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2-FUNCTIONALS AND SOME EXTENSION THEOREMS IN LINEAR SPACES

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The concepts of convex 2-functionals and sublinear 2-functionals are introduced and a theorem on the extension of linear 2-functionals dominated by certain convex 2-functionals in linear spaces has been established. Some consequences of this extension theorem for 2-normed spaces and a result on the minimal extension of a sublinear 2-functional in real linear spaces have also been obtained.

1. INTRODUCTION

Let E be a real linear space and M,L two linear subspaces of E. It has been shown by Gähler (1969) that a linear 2-functional with domain $M\times L$ does not have a linear extension on $E\times E$. However, White (1969) has proved that in a 2-normed space E a bounded linear 2-functional with domain $M\times[x]$ (or respectively $[x]\times M$) where $[x]$ is the linear subspace generated by an $x\in E$ has an extension to $E\times[x]$ (or respectively $[x]\times E$). A slight generalisation of this result is contained in Ehret (1969). Recently Milman (1963, 1969, 1978) has obtained some interesting results on the extension of functionals. The main result in Section 2 of this paper is on the extension of a linear 2-functional dominated by certain convex 2-functionals. In Section 3 we introduce the concept of a sub-linear 2-functional and obtain a result on its minimal extension.

2. EXTENSION OF LINEAR 2-FUNCTIONALS

A real valued mapping f on $A\times C$ where A,C are subsets of a set X is called a 2-functional with domain $A\times C$. A 2-functional with domain $A\times A$ is briefly called a 2-functional on A. A 2-functional f with domain $M\times L$ where M and L are linear subspaces of a real linear space E is said to be a linear 2-functional if for all $x,x'\in M$, $y,y'\in L$,

$$f(x+x',\, y+y') = f(x,y)+f(x,y') + f(x',y) + f(x',y') \qquad \ldots(2.1)$$

and

$$f(\alpha x,\beta y) = \alpha\beta\, f(x,y) \qquad \ldots(2.2)$$

for all α,β in R.

A 2-functional p with domain $M\times L$ where M and L are linear subspaces of a real linear space E is convex 2-functional if

$$p(\alpha\lambda x+(\alpha-\alpha\lambda)x',\, \beta\mu y+(\beta-\beta\mu)y') \leqslant \alpha\beta\,|\,\lambda\mu\,|\,p(x,y)+\alpha\,|\,\lambda\,|\,(\beta-\beta\mu)p(x,y') + (\alpha-\alpha\lambda)\beta\,|\,\mu\,|\,p(x',y)+(\alpha-\alpha\lambda)(\beta-\beta\mu)p(x',y') \qquad \ldots(2.3)$$

for all $|\lambda| \leqslant 1$, $|\mu| \leqslant 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$ and $x, x' \in M, y, y' \in L$.

Theorem 2.1—Let E be a real linear space and M, L two linear subspaces of E where L is of finite dimension with a set of basis vectors $\{e_i\}_{i=1}^n$. If p is a non negative convex 2-functional on E and f a linear 2-functional on $M \times L$ such that for all $(x,y) \in M \times L$, $\alpha_i \in R$,

$$p(x, \sum_{i=1}^{n} \alpha_i e_i) = \sum_{i=1}^{n} p(x, \alpha_i e_i) \qquad \ldots(2.4)$$

and

$$f(x,y) \leqslant p(x,y), \qquad \ldots(2.5)$$

then there exists a linear 2-functional F on $E \times L$ such that $F(x,y) = f(x,y)$ for all $(x,y) \in M \times L$ and $F(x,y) \leqslant p(x,y)$ for all $(x,y) \in E \times L$.

