Indian J. pure appl. Math., **13**(8) : 907–911, August 1982

ON AN INVERSION FORMULA

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(*Received* 25 *March* 1981; *after revision* 17 *August* 1981)

This paper incorporates an inverse series relation which, when particularized, yields the corresponding relation for an interesting generalization of several known polynomials, viz. $g_n^c(x,r,s)$ introduced recently by Panda (1977). [see also Srivastava (1980)].

1. INTRODUCTION

Having been motivated by earlier works of Rainville (1960, p.137, Theorem 48), who introduced the class of polynomials $\{f_n(x)\}$ by means of the generating relation

$$(1-t)^{-c}\, G\left[\frac{-4xt}{(1-t)^2}\right] = \sum_{n=0}^{\infty} f_n(x)t^n \qquad \text{...(1.1)}$$

where

$$G(z) = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_n \neq 0, \qquad \text{...(1.2)}$$

Panda (1977) initiated the study of a new class of polynomials which provides a generalization of several known polynomial systems belonging to the families of the classical Jacobi, Hermite and Laguerre polynomials. In terms of the power series (1.2) she defined the class of polynomials $\{g_n^c(x,r,s) \mid n = 0,1,2,\ldots\}$ by the relation

$$(1-t)^{-c}\, G\left[\frac{xt^s}{(1-t)^r}\right] = \sum_{n=0}^{\infty} g_n^c(x,r,s)t^n \qquad \text{...(1.3)}$$

where c, like in (1.1), is an arbitrary parameter, r is an integer, positive or negative, and $s = 1,2,3,\ldots$. The explicit representation of $g_n^c(x,r,s)$ is as given below [Panda 1977, p.105, eqn. (4)] :

$$g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk}}{(n-sk)!}\gamma_k\, x^k. \qquad \text{...(1.4)}$$

See also Srivastava (1980) for a new class of generating functions for the polynomials $g_n^c(x,r,s)$.

It is worth mentioning here that all the results for $f_n(x)$ included in the theorem of Rainville referred to above can be obtained as particular cases of the substantially more general results given by Panda (1977), except the interesting expansion of x^n in terms of $f_n(x)$, for which there is no analogous result in Panda (1977). Although the expansion of x^n in terms of the special case $s = 1$ of $g_n^c(x,r,s)$ is taken care of by a result given by Singhal and Savita Kumari (1982), but for other values of s, such an expansion (which may be viewed as the inversion formula of (1.4)) does not seem to have been given thus far. Our attempt in this direction leads us to a general result which is given in the form of a theorem in section 2 of this paper.

2. The Inverse Relation

The theorem that we prove in this section may be stated in the form :

If

$$a_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} b_k, \qquad \ldots(2.1)$$

then

$$b_n = \sum_{k=0}^{ns} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} a_k, \qquad \ldots(2.2)$$

where p and q are arbitrary parameters and s is a positive integer.

Writing (2.1) and (2.2) in the forms

$$a_n = \sum_{k=0}^{[n/s]} a_{nk}\, b_k, \quad b_n = \sum_{k=0}^{sn} b_{nk}\, a_k,$$

it is easy to observe that the validity of the above theorem is established if the following orthogonal relation holds true.

$$\delta_{nm} = \sum_{k=sm}^{sn} b_{nk}\, a_{km} = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \qquad \ldots(2.3)$$

In order to prove (2.3), we employ the method which runs parallel to that of Riordan (1968, pp. 50–51).

We first note that the expression $\Sigma\, b_{nk}a_{km}$ for δ_{nm} may be written as

$$\delta_{nm} = \sum_{k=sm}^{sn} (-1)^{k-sm} \binom{p+qsm-sm}{k-sm} \times \left\{ \binom{p+qsn-k}{sn-k} - q \binom{p+qsn-k-1}{sn-k-1} \right\}, \qquad \ldots(2.4)$$

which we abbreviate as

$$\delta_{nm} = F_{nm} + q\, G_{nm}, \qquad \ldots(2.5)$$

with

$$F_{nm} = \sum_{k=sm}^{sn} (-1)^{k-sm} \binom{p+qsm-sm}{k-sm} \binom{p+qsn-k}{sn-k} \qquad \ldots(2.6)$$

$$= \sum_{j=0}^{sn-sm} (-1)^{j} \binom{A+(sn-sm)\,q-j}{sn-sm-j} \binom{A}{j}, \quad A = p+qsm-sm$$

