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ON NEW CRITERIA FOR *P*-VALENT FUNCTIONS

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Goel and Sohi (1980) defined the special classes $K_p\ (n-1)$ of p-valent functions and showed some results for these classes. It is the purpose of this paper to define the generalized classes of $K_p\ (n-1)$ by using the fractional calculus and show some results for these classes and their subclasses.

1. INTRODUCTION

Let A_p denote the class of functions

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

which are analytic in the unit disk $U = \{ | z | < 1\}$, where p is a positive integer. Goel and Sohi (1980 a) defined the classes $K_p\ (n-1)$ (n is any integer greater than $-p$) of functions $f(z) \in A_p$ satisfying the following condition

$$\operatorname{Re}\left[\frac{\{z^n f(z)\}^{(n+p)}}{\{z^{n-1} f(z)\}^{(n+p-1)}}\right] > \frac{n+p}{2} \quad (z \in U)$$

and showed $K_p\ (n) \subset K_p\ (n-1)$ for each n and p.

Let $f * g\ (z)$ denote the Hadamard product of two functions

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \dots$$

and

$$g(z) = z^p + b_{p+1} z^{p+1} + b_{p+2} z^{p+2} + \dots$$

in the class A_p, that is,

$$f * g\ (z) = z^p + a_{p+1} b_{p+1} z^{p+1} + a_{p+2} b_{p+2} z^{p+2} + \dots$$

And let

$$D^{n+p-1} f(z) = \left\{\frac{z^p}{(1-z)^{n+p}}\right\} * f(z), \qquad \dots(1)$$

then we have

$$D^{n+p-1} f(z) = \frac{z^p \{z^{n-1} f(z)\}^{(n+p-1)}}{(n+p-1)!}.$$

With this notation (1), we have that the necessary and sufficient condition for a function $f(z) \in A_p$ to be in the class $K_p\ (n-1)$ is

$$\operatorname{Re}\left\{\frac{D^{n-p} f(z)}{D^{n+p-1} f(z)}\right\} > \frac{1}{2} \quad (z \in U).$$

Let $R_p(n-1)$ denote the class of functions $f(z) \in A_p$ satisfying the condition

$$\text{Re}\left\{\frac{D^{n+p} f(z)}{D^{n+p-1} f(z)}\right\} > \frac{n+p}{n+p+1} \qquad (z \in U).$$

Then $R_p(n) \subset R_p(n-1)$ holds for every $p \in N$ and $n > -p$ and $R_p(n-1) \subset K_p(n-1)$ holds for each $p \in N >$ and $n > -p$.

Remark 1 : In particular, for $p = 1$, Ruscheweyh (1975) defined the classes $K_1(n)$ and Singh and Singh (1979) defined the classes $R_1(n)$. Furthermore, the symbol $D^n f(z)$ was named the nth order Ruscheweyh derivative of $f(z)$ by Al-Amiri (1980).

Remark 2 : Recently, Goel and Sohi (1980 b) defined the classes $T_{n+p-1}(\alpha)$ of functions $f(z) \in A_p$ satisfying

$$\text{Re}\left[\frac{\{D^{n+p-1} f(z)\}}{p\, z^{p-1}}\right] > \alpha \qquad (z \in U)$$

for some $0 \leqq \alpha < 1$ and showed some results for these classes.

2. The Generalized Classes

There are many definitions of the fractional calculus. Owa (1978) gave the following definitions for the fractional calculus.

Definition 1— The fractional integral of order α is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{1-\alpha}},$$

where $\alpha > 0$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^{-\alpha} f(z).$$

Definition 2 — The fractional derivative of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)\, d\zeta}{(z-\zeta)^{\alpha}},$$

where $0 < \alpha < 1$, $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$. Moreover,

$$f(z) = \lim_{\alpha \to 0} D_z^{\alpha} f(z).$$

Definition 3 —The fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^{\alpha} f(z),$$

where $0 < \alpha < 1$ and $n \in N \cup \{0\}$.

Remark 3 : For other definitions of the fractional calculus, see Agarwal (1969, 1976), Al-Salam (1966), Osler (1970), Ross (1975), Nishimoto (1976) and Saigo (1978).