PROOF : Let P denote the collection of all ordered pairs (M_λ, F_λ) where M_λ is a linear subspace in E and $M \subseteq M_\lambda$ and F_λ is a linear 2-functional which is an extension of f to the domain $M_\lambda \times L$ such that $F_\lambda(x,y) \leqslant p(x,y)$ for all $(x,y) \in M_\lambda \times L$. Partially order P by setting $(M_\lambda, F_\lambda) \leqslant (M_\mu, F_\mu)$ iff $M_\lambda \subseteq M_\mu$ and F_μ is an extension of F_λ. P is non-empty as $(M,f) \in P$. Let $Q = \{(M_i, F_i\})_{i \in \Delta}$ be a totally ordered set in P and define a linear 2-functional on $(\bigcup_{i \in \Delta} M_i) \times L$ by $\phi(x,y) = F_i(x,y)$ for all $(x,y) \in M_i \times L$. Then $(\bigcup_{i \in \Delta} M_i, \phi) \in P$ and Q has an upper bound $(\bigcup_{i \in \Delta} M_i, \phi)$. An application of Zorn's lemma gives a maximal member, say, (N,F) in P. We claim that the domain of F is all of $E \times L$.

If possible, let $x_0 \in E - N$ and define $H = \{x + \beta x_0 : x \in N, \beta$ arbitrary real number$\}$. Then H is a linear subspace of E and $N \subsetneq H$. With arbitrarily chosen fixed real numbers r_i define G on $H \times L$ by

$$G(x+\beta x_0, y) = F(x, \sum_{i=1}^{n} \alpha_i e_i) + \beta \sum_{i=1}^{n} \alpha_i r_i$$

where $y = \sum_{i=1}^{n} \alpha_i e_i$. Evidently G is a well-defined linear 2-functional and extends F if we can choose r_i in such a way that $G(x,y) \leqslant p(x,y)$ for all $(x,y) \in H \times L$.

For $\alpha > 0$, $\beta < 0$ let $\delta = (\alpha^{-1} - \beta^{-1})^{-1}$ and then for all $x, x' \in N$ we have due to (2.1), (2.2) and (2.3),

$$\begin{aligned} \delta\,[\alpha^{-1}F(x,e_i) - \beta^{-1}F(x',e_i)] &= F(\delta(\alpha^{-1}x - \beta^{-1}x'), e_i) \\ &\leqslant p(\alpha^{-1}\delta x - \beta^{-1}\delta x', e_i) \\ &= p(\alpha^{-1}\delta(x+\alpha x_0) - \beta^{-1}\delta(x'+\beta x_0), e_i) \\ &\leqslant \alpha^{-1}\delta p(x+\alpha x_0, e_i) - \beta^{-1}\delta p(x'+\beta x_0, e_i) \end{aligned}$$

and hence

$$\beta^{-1}p(x'+\beta x_0, e_i) - \beta^{-1}F(x',e_i) \leqslant \alpha^{-1}p(x+\alpha x_0, e_i) - \alpha^{-1}F(x,e_i).$$

As x and x' vary over N, the above inequality implies that there exists r_i such that $S_i \leqslant r_i \leqslant I_i$ where

$$S_i = \sup\{\beta^{-1}p(x'+\beta x_0, e_i) - \beta^{-1}F(x', e_i) : \beta<0, x'\in N\}$$

and

$$I_i = \inf\{\alpha^{-1}p(x+\alpha x_0, e_i) - \alpha^{-1}F(x, e_i) : \alpha>0,\ x\in N\}.$$

Consequently for every i we have

$$\beta^{-1}p(x+\beta x_0, e_i) - \beta^{-1}F(x, e_i) \leqslant r_i \leqslant \alpha^{-1} p(x+\alpha x_0, e_i) - \alpha^{-1}F(x, e_i) \qquad \ldots(2.6)$$

for all $x\in N$ and $\alpha>0$, $\beta<0$.

Note that $\lambda = \mu = 1$ in (2.3) implies

$$p(\alpha x, \beta y) \leqslant \alpha\beta\, p(x,y) \qquad \ldots(2.7)$$

for $\alpha\geqslant 0$, $\beta\geqslant 0$ and $x,y\in E$. Since p is non-negative on E, $p(0,y) = p(x,0) = 0$ for all $x,y\in E$ and therefore $\lambda = \mu = -1$, $x' = y' = 0$ in (2.3) yield

$$p(-\alpha x, -\beta y) \leqslant \alpha\beta\, p(x,y) \qquad \ldots(2.8)$$

for all $\alpha \geqslant 0$, $\beta \geqslant 0$. For $\beta\alpha_i>0$ putting $\alpha = 1$ and $x = x/\beta$ in (2.6), it follows that

