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LAMINAR FORCED FREE CONVECTION IN HORIZONTAL TUBES OF VARYING CROSS-SECTION AT LOW RAYLEIGH NUMBERS

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The effects of buoyancy forces on the steady laminar viscous flow in a horizontal tube of varying cross-section are analysed. Closed form solutions for the flow and temperature fields are obtained, for the case of uniform wall temperature at low Rayleigh numbers. It is found that while the variation of the tube cross-section influences the momentum flux, the buoyancy effects govern the heat flux. Further, it is found that the oscillatory nature of the Nusselt number is diminished by the free convective force. The pressure distribution obtained shows its non-uniform dependence on the axial distance.

1. INTRODUCTION

The secondary motion arising due to the buoyancy forces in the flow field of a viscous fluid, increases the rate of heat transfer with increasing axial distance. Hence the results obtained by neglecting the free convective effects are applicable only to those situations of very small heat transfer. These convective forces dominate in aircraft propulsion, nuclear power plants etc. Further, there are rheologically complex fluids whose density and temperature variations are favourable to free convection. Due to the complexity of the governing equations of motion and energy most of the work done including convective effects are experimental and the theoretical models are based on numerical techniques. Morton (1959) and Casal and Gill (1962) have analysed this problem for straight circular tube flows and have presented closed form solutions, neglecting the axial dependence of the flow variables.

In this paper, we present analytical expressions for the flow field of the secondary flow due to the buoyancy forces and for the temperature distribution due to the modified field of flow, as a next approximation over the forced convection solution, in ducts of non-uniform cross-section. We assume the wall of the tube to be varying along the axial direction by representing the radius 'a' of the tube by an arbitrary function $s(\epsilon X/a_0)$ where $\epsilon \ll 1$ gives the slow variation of the wall along the axial direction X and a_0 is the mean radius of the tube. This represents tubes of membrane oxygenators, heat exchangers in biomedical apparatus and also constricted and taper tubes. Apart from the aforementioned instances of applications, the present analysis might also help us gain more accurate insight into the momentum and heat transfer phenomena in stenotic arteries and blood vessels, (Manton 1971, 1975; Chow and

Soda 1973; Lew and Fung 1970; Rao and Devanathan 1973), however small the convective effects be.

The solution has been presented for low values of the governing parameter, Rayleigh number defined by

$$Ra = g\ \beta\tau a_0^4/\nu K$$

where g is the acceleration due to gravity, β is the coefficient of cubical expansion, ν the coefficient of kinematic viscosity and K the thermal diffusivity of the fluid and τ a constant axial temperature gradient. Nusselt number has been calculated. The results are compared with that of Morton (1959), Manton (1971) and Chow and Soda (1973). It is found that while the non-uniformity of the cross-section plays a dominant role on the momentum flux, the heat flux is greatly influenced by the free convection.

2. Formulation of the Problem

We consider the fully developed laminar steady motion of a viscous incompressible fluid in a horizontal tube of radius 'a', with non-uniform cross-section, the wall of which is heated uniformly so that a constant temperature gradient τ is maintained along the axial direction. Cylindrical polar coordinates (R, ϕ, X) are used with ϕ measured anti-clockwise from the upward vertical and X along the axis of the tube ; the corresponding velocities are (U,V,W). We neglect the density variations insofar as they give rise to a gravitational force and neglect the temperature dependence of ν and K. These assumptions are well justified since the wall temperature increases slowly with distance along the tube axis. In view of the smaller velocities and comparatively larger temperature differences involved we can also neglect the dissipation terms in the energy equation. Hence the governing equations are the equations of continuity and momentum balance along with buoyancy terms and the energy balance equation.

As mentioned earlier, the non-uniformity of the tube cross-section is introduced based on the long wave length approximation. This is achieved by taking the radius of the tube ($R=a$) to vary slowly along the axial direction i.e.,

$$a = a_0 s\ (\epsilon X/a_0). \qquad \ldots(2.1)$$

The basic equations are rendered dimensionless by the following scheme:

$$\left.\begin{aligned} &r = R/a_0, \quad x = \epsilon X/a_0 \\ &(u,v,w) = (a_0^2/\psi_0)\,(U,V,W) \\ &p = (a_0^4/\rho\psi_0^2)\ P \text{ and } T_w - T = (\tau a_0 \nu/K)\theta \end{aligned}\right\} \qquad \ldots(2.2)$$

where ψ_0 is the stream constant at the wall of the tube and T_w is the specified wall temperature which is taken as

$$T_w = T_0 + \tau X \qquad \ldots(2.3)$$

where T_0 is the temperature at the cross-section containing the origin. Thus the governing equations are given by