Let $K_p(\alpha-1)$ and $K_p(-\alpha-1)$ denote the classes of functions $f(z) \in A_p$ satisfying the following conditions

$$\mathrm{Re}\left[\frac{D_z^{p+\alpha}\{z^\alpha f(z)\}}{D_z^{p+\alpha-1}\{z^{\alpha-1}f(z)\}}\right] > \frac{p+\alpha}{2} \qquad (z \in U)$$

for $0 < \alpha < 1$ and $p \in N$ and

$$\mathrm{Re}\left[\frac{D_z^{p-\alpha}\{z^{-\alpha} f(z)\}}{D_z^{p-\alpha-1}\{z^{-\alpha-1}f(z)\}}\right] > \frac{p-\alpha}{2} \qquad (z \in U)$$

for $0 < \alpha < p$ and $p \in N$, respectively. And let

$$D^{p+\alpha-1} f(z) = \left\{\frac{z^p}{(1-z)^{p+\alpha}}\right\} * f(z) \qquad (z \in U) \tag{2}$$

for $p \in N$ and α is any real number greater than $-p$. From this, we see that the necessary and sufficient condition for a function $f(z) \in A_p$ to be in the class $K_p(\alpha-1)$ $(0 < \alpha < 1, p \in N)$ is

$$\mathrm{Re}\left\{\frac{D^{p+\alpha} f(z)}{D^{p+\alpha-1} f(z)}\right\} > \frac{1}{2} \qquad (z \in U)$$

and the necessary and sufficient condition for a function $f(z) \in A_p$ to be in the class $K_p(-\alpha-1)$ $(0 < \alpha < p, p \in N)$ is

$$\mathrm{Re}\left\{\frac{D^{p-\alpha} f(z)}{D^{p-\alpha-1} f(z)}\right\} > \frac{1}{2} \qquad (z \in U).$$

Furthermore, let $R_p(\alpha-1)$ and $R_p(-\alpha-1)$ denote the classes of functions $f(z) \in A_p$ satisfying the following conditions

$$\mathrm{Re}\left\{\frac{D^{p+\alpha} f(z)}{D^{p+\alpha-1} f(z)}\right\} > \frac{p+\alpha}{1+p+\alpha} \qquad (z \in U)$$

for $0 < \alpha < 1$ and $p \in N$ and

$$\mathrm{Re}\left\{\frac{D^{p-\alpha} f(z)}{D^{p-\alpha-1} f(z)}\right\} > \frac{p-\alpha}{1+p-\alpha} \qquad (z \in U)$$

for $0 < \alpha < p$ and $p \in N$, respectively

Remark 4 : Ruscheweyh (1977) defined the classes $\mathscr{R}_\alpha \in A_1$ of prestar-like functions $f(z)$ of order α, that is, a function $f(z) \in A_1$ is called prestarlike of order α $(\alpha \leq 1)$ if, and only if,

$$\mathrm{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \frac{1}{2} \qquad (z \in U)$$

for $\alpha = 1$ and

$$\left\{\frac{z}{(1-z)^{2(1-\alpha)}}\right\} * f(z) = \in \mathcal{S}_\alpha \qquad (z \in U)$$

for $\alpha < 1$, where $\mathcal{S}_\alpha$ denotes the class of starlike functions $f(z) \in A_1$ of order α. Moreover, Ruscheweyh (1977) showed some results for these classes $\mathcal{R}_\alpha$ of prestar-like functions $f(z)$ of order α.

Theorem 1— Let a function

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be in the class A_p. Then we have

$$D^{p+\alpha-1} f(z) = \frac{z^p}{\Gamma(p+\alpha)} D_z^{p+\alpha-1} \{z^{\alpha-1} f(z)\}$$

and

$$D^{p-1} f(z) = \lim_{\alpha \to 0} D^{p+\alpha-1} f(z)$$

for $0 < \alpha < 1$ and $z \in U$.