$$\begin{aligned} G(x+\beta x_0, \alpha_i e_i) &= \alpha_i F(x,e_i) + \alpha_i\beta r_i \\ &= \alpha_i\beta\,[F(x/\beta,e_i) + r_i] \\ &\leqslant \alpha_i\beta p\,(x/\beta + x_0, e_i) \\ &= \alpha_i\beta p\left(\frac{x+\beta x_0}{\beta}, \frac{\alpha_i e_i}{\alpha_i}\right) \\ &\leqslant p\,(x+\beta x_0, \alpha_i e_i) \end{aligned}$$

on account of (2.7) and (2.8). For $\beta\alpha_i<0$ putting $\beta = -1$ and $x = -x/\beta$ in (2.6) the first two terms imply that

$$\begin{aligned} G\,(x + \beta x_0, \alpha_i e_i) &= \alpha_i F(x, e_i) + \alpha_i\beta r_i \\ &= \alpha_i\beta\,[F(x/\beta, e_i) + r_i] \\ &\leqslant (-\alpha_i\beta)\, p\left(\frac{x+\beta x_0}{-\beta}, \frac{\alpha_i e_i}{\alpha_i}\right) \\ &\leqslant p\,(x+\beta x_0, \alpha_i e_i) \end{aligned}$$

on account of (2.7) and (2.8). For $\alpha_i\beta = 0$ we have

$$G\,(x+\beta x_0, \alpha_i e_i) = \begin{cases} F\,(x, \alpha_i e_i) & \text{if } \beta = 0 \\ F\,(x, 0) & \text{if } \alpha_i = 0 \end{cases}$$

and therefore

$$G\,(x+\beta x_0, \alpha_i e_i) \leqslant p\,(x+\beta x_0, \alpha_i e_i)$$

as $(N,F) \in P$. Consequently it follows from (2.4) that for all $x + \beta x_0\in H$ and $\sum_{i=1}^{n} \alpha_i e_i$ in L,

$$G\,(x+\beta x_0, \sum_{i=1}^{n} \alpha_i e_i) = F\,(x, \sum_{i=1}^{n} \alpha_i e_i) + \beta \sum_{i=1}^{n} \alpha_i r_i$$

(equation continued on p. 915)

$$= \sum_{i=1}^{n} F(x, \alpha_i e_i) + \beta\, \alpha_i r_i$$

$$= \sum_{i=1}^{n} G(x+\beta x_0, \alpha_i e_i)$$

$$\leqslant \sum_{i=1}^{n} p(x+\beta x_0, \alpha_i e_i)$$

$$= p(x+\beta x_0, \sum_{i=1}^{n} \alpha_i e_i).$$

Thus for every i if r_i is taken from the interval $[S_i, I_i]$, then $G(x,y) \leqslant p(x,y)$ for all $(x,y) \in H \times L$ and therefore $(H,G) \in P$ such that $(N,F) \underset{\neq}{<} (H,G)$. This contradicts the maximality of F. Hence $N = E$ and the theorem follows.

Proceeding similarly as above we can prove the following:

Theorem 2.2 – Let E be a real linear space and M, L two linear subspaces of E where L is of finite dimension with basis vectors $\{e_i\}_{i=1}^{n}$. If p is a non-negative convex 2-functional on E and f a linear 2-functional on $L \times M$ such that for all $x \in L$, $y \in M$, $\alpha_i \in R$,

$$p\left(\sum_{i=1}^{n} \alpha_i e_i, y\right) = \sum_{i=1}^{n} p(\alpha_i e_i, y) \qquad \text{...(2.9)}$$

and $f(x,y) \leqslant p(x,y)$, then there exists a linear 2-functional F which extends f from $L \times M$ to whole of $L \times E$ such that $F(x,y) \leqslant p(x,y)$ on $L \times E$.