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \phi} + \epsilon \frac{\partial(wr)}{\partial x} = 0 \qquad \ldots(2.4)$$

$$u \frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{v^2}{r} + \epsilon w \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial r}$$

$$+ \frac{1}{Re}\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right] - (Ra/Re^2)\, \theta \cos \phi \qquad \ldots(2.5)$$

$$u \frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + \frac{uv}{r} + \epsilon\, w \frac{\partial v}{\partial x} = - \frac{1}{r}\frac{\partial p}{\partial \phi}$$

$$+ \frac{1}{Re}\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \phi}\right] + (Ra/Re^2)\, \theta \sin\phi \qquad \ldots(2.6)$$

$$u \frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \phi} + \epsilon\, w \frac{\partial w}{\partial x} = - \epsilon \frac{\partial p}{\partial x} + \frac{1}{Re} \nabla^2 w \qquad \ldots(2.7)$$

and

$$u \frac{\partial \theta}{\partial r} + \frac{v}{r}\frac{\partial \theta}{\partial \phi} + w\left(\epsilon \frac{\partial \theta}{\partial x} - \frac{1}{Pr}\right) = \frac{1}{Pr.\, Re} \nabla^2 \theta \qquad \ldots(2.8)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \epsilon^2 \frac{\partial^2}{\partial x^2}\,.$$

The dimensionless parameters involved in (2.4) to (2.8) are

$$\left.\begin{aligned} Re &- \text{(Reynolds number)} = \psi_0/a_0\nu \\ Ra &- \text{(Rayleigh number)} = g\beta\tau a_0^4/K\nu \\ Pr &- \text{(Prandtl number)} = \nu/K. \end{aligned}\right\} \qquad \ldots(2.9)$$

It is important to note here that the eqns. (2.4) to (2.8) are asymmetrical and also the dependence of the variables u,v,w,p and θ on the axial distance is included, while it is neglected in the analyses of Morton (1959), Yao and Berger (1978), Casal and Gill (1962) etc.

The corresponding boundary conditions are

$$\left.\begin{aligned} &u,v,w \text{ and } \theta \text{ are zero on } r = s(x) \\ &u,v,w \text{ and } \theta \text{ are finite on } r = 0. \end{aligned}\right\} \qquad \ldots(2.10)$$

Though the solutions of eqns. (2.4) to (2.8) subject to (2.10) are a matter of considerable difficulty, successive approximations to the solution can be determined by expanding u,v,w,p and θ as power series in terms of the two parameters Ra and ϵ which are very small.

3. Solutions

We can consider the solution of (2.4) to (2.8) as consisting of two parts, one due to the variation of the tube cross-section and the other due to the buoyancy effect. Accordingly, a suitable perturbation scheme for the solution would be

$$Y(r,\phi,x) = Y_{00} + (\epsilon Y_{10} + Ra Y_{11})$$

$$+ (\epsilon^2 Y_{20} + \epsilon Ra Y_{21} + Ra^2 Y_{22}) + \ldots \qquad \ldots(3.1)$$

where the Y_{nm} take the variables u,v,w,p and θ. Substituting (3.1) into (2.4) to (2.8) and separating the coefficients of equal powers of the parameters yield a number of equations. These equations can be grouped again into three parts namely (i) those that govern the momentum and heat transfer in tubes of varying cross-section (ii) those that yield the free convection effects and (iii) those that give the interaction of the two factors. In view of this, eqn. (3.1) can be written as

$$Y(r,\phi,x) = (Y_0(r,\phi,x),\ Y_1(r,\phi,x)\ Y_2(r,\phi,x)) \qquad \text{...(3.2)}$$

where Y_0, Y_1 and Y_2 are solutions respectively of the equations

$$V_0 \equiv 0 \qquad \text{...(3.3a)}$$

$$\frac{\partial}{\partial r}(ru_0) + \epsilon \frac{\partial}{\partial x}(rw_0) = 0 \qquad \text{...(3.3b)}$$

$$u_0 \frac{\partial u_0}{\partial r} + \epsilon w_0 \frac{\partial u_0}{\partial x} = - \frac{\partial p_0}{\partial r} + \frac{1}{Re}\left[\nabla^2 u_0 - \frac{u_0}{r^2}\right] \qquad \text{...(3.3c)}$$