PROOF : By using (2) for $0 < \alpha < 1$ and $p \in N$, we have

$$D^{p+\alpha-1} f(z) = \left\{\frac{z^p}{(1-z)^{p+\alpha}}\right\} * \left(z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}\right)$$

$$= z^p + \sum_{n=1}^{\infty} \frac{(p+\alpha)(p+\alpha+1)\ldots(p+\alpha+n-1)}{n!} a_{p+n} z^{p+n}.$$

On the other hand, we have

$$\frac{z^p}{\Gamma(p+\alpha)} D_z^{p+\alpha-1}\{z^{\alpha-1} f(z)\}$$

$$= \frac{z^p}{\Gamma(p+\alpha)} D_z^{p+\alpha-1}\left(z^{p+\alpha-1} + \sum_{n=1}^{\infty} a_{p+n} z^{p+n+\alpha-1}\right)$$

$$= z^p + \sum_{n=1}^{\infty} \frac{\Gamma(p+n+\alpha)}{n!\,\Gamma(p+\alpha)} a_{p+n} z^{p+n}$$

$$= z^p + \sum_{n=1}^{\infty} \frac{(p+\alpha)(p+\alpha+1)\ldots(p+\alpha+n-1)}{n!} a_{p+n} z^{p+n}.$$

This completes the proof of the theorem.

Theorem 2 — Let a function

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be in the class A_p. Then we have

$$D^{p-\alpha-1} f(z) = \frac{z^p}{\Gamma(p-\alpha)} \; D_z^{p-\alpha-1} \{z^{-\alpha-1} f(z)\}$$

and

$$D^{p-1} f(z) = \lim_{\alpha \to 0} D^{p-\alpha-1} f(z)$$

for $0 < \alpha < p$ and $p \in N$.

The proof of Theorem 2 is obtained by using the same technique as in the proof of Theorem 1.

Theorem 3 — $K_p(\alpha) \subset K_p(\alpha-1)$ for $0 < \alpha < 1$ and $p \in N$.

Theorem 4 — $K_p(-\alpha) \subset K_p(-\alpha-1)$ for $0 < \alpha < p$ and $p \in N$.

The proofs of Theorem 3 and Theorem 4 are obtained by using the same technique as in the proof of $K_p(n) \subset K_p(n-1)$ in Goel and Sohi (1980).

3. Some Results for Generalized Classes

Theorem 5 — Let a function

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

be in the class A_p and satisfy the following condition

$$\sum_{n=1}^{\infty} (n+p+k-1)(n+p+k-2) \ldots (n+k)(2n+2p+2k+1) \; | a_{p+n} | \leqq p\,!$$

for some $k > 0$. Then the function $f(z)$ belongs to the class $K_p(\alpha-1)$ for $0 < \alpha < 1$ and $0 < \alpha \leqq k$.

Proof : The hypothesis of the theorem

$$\sum_{n=1}^{\infty} (n+p+k-1)(n+p+k-2) \ldots (n+k)(2n+2p+2k+1) \; | a_{p+n} | \leqq p\,!$$

implies the inequality

$$\frac{1 - \displaystyle\sum_{n=1}^{\infty} \frac{(n+p+k)(n+p+k-1) \ldots (n+k)}{p\,!} \, | a_{p+n} |}{1 + \displaystyle\sum_{n=1}^{\infty} \frac{(n+p+k-1)(n+p+k-2) \ldots (n+k)}{p\,!} \, | a_{p+n} |} \geqq \frac{1}{2}.$$

Hence we have

$$\mathrm{Re} \left\{ \frac{D^{p+\alpha} f(z)}{D^{p+\alpha-1} f(z)} \right\} = \mathrm{Re} \left\{ \frac{z^p + \displaystyle\sum_{n=1}^{\infty} \frac{\Gamma(p+n+\alpha+1)}{n\,!\,\Gamma(p+\alpha+1)} a_{p+n} z^{p+n}}{z^p + \displaystyle\sum_{n=1}^{\infty} \frac{\Gamma(p+n+\alpha)}{n\,!\,\Gamma(p+\alpha)} a_{p+n} z^{p+n}} \right\}$$