It is noteworthy that in case L is one-dimensional the conditions (2.4) and (2.9) are automatically satisfied. If $(E, \|.,.\|)$ is a 2-normed space and f a bounded linear 2-functional with its norm $\| f \| = \text{g.l.b.}\ \{K : |f(x,y)| \leqslant K \|x,y\|, (x,y) \in D(f)\}$ then by taking $p(x,y) = \| f \|\, \| x,y \|$ in Theorem 2.1 and using Theorem 2.2 of White (1969) we obtain the following result due to White (1969) :

Theorem 2.3—Let E be a real 2-normed space and M, $[y]$ linear manifolds of E where $[y]$ is the subspace generated by $y \in E$. If f is a bounded linear 2-functional with domain $M \times [y]$, then there exists a bounded linear 2-functional F on $E \times [y]$ such that $F(x, \alpha y) = f(x, \alpha y)$ for all $(x, \alpha y) \in M \times [y]$ and $\| F \| = \| f \|$.

With the help of the above theorem we obtain some more results.

Theorem 2.4—Let M be a linear manifold of a real 2-normed space E. If for any two vectors $x_0, y \in E - M$, $r = \inf\{\| x - x_0, y \| : x \in M\} > 0$ and $y \notin \{x + \alpha x_0 : x \in M, \alpha$ arbitrary real number$\}$, then there exists a linear 2-functional F on $E \times [y]$ such that $F(x,y) = 0$ for all $x \in M$, $F(x_0, y) = 1$ and $\| F \| = 1/r$.

Proof : Write $H = \{x + \alpha x_0 : x \in M, \alpha$ arbitrary number$\}$ and define $f : H \times [y] \rightarrow R$ by $f(x + \alpha x_0, \beta y) = \alpha\beta$. Then f is a linear 2-functional on $H \times [y]$ such that $f(x_0, y) = 1$ and $f(x, y) = 0$ for all $x \in M$. In view of Theorem 2.1 of White (1969)

$$\| f \| = \sup \left\{ \frac{|f(x+\alpha x_0, \beta y)|}{\| x+\alpha x_0, \beta y \|} : x + \alpha x_0 \in H, \| x + \alpha x_0, \beta y \| \neq 0 \right\}$$

(*equation continued on p.* 916)

$$= \sup\left\{\frac{|f(x+\alpha x_0, y)|}{\| x+\alpha x_0, y \|} : x \in M, \alpha \neq 0\right\}$$

$$= \sup\left\{\frac{1}{\| x_0 - w, y \|} : w = -\frac{x}{\alpha} \in M\right\}$$

$$= 1/r.$$

By Theorem 2.3 there exists a linear 2-functional F on $E \times [y]$ such that $\| F \| = \| f \| = 1/r$, $F(x,y) = 0$ for all $x \in M$ and $F(x_0,y) = 1$.

Theorem 2.5—Let M be a linear manifold of a real 2-normed space E. For any two vectors x_0, y in E-M let $r = \inf\{ \| x_0 - x, y \| : x \in M\} > 0$ and $y \notin \{x + \alpha x_0 : x \in M, \alpha \text{ arbitrary}\}$. Then there exists a linear 2-functional F on $E \times [y]$ such that $F(x,y) = 0$ for all $x \in M$, $F(x_0,y) = r$ and $\| F \| = 1$.

PROOF : Define $f : H \times [y] \to R$ by $f(x + \alpha x_0, \beta y) = \alpha\beta\, r$ and the rest of the proof is now similar to that of Theorem 2.4 with suitable modifications.

It is interesting to note that the case $M = \{0\}$ of Theorem 2.5 is contained in Theorem 2.8 of White (1969). Moreover, the above theorems report that every real 2-normed space possesses bounded (and hence 2-continuous) linear 2-functionals which do not vanish identically and they also give a characterization for linear dependence of two non-zero vectors as shown in the following:

Theorem 2.6—Let E be a real 2-normed space and let $f(x_0,y) = 0$ for all bounded linear 2-functionals f on $E \times [y]$. Then x_0 is linearly dependent on y.

PROOF : Assume that x_0 and y are linearly independent. Then by Theorem 2.5 with $M = \{0\}$ there exists a bounded linear 2-functional F on $E \times [y]$ such that $F(x_0,y) = \| x_0, y \| > 0$ which contradicts the hypothesis. Hence x_0 is linearly dependent on y.