$$u_0 \frac{\partial w_0}{\partial r} + \epsilon w_0 \frac{\partial w_0}{\partial x} = - \epsilon \frac{\partial p_0}{\partial x} + \frac{1}{Re}\nabla^2 w_0 \qquad \text{...(3 3d)}$$

$$u_0 \frac{\partial \theta_0}{\partial r} + w_0\left(\epsilon \frac{\partial \theta_0}{\partial x} - \frac{1}{Pr}\right) = \frac{1}{Pr.Re}\nabla^2\theta_0 \qquad \text{...(3.3e)}$$

$$\frac{\partial}{\partial r}(ru_1) + \frac{\partial v_1}{\partial \phi} = 0 \qquad \text{...(3.4a)}$$

$$u_1 \frac{\partial u_1}{\partial r} + \frac{v_1}{r}\frac{\partial u_1}{\partial \phi} - \frac{v_1^2}{r} = - \frac{\partial p_1}{\partial r}$$

$$+ \frac{1}{Re}\left[\nabla^2 u_1 - \frac{u_1}{r^2} - \frac{2}{r^2}\frac{\partial v_1}{\partial \phi}\right] - \frac{1}{Re^2}\theta_1 \cos\phi \qquad \text{...(3.4b)}$$

$$u_1 \frac{\partial v_1}{\partial r} + \frac{v_1}{r}\frac{\partial v_1}{\partial \phi} + \frac{u_1 v_1}{r}$$

$$= -\frac{1}{r}\frac{\partial p_1}{\partial \phi} + \frac{1}{Re}\left[\nabla^2 v_1 - \frac{v_1}{r^2} + \frac{2}{r^2}\frac{\partial u_1}{\partial \phi}\right] + \frac{1}{Re^2}\theta_1 \sin\phi \qquad \text{...(3.4c)}$$

$$u_1 \frac{\partial w_1}{\partial r} + \frac{v_1}{r}\frac{\partial w_1}{\partial \phi} = \frac{1}{Re}\nabla^2 w_1 \qquad \text{...(3.4d)}$$

$$u_1 \frac{\partial \theta_1}{\partial r} + \frac{v_1}{r}\frac{\partial \theta_1}{\partial \phi} - \frac{w_1}{Pr} = \frac{1}{Pr.Re}\nabla^2\theta_1 \qquad \text{...(3.4e)}$$

and

$$\frac{\partial}{\partial r}(ru_{21}) + \frac{\partial v_{21}}{\partial \phi} + \frac{\partial}{\partial x}(rw_{11}) = 0 \qquad \text{...(3.5a)}$$

$$u_{10}\frac{\partial u_{11}}{\partial r} + u_{11}\frac{\partial u_{10}}{\partial r} + w_{00}\frac{\partial u_{11}}{\partial x} + \frac{v_{11}}{r}\frac{\partial u_{10}}{\partial \phi}$$

$$= -\frac{\partial p_{21}}{\partial r} + \frac{1}{Re}\left[\nabla^2 u_{21} - \frac{u_{21}}{r^2} - \frac{2}{r^2}\frac{\partial v_{21}}{\partial \phi}\right] - \frac{\theta_{10}}{Re^2}\cos\phi. \qquad \text{...(3.5b)}$$

$$u_{10}\frac{\partial v_{11}}{\partial r}+w_{00}\frac{\partial v_{11}}{\partial x}+\frac{u_{10}v_{11}}{r}$$

$$=-\frac{1}{r}\frac{\partial p_{21}}{\partial \phi}+\frac{1}{Re}\left[\nabla^2 v_{21}-\frac{v_{21}}{r^2}+\frac{2}{r^2}\frac{\partial u_{21}}{\partial \phi}\right]+\frac{\theta_{10}}{Re^2}\sin\phi \qquad \ldots(3.5c)$$

$$u_{10}\frac{\partial w_{11}}{\partial r}+u_{11}\frac{\partial w_{10}}{\partial r}+u_{21}\frac{\partial w_{00}}{\partial r}+\frac{\partial}{\partial x}(w_{00}w_{11})$$

$$=-\frac{\partial p_{11}}{\partial x}+\frac{1}{Re}\nabla^2 w_{21} \qquad \ldots(3.5d)$$

$$u_{10}\frac{\partial \theta_{11}}{\partial r}+u_{11}\frac{\partial \theta_{10}}{\partial r}+u_{21}\frac{\partial \theta_{00}}{\partial r}+w_{00}\frac{\partial \theta_{11}}{\partial x}$$

$$+w_{11}\frac{\partial \theta_{00}}{\partial x}-\frac{w_{21}}{Pr}=\frac{1}{Pr.\ Re}\nabla^2\theta_{21}. \qquad \ldots(3.5e)$$