Now, in view of the relation

$$\binom{-n}{m} = (-1)^m \binom{n+m-1}{m}$$

and the Vandermonde's convolution

$$\binom{n}{m} = \sum_{k=0} \binom{n-p}{m-k} \binom{p}{k},$$

the last expression for F_{nm} can successively be put as

$$F_{nm} = \sum_{j=0}^{sn-sm} (-1)^{sn-sm} \binom{-1-A-(q-1)\,(sn-sm)}{sn-sm-j} \binom{A}{j} \qquad \ldots(2.7)$$

$$F_{nm} = (-1)^{sn-sm} \binom{-1-(q-1)\,(sn-sm)}{sn-sm} \qquad \ldots(2.8)$$

$$F_{nm} = \binom{qsn-qsm}{sn-sm}. \qquad \ldots(2.9)$$

Likewise, the expression

$$G_{nm} = \sum_{k=sm}^{sn-1} (-1)^{k-sm+1} \binom{p+qsm-sm}{k-sm} \binom{p+qsn-k-1}{sn-k-1} \qquad \ldots(2.10)$$

can be simplified to give

$$G_{nm} = (-1) \binom{qsn-qsm-1}{sn-sm-1}, \qquad \ldots(2.11)$$

so that

$$q\, G_{nm} = (-1) \binom{qsn-qsm}{sn-sm}. \qquad \ldots(2.12)$$

Equations (2.9) and (2.12), when combined with (2.5), yield

$$\delta_{nm} = 0, \text{ for } n \neq m$$

whereas, for $n = m$, (2.4) evidently gives

$$\delta_{nm} = 1.$$

This completes the proof of the theorem.

An alternate form of (2.1) and (2.2) is:

If

$$A_n = \sum_{k=0}^{[n/s]} \binom{p + qsk - sk}{n - sk} B_k, \qquad \ldots(2.13)$$

then

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \frac{p + qk - k}{p + qsn - k} \binom{p + qsn - k}{sn - k} A_k . \qquad \ldots(2.14)$$

This is the result of the substitution $A_n = (-1)^n a_n$, $B_n = (-1)^{sn} b_n$ in (2.1) and (2.2).

Yet another variation of (2.13) and (2.14) may be obtained by multiplying A_n by $p + qn - n$ and B_n by $p + qsn - sn$, we thus get:

If

$$A_n = \sum_{k=0}^{[n/s]} \left\{ \binom{p + qsk - sk}{n - sk} + q \binom{p + qsk - sk}{n - sk - 1} \right\} B_k, \qquad \ldots(2.15)$$

then

$$B_n = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{p + qsn - k}{sn - k} A_k . \qquad \ldots(2.16)$$

3. Expansion of x^n in Terms of $g_n^c(x,r,s)$

The explicit representation (1.4) can be written in the form

$$g_n^c(x,r,s) = \sum_{k=0}^{[n/s]} \binom{-c-rk}{n - sk} (-1)^{n-sk} \gamma_k x^k \qquad \ldots(3.1)$$

which on being compared with (2.1), with $p = -c$, $q = -(r-s)/s$, readily yields the expansion (or the inversion formula of (1.4)) formula

$$x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{(c + rk/s)}{(c + (r-s)n+k)} \binom{-c-(r-s)n-k}{sn - k} g_k^c(x,r,s) \qquad \ldots(3.2)$$

The inversion formula (3.2) can also be expressed in the alternate form:

$$x^n = \frac{1}{\gamma_n} \sum_{k=0}^{sn} \frac{(-c-rk/s)}{(sn - k)!} (1-c-rn)_{sn-k-1}\, g_k^c(x,r,s) \qquad \ldots(3.3)$$

which can be further transformed to a more elegant form:

$$x^n = \frac{1}{\Upsilon_n} \sum_{k=0}^{sn} \frac{(-1)^{sn-k}(c + rk/s)\,(c)_{rn}}{(sn - k)!\,(c)_{(r-s)n+k+1}}\; g_k^c(x,r,s). \quad \ldots(3.4)$$

When $s = 1$, (3.4) would evidently reduce to

$$x^n = \frac{(c)_{rn}}{\Upsilon_n} \sum_{k=0}^{n} \frac{(-1)^{n-k}(c + rk)}{(n-k)!\,(c)_{(r-1)n+k+1}}\; g_k^c\,(x,r,1), \quad \ldots(3.5)$$

which corresponds to the result obtained by the authors [Singhal and Savita Kumari 1982, eqn. (17)].

The relation (3.5) can be further particularized by taking $r = 2$ and replacing x by $-4x$; (3.5) would thus simplify to

$$x^n = \frac{(c)_{2n}}{2^{2n}\,\Upsilon_n} \sum_{k=0}^{n} \frac{(-1)^k\,(c + 2k)}{(n-k)!\,(c)_{n+k+1}}\; f_k(x) \quad \ldots(3.6)$$

which was obtained by Rainville [1960, p.137, eqn. (4)] in a different manner.

ACKNOWLEDGEMENT

Thanks are due to Prof. H.M. Srivastava for his kind and valuable suggestions.

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