Another interesting application of Theorem 2.5 is found in the following:

Theorem 2.7—For an arbitrary non-zero vector y in a real 2-normed space $(E, \| \cdot, \cdot \|)$ let I be an indexed set such that y is linearly independent on $\{x_i : i \in I\}$ and for every $\epsilon > 0$, $x \in E$ there exists an $i \in I$ such that $\| x_i - x, y \| < \epsilon$. Then there exist bounded linear 2-functionals F_i, $i \in I$ on $E \times [y]$ such that $\| x,y \| = \sup_{i \in I} | F_i(x,y) |$.

PROOF : By Theorem 2.5 with $M = \{0\}$ it follows that for every $i \in I$ there is a linear 2-functional F_i on $E \times [y]$ such that $\| F_i \| = 1$ and $F_i(x_i, y) = \| x_i, y \|$. For every $x \in E$, $\| x,y \| \neq 0$ and $i \in I$ we have

$$1 = \| F_i \| \geqslant \frac{| F_i(x,y) |}{\| x,y \|}$$

so that $\| x,y \| \geqslant | F_i(x,y) |$ and therefore $\| x,y \| \geqslant \sup_{i \in I} | F_i(x,y) |$.

Also,

$$\Big| | F_i(x_i,y) | - \| x,y \| \Big| = \Big| \| x_i,y \| - \| x,y \| \Big| \leqslant \| x_i - x, y \|$$

and

$$\Big| | F_i(x,y) | - | F_i(x_i,y) | \Big| \leqslant \Big| F_i(x,y) - F_i(x_i,y) \Big| \leqslant \| x_i - x, y \|$$

so that $| \, | F_i(x,y) | - \| x,y \| \, | \leqslant 2 \| x_i - x,y \|$. As for every $\epsilon > 0$ and $x \in E$ there is an $i \in I$ such that $\| x_i - x,y \| < \epsilon$, it follows that $\| x,y \| \leqslant \sup_{i \in I} | F_i(x,y) |$. If $\| x,y \| = 0$ then x and y are linearly dependent and hence $F(x,y) = 0$ for every bounded 2-functional F on E. Consequently for every $x \in E$ we have

$$\| x,y \| = \sup_{i \in I} | F_i(x,y) |$$

and the theorem is thus established.

Finally we note that results analoguous to Theorem 2.3 through 2.7 can be obtained from Theorem 2.2.

3. Minimal Extension of Sublinear 2-Functionals

By a cone C in a real linear space E we mean a subset C of E such that $C + C \subseteq C$ and $\lambda C \subseteq C$ for $\lambda \geqslant 0$. A 2-functional f with domain $A \times C$ where A and C are cones in E is called a sublinear 2-functional if for $x, x' \in A$, $y, y' \in C$, $\alpha \geqslant 0$, $\beta \geqslant 0$.

$$f(x+x', y+y') \leqslant f(x,y) + f(x,y') + f(x',y) + f(x',y') \qquad \text{...(3.1)}$$

and

$$f(\alpha x, \beta y) = \alpha\beta \, f(x, y). \qquad \text{...(3.2)}$$

Let M_1 and M_2 be two cones in E and f_1, f_2 sublinear 2-functionals on $M_1 \times [y]$ and $M_2 \times [y]$ respectively. Let f_0 be a sublinear 2-functional on $M_0 \times [y]$ where M_0 is a linear subspace of E. With these notations we have

Theorem 3.1—Let

$$M_0 - M_1 + M_2 = E, \qquad \text{...(3.3)}$$

$$x_2 \in M_2, \; x_1 \in M_1 \Rightarrow x_2 - x_1 \in M_0, \qquad \text{...(3.4)}$$

and

$$f_0(x_2 - x_1, \alpha y) \leqslant f_1(x_1, \alpha y) + f_2(x_2, \alpha y). \qquad \text{...(3.5)}$$

Then there exists a minimal sublinear 2-functional f which extends f_0 from $M_0 \times [y]$ to $E \times [y]$ such that

$$f(x_2 - x_1, \alpha y) \leqslant f_1(x_1, \alpha y) + f_2(x_2, \alpha y). \qquad \text{...(3.6)}$$