(*equation continued on p.* 925)

$$> \frac{1-\sum_{n=1}^{\infty}\frac{(n+p+\alpha)\,(n+p+\alpha-1)\,\ldots\,(n+\alpha)}{p\,!}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{(n+p+\alpha-1)\,(n+p+\alpha-2)\ldots(n+\alpha)}{p\,!}\,|\,a_{p+n}\,|}$$

$$\geqq \frac{1-\sum_{n=1}^{\infty}\frac{(n+p+k)\,(n+p+k-1)\,\ldots\,(n+k)}{p\,!}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{(n+p+k-1)\,(n+p+k-2)\,\ldots\,(n+k)}{p\,!}\,|\,a_{p+n}\,|}$$

$$\geqq \frac{1}{2}$$

for $0 < \alpha < 1$ and $0 < \alpha \leqq k$. Thus we have the theorem.

The following theorems are obtained by using the same technique as in the proof of Theorem 5.

Theorem 6 — Let a function

$$f(z) = z^{p} + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}$$

be in the class A_p and satisfy the following condition

$$\sum_{n=1}^{\infty} (n+p+\alpha-1)\,(n+p+\alpha-2)\,\ldots\,(n+\alpha)\,\{\,(p+\alpha+1)\,n+2\,(p+\alpha) +2\,p\,\alpha+2\,(p^{2}+\alpha^{2})\,\}\;|\,a_{p+n}\,|\leqq p!$$

for $0 < \alpha < 1$. Then the function $f(z)$ belongs to the class $R_p\,(\alpha-1)$.

Theorem 7 — Let a function

$$f(z) = z^{p} + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}$$

be in the class A_p and satisfy the following condition

$$\sum_{n=1}^{\infty} (n+p-k-1)\,(n+p-k-2)\,\ldots\,(n+1-k)\,(2n+3p-2k)\;|\,a_{p+n}\,|\leqq p\,!$$

for some $k > 0$. Then the function $f(z)$ belongs to the class $K_p\,(-\alpha-1)$ for $0 < \alpha < 1$ and $k \leqq \alpha < 1$.

Theorem 8 — Let a function

$$f(z) = z^{p} + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n}$$

be in the class A_p and satisfy the following condition

$$\sum_{n=1}^{\infty} (n+p-\alpha-1)\,(n+p-\alpha-2)\,\ldots\,(n+1-\alpha)\,\{(1+p+\alpha)\,n +p-\alpha-3p\alpha+2p^{2}-\alpha^{2}\}\;|\,a_{p+n}\,|\leqq p\,!$$

for $0 < \alpha < 1$. Then the function $f(z)$ belongs to the class $R_p\,(-\alpha-1)$.

4. The Subclasses of the Generalized Classes

Let $\tilde{K}_p(\alpha-1)$, $\tilde{K}_p(-\alpha-1)$, $\tilde{R}_p(\alpha-1)$ and $\tilde{R}_p(-\alpha-1)$ denote the classes of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

in the class A_p and satisfy the following conditions

$$\sum_{n=1}^{\infty} (n+p+\alpha-1)(n+p+\alpha-2)\dots(n+\alpha)(2n+2p+2\alpha+1)\,|a_{p+n}| \leqq p\,!$$

for $0<\alpha<1$ and $p \in N$,

$$\sum_{n=1}^{\infty} (n+p-\alpha-1)(n+p-\alpha-2)\dots(n+1-\alpha)(2n+3p-2\alpha)\,|a_{p+n}| \leqq p\,!$$

for $0<\alpha<p$ and $p \in N$,

$$\sum_{n=1}^{\infty} (n+p+\alpha-1)(n+p+\alpha-2)\dots(n+\alpha)\{(p+\alpha+1)n+2(p+\alpha)$$
$$+2p\alpha+2(p^2+\alpha^2)\}\,|a_{p+n}| \leqq p\,!$$

for $0<\alpha<1$ and $p \in N$ and

$$\sum_{n=1}^{\infty} (n+p-\alpha-1)(n+p-\alpha-2)\dots(n+1-\alpha)\{(p-\alpha+1)n$$
$$+p-\alpha-3p\alpha+2p^2-\alpha^2\}\,|a_{p+n}| \leqq p\,!$$

for $0<\alpha<p$ and $p \in N$, respectively. Then we have

$\tilde{K}_p(\alpha-1) \subset K_p(\alpha-1)$, $\tilde{K}_p(-\alpha-1) \subset K_p(-\alpha-1)$,

$\tilde{R}_p(\alpha-1) \subset R_p(\alpha-1)$ and $\tilde{R}_p(-\alpha-1) \subset R_p(-\alpha-1)$.

