(n-k+2p+1)t \sin \frac{1}{2}(n-k+1)t}{\sin^2 t/2}\right| dt\right]$$

$$= O\left[\frac{1}{R_n}\int_{\pi/n}^{\pi} |\phi(t)| \left| \sum_{k=0}^{n} p_k q_{n-k} \frac{\sin \frac{1}{2}(n-k+2p+1)t}{\sin t/2}\right| dt\right]$$

$$= O\left[\frac{1}{R_n}\int_{\pi/n}^{\pi} t^{\alpha-1} \left| \sum_{k=0}^{n} p_k q_{n-k} \sin (n-k+2p+1)t/2\right| dt\right]$$

$$= O\left[\frac{1}{R_n}\int_{\pi/n}^{\pi} t^{\alpha-1} R\left(\frac{1}{t}\right) dt\right] \text{ (by lemma)}$$

$$= O\left[\frac{1}{R_n}\int_{1}^{n} \frac{R(y)}{y^{\alpha+1}}\, dy\right]$$

$$= O\left[\frac{1}{R_n}\left(\frac{R_n}{n^{\alpha}}\right)\right]$$

(*equation continued on p.* 903)

$$= O\left[\left(\frac{1}{n}\right)^{\alpha}\right].$$

This completes the proof of the theorem.

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