Proof : Let S_{f_0} be the collection of all sublinear extensions f of f_0 from $M_0 \times [y]$ to $E \times [y]$ satisfying (3.6). Note that $f \in S_{f_0}$ is a minimal extension of f_0 if for $g \in S_{f_0}$, $g(x, \alpha y) \leqslant f(x, \alpha y)$ for all $(x, \alpha y) \in E \times [y]$ implies that $g = f$. Let T_{f_0} be the set of minimal extensions of f_0. Define

$$f_0^*(x, \alpha y) = \inf_{x = x_0 - x_1 + x_2} [f_0(x_0, \alpha y) + f_1(x_1, \alpha y) + f_2(x_2, \alpha y)] \qquad \text{...(3.7)}$$

for all $(x, \alpha y) \in E \times [y]$. As f_i $(i = 0,1,2)$ satisfies (3.1) f_0^* satisfies (3.1). Also, for $\lambda \geqslant 0$, $\mu \geqslant 0$ and $(x, \alpha y) \in E \times [y]$ we have

$$f_0^*(\lambda x, \mu\alpha y) = \inf_{\lambda x = \lambda x_0 - \lambda x_1 + \lambda x_2} [f_0(\lambda x_0, \mu\alpha y) + f_1(\lambda x_1, \mu\alpha y) + f_2(\lambda x_2, \mu\alpha y)]$$

(*equation continued on p.* 918)

$$= \inf_{x=x_1-x_1+x_2} [\lambda\mu \{ f_0 (x_0,\alpha y) + f_1 (x_1,\alpha y) + f_2 (x_2,\alpha y)\}]$$
$$= \lambda\mu \inf_{x=x_0-x_1+x_2} [f_0 (x_0,\alpha y) + f_1 (x_1,\alpha y) + f_2 (x_2,\alpha y)]$$
$$= \lambda\mu f_0^* (x,\alpha y)$$

and thus f_0^* is a sublinear 2-functional on $E\times[y]$. We shall prove that $f_0^* \in S_{f_0}$. By (3.3), $-x = y_0 - y_1 + y_2$ where $y_i \in M_i$ $(i=0,1,2)$ and $x=x_0-x_1+x_2$ imply that $-(x_0+y_0) = (x_2+y_2) - (x_1+y_1)$ and therefore in view of (3.1), (3.2) and (3.5) we have

$$- f_0 (x_0,\alpha y) - f_0 (y_0,\alpha y) \leqslant - f_0 (x_0+y_0,\alpha y) \leqslant f_0 (-(x_0+y_0), \alpha y)$$
$$\leqslant f_1 (x_1,\alpha y) + f_1 (y_1,\alpha y) + f_2 (x_2,\alpha y) + f_2 (y_2,\alpha y)$$

i.e., $- [f_0 (y_0,\alpha y) + f_1 (y_1,\alpha y) + f_2 (y_2,\alpha y)] \leqslant f_0 (x_0,\alpha y) + f_1 (x_1,\alpha y) + f_2 (x_2,\alpha y)$ for all $\alpha y \in [y]$. Keep y_0,y_1,y_2 fixed and let x_0,x_1,x_2 vary. Then from (3.7) it follows that

$$- \infty < - [f_0 (y_0,\alpha y) + f_1 (y_1,\alpha y) + f_2 (y_2,\alpha y)] \leqslant f_0^* (x,\alpha y).$$

This proves that f_0^* is finite. By the property of infimum we also have

$$f_0^* (x_0,\alpha y) \leqslant f_0 (x_0 \alpha y),\ f_0^* (- x_1,\alpha y) \leqslant f_1 (x_1,\alpha y),\ f_0^* (x_2,\alpha y) \leqslant f_2 (x_2,\alpha y) \quad \dots(3.8)$$

for all $(x_i,\alpha y) \in M_i \times [y]$. In order to show that f_0^* is an extension of f_0 we need only show that $f_0 (x,\alpha y) \leqslant f_0^* (x,\alpha y)$ for all $(x,\alpha y) \in M_0 \times [y]$. Since for $(x_0,\alpha y) \in M_0 \times [y]$

$$f_0 (x_0,\alpha y) \leqslant f_0 (x-x,\alpha y) + f_0 (x_0,\alpha y),$$

taking decomposition $x=x_0-x_1+x_2$ we have $x - x_0 = x_2 - x_1$ and therefore (3.5) implies

$$f_0 (x-x_0,\alpha y) \leqslant f_1 (x_1,\alpha y) + f_2 (x_2,\alpha y)$$

so that

$$f_0 (x,\alpha y) \leqslant f_0 (x_0,\alpha y) + f_1 (x_1,\alpha y) + f_2 (x_2,\alpha y).$$