Theorem 9 — $\tilde{K}_p(\alpha+\delta-1) \subset \tilde{K}_p(\alpha-1)$ for $0<\alpha<1$, $\delta \geqq 0$ and $0<\alpha+\delta<1$.

Proof : Let a function

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$$

be in the class $\tilde{K}_p(\alpha+\delta-1)$ for $0<\alpha<1$, $\delta \geqq 0$ and $0<\alpha+\delta<1$. Then we obtain

$$\sum_{n=1}^{\infty} (n+p+\alpha+\delta-1)(n+p+\alpha+\delta-2)\dots(n+\alpha+\delta)$$
$$(2n+2p+2\alpha+2\delta+1)\,|a_{p+n}| \leqq p\,!$$

which implies

$$\frac{1-\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta)\,(n+p+\alpha+\delta-1)\,\ldots\,(n+\alpha+\delta)}{p\,!}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta-1)\,(n+p+\alpha+\delta-2)\,\ldots\,(n+\alpha+\delta)}{p\,!}\,|\,a_{p+n}\,|}\geqq\frac{1}{2}\,.$$

Consequently we obtain

$$\text{Re}\left\{\frac{D^{p+\alpha}f(z)}{D^{p+\alpha-1}f(z)}\right\}=\text{Re}\left\{\frac{z^p+\sum_{n=1}^{\infty}\frac{\Gamma\,(p+n+\alpha+1)}{n\,!\,\Gamma\,(p+\alpha+1)}\,a_{p+n}\,z^{p+n}}{z^p+\sum_{n=1}^{\infty}\frac{\Gamma\,(p+n+\alpha)}{n\,!\,\Gamma\,(p+\alpha)}\,a_{p+n}\,z^{p+n}}\right\}$$

$$\geqq\frac{1-\sum_{n=1}^{\infty}\frac{\Gamma\,(p+n+\alpha+1)}{n\,!\,\Gamma(p+\alpha+1)}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{\Gamma\,(p+n+\alpha)}{n\,!\,\Gamma\,(p+\alpha)}\,|\,a_{p+n}\,|}$$

$$>\frac{1-\sum_{n=1}^{\infty}\frac{(n+p+\alpha)\,(n+p+\alpha-1)\,\ldots\,(n+\alpha)}{p\,!}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{(n+p+\alpha-1)\,(n+p+\alpha-2)\,\ldots\,(n+\alpha)}{p\,!}\,|\,a_{p+n}\,|}$$

$$\geqq\frac{1-\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta)\,(n+p+\alpha+\delta-1)\,\ldots\,(n+\alpha+\delta)}{p\,!}\,|\,a_{p+n}\,|}{1+\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta-1)\,(n+p+\alpha+\delta-2)\ldots(n+\alpha+\delta)}{p\,!}\,|\,a_{p+n}\,|}$$

$$\geqq\frac{1}{2}$$

for $0<\alpha<1$, $\delta\geqq 0$ and $0<\alpha+\delta<1$. This completes the proof of the theorem.

Theorem 10 — $\widetilde{R}_p\,(\alpha+\delta-1)\subset\widetilde{R}_p\,(\alpha-1)$ for $0<\alpha<1$, $\delta\geqq 0$ and $0<\alpha+\delta<1$.

PROOF : Let a function

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{p+n}\,z^{p+n}$$

be in the class $\widetilde{R}_p\,(\alpha+\delta-1)$ for $0<\alpha<1$, $\delta\geqq 0$ and $0<\alpha+\delta<1$.