Hence $f_0 (x,\alpha y) \leqslant f_0^* (x,\alpha y)$ in view of (3.7) and therefore $f_0 (x,\alpha y) = f_0^* (x,\alpha y)$ for all $(x,\alpha y) \in M_0 \times [y]$ by virtue of (3.8). Also, for $(x_i,\alpha y) \in M_i \times [y]$ it follows from (3.8) that

$$f_0^* (x_2-x_1, \alpha y) \leqslant f_0^* (-x_1, \alpha y) + f_0^* (x_2, \alpha y) \leqslant f_1 (x_1, \alpha y) + f_2 (x_2, \alpha y)$$

and hence f_0^* satisfies (3.6) and thus $f_0^* \in S_{f_0}$.

We shall now prove that $S_{f_0} = T_{f_0^*}$. Let $f \in T_{f_0^*}$. In view of the definition of $T_{f_0^*}$ it follows that $f \in S_{f_0}$ and $T_{f_0^*} \subseteq S_{f_0}$. Conversely if $f \in S_{f_0}$ then for $x_i \in M_i$ and $x=x_0-x_1+x_2$ we have

$$f (x,\alpha y) \leqslant f (x_0, \alpha y) + f (x_2-x_1, \alpha y)$$
$$= f_0 (x_0,\alpha y) + f (x_2-x_1, \alpha y)$$
$$\leqslant f_0 (x_0, \alpha y) + f_1 (x_1, \alpha y) + f_2 (x_2, \alpha y)$$

so that $f(x, \alpha y) \leqslant f_0^*(x, \alpha y)$ for all $(x, \alpha y) \in E \times [y]$. Consequently $f \in T_{f_0^*}$ and hence $S_{f_0} = T_{f_0^*}$ and f_0^* is the greatest sublinear 2-functional in S_{f_0}.

To ensure the existence of a minimal sublinear extension f_0 it remains to show that T_{f_0} is non-empty. Let $\{g_r\}_{r \in \Delta}$ be a totally ordered set in S_{f_0} ordered in the decreasing sense. For given $(x, \alpha y) \in E \times [y]$, fixed $r_1 \in \Delta$ and any $r \geqslant r_1$ we have $g_r \leqslant g_{r_1}$ and

$$-g_{r_1}(-x, \alpha y) \leqslant -g_r(-x, \alpha y) \leqslant g_r(x, \alpha y)$$

which implies

$$-g_{r_1}(-x, \alpha y) \leqslant \inf_{r \in \Delta} g_r(x, \alpha y).$$

Thus the 2-functional $g_\Delta(x, \alpha y) = \inf_{r \in \Delta} g_r(x, \alpha y)$ for $(x, \alpha y) \in E \times [y]$ is finite. If $x, x' \in E$ we have

$$g_\Delta(x+x', \alpha y+\alpha' y) \leqslant g_r(x+x', \alpha y+\alpha' y)$$
$$\leqslant g_r(x, \alpha y + g_r(x', \alpha y) + g_r(x, \alpha' y) + g_r(x', \alpha' y)$$

for all $\alpha y, \alpha' y \in [y]$. Taking infimum on the right we see that g_Δ satisfies (3.1) on $E \times [y]$. Evidently g_Δ satisfies (3.2) and therefore g_Δ is an extension of f_0 from $M_0 \times [y]$ to $E \times [y]$ such that

$$g_\Delta(x, \alpha y) \leqslant g_r(x, \alpha y) \leqslant f_0^*(x, \alpha y).$$

Then $g_\Delta \in S_{f_0}$ and $\{g_r\}_{r \in \Delta}$ has a minimum in S_{f_0}. By Zorn's lemma there exists a minimal sublinear 2-functional f in S_{f_0}, i.e. $f \in T_{f_0}$. It follows that T_{f_0} is non-empty and the theorem is established.

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