Then the condition

$$\sum_{n=1}^{\infty} (n+p+\alpha+\delta-1)(n+p+\alpha+\delta-2)\dots(n+\alpha+\delta)\ \{(p+\alpha+\delta+1)n + 2(p+\alpha+\delta) + 2p(\alpha+\delta)+2p^2+2(\alpha+\delta)^2\}\ |a_{p+n}| \leqq p\,!$$

implies the inequality

$$\frac{1-\displaystyle\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta)(n+p+\alpha+\delta-1)\dots(n+\alpha+\delta)}{p\,!}|a_{p+n}|}{1+\displaystyle\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta-1)(n+p+\alpha+\delta-2)\dots(n+\alpha+\delta)}{p\,!}|a_{p+n}|} \geqq \frac{p+\alpha+\delta}{1+p+\alpha+\delta}.$$

Accordingly we have

$$\mathrm{Re}\left\{\frac{D^{p+\alpha}f(z)}{D^{p+\alpha-1}f(z)}\right\}=\mathrm{Re}\left\{\frac{z^p+\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha+1)}{n\,!\,\Gamma(p+\alpha+1)}a_{p+n}z^{p+n}}{z^p+\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha)}{n\,!\,\Gamma(p+\alpha)}a_{p+n}z^{p+n}}\right\}$$

$$\geqq \frac{1-\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha+1)}{n\,!\,\Gamma(p+\alpha+1)}|a_{p+n}|}{1+\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha)}{n\,!\,\Gamma(p+\alpha)}|a_{p+n}|}$$

$$\geqq \frac{1-\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha+\delta+1)}{n\,!\,\Gamma(p+\alpha+\delta+1)}|a_{p+n}|}{1+\displaystyle\sum_{n=1}^{\infty}\frac{\Gamma(p+n+\alpha+\delta)}{n\,!\,\Gamma(p+\alpha+\delta)}|a_{p+n}|}$$

$$> \frac{1-\displaystyle\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta)(n+p+\alpha+\delta-1)\dots(n+\alpha+\delta)}{p\,!}|a_{p+n}|}{1+\displaystyle\sum_{n=1}^{\infty}\frac{(n+p+\alpha+\delta-1)(n+p+\alpha+\delta-2)\dots(n+\alpha+\delta)}{p\,!}|a_{p+n}|}$$

$$\geqq \frac{p+\alpha+\delta}{1+p+\alpha+\delta}$$

$$\geqq \frac{p+\alpha}{1+p+\alpha}$$

for $0 < \alpha < 1$, $\delta \geqq 0$ and $0 < \alpha + \delta < 1$. Thus we have the theorem.

Theorem 11 — $\tilde{K}_p(-\alpha+\delta-1) \subset \tilde{K}_p(-\alpha-1)$ for $0 < \alpha < 1$, $\delta \geqq 0$ and $0 < \alpha - \delta < 1$.

The proof of Theorem 11 is obtained by using the same technique as in the proof of Theorem 9.

Theorem 12 — $\tilde{R}^p(-\alpha+\delta-1) (\subset \tilde{R}_p(-\alpha-1)$ for $0 < \alpha < 1$, $\delta \geqq 0$ and $0 < \alpha - \delta < 1$.

The proof of Theorem 12 is obtained by using the same technique as in the proof of Theorem 10.

Finally, from the definitions of the classes $\tilde{K}_p(\alpha-1)$, $\tilde{K}_p(-\alpha-1)$, $\tilde{R}_p(\alpha-1)$ and $\tilde{R}_p(-\alpha-1)$, we have the following results.

Remark 5 : Let the functions $f(z)$ and $g(z)$ be in the same class $\tilde{K}_p(\alpha-1)$. Then the Hadamard product $f * g(z)$ belongs to the same class.

Remark 6 : Let the functions $f(z)$ and $g(z)$ be in the same class $\tilde{K}_p(-\alpha-1)$. Then the Hadamard product $f * g(z)$ belongs to the same class.

Remark 7 : Let the functions $f(z)$ and $g(z)$ be in the same class $\tilde{R}_p(\alpha-1)$. Then the Hadamard product $f * g(z)$ belongs to the same class.

Remark 8 : Let the functions $f(z)$ and $g(z)$ be in the same class $\tilde{R}_p(-\alpha-1)$. Then the Hadamard product $f * g(z)$ belongs to the same class.

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