It is to be noted here that we have given the explicit expanded form for eqn. (3.5) alone, in terms of the velocity while similar expressions for eqn. (3.3) and (3.4) are presented in a compact form. To solve the equations, we introduce stream function between any two of the velocity components as follows;

$$u_0=-\frac{\epsilon}{r}\frac{\partial \psi}{\partial x} \text{ and } w_0=\frac{1}{r}\frac{\partial \psi}{\partial r} \qquad \ldots(3.6)$$

$$u_1=\frac{1}{r}\frac{\partial \chi}{\partial \phi} \qquad \text{and } v_1=-\frac{\partial \chi}{\partial r} \qquad \ldots(3.7)$$

and

$$u_{21}=\frac{1}{r}\frac{\partial \pi}{\partial \phi} \text{ and } V_{21}=-\frac{\partial \pi}{\partial r} \qquad \ldots(3.8a)$$

where

$$V_{21}=v_{21}+\frac{Re}{1440}ss_x\eta^4(\eta^6-15\eta^4+35\eta^2-25)\sin\phi \qquad \ldots(3.8b)$$

with $\eta=r/s$ and the suffix denotes differentiation w.r.t. that variable. In view of (3.1), we write

$$\psi=\psi_{00}+\epsilon\psi_{10}+\epsilon^2\psi_{20}+\ldots$$

i.e. $$=\sum_0^\infty \epsilon^n\psi_{no}$$

$$\theta_0=\theta_{00}+\epsilon\theta_{10}+\epsilon^2\theta_{20}+\ldots=\sum_0^\infty \epsilon^n\theta_{no}$$

$$\chi=Ra\chi_{11}+Ra^2\chi_{22}+\ldots=\sum_1^\infty Ra^n\chi_{nn} \qquad \ldots(3.9)$$

$$w_1=w_{00}+\sum_1^\infty Ra^n w_{nn}$$

$$\theta_1=\theta_{00}+\sum_1^\infty Ra^n\theta_{nn}.$$

But the zeroth order velocity and temperature distributions are determined from (3.3) and are substituted in w_1 and θ_1. The corresponding boundary conditions are

$$\left.\begin{aligned} \frac{\partial}{\partial r}\psi_{no} &= 0 \\ \psi_{00} &= 1 \\ \psi_{no} &= 0 \text{ for } n \neq 0 \\ \theta_{no} &= 0 \\ \psi_{no} &= 0 \text{ on } r = 0 \end{aligned}\right\} \text{ on } r = s(x) \qquad \ldots(3.10)$$

$\frac{1}{r}\frac{\partial}{\partial x}\psi_{no}$, $\frac{\partial}{\partial r}\left(\frac{1}{r}\psi_{no}\right)$ and $\theta_{no} \to 0$ as $r \to 0$ u_1, v_1, w_1 and θ_1 are zero on $r = s(x)$ and are finite at $r = 0$. ...(3.11)

and

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \pi}{\partial \phi} &= 0 \\ \frac{\partial \pi}{\partial r} &= \frac{1}{360} Re\, ss_x \sin\phi \\ w_{21} &= 0 \\ \theta_{2_1} &= 0 \end{aligned}\right\} \text{ on } r = s(x) \qquad \ldots(3.12)$$

and

$\frac{1}{r}\frac{\partial \pi}{\partial \phi}$, $\frac{\partial \pi}{\partial r}$, w_{21} and θ_{2_1} are finite on $r = 0$.

Substituting (3.9) into (3.3) to (3.5) and using (3.6)-(3.8) we obtain the required solutions satisfying (3.10)-(3-12). Leaving the details, we present the solutions alone.

$$\psi_{00} = \eta^2 (2-\eta^2)$$

$$\psi_{10} = \frac{1}{36} Re\, s^4 (1/s^4)_x \eta^2 [\eta^6 - 6\eta^4 + 9\eta^2 - 4]$$

$$\psi_{20} = Re^2 \sum_{n=1}^{6} (-1)^{n+1} a_n \eta^{2n} + \frac{1}{12} s^6 (1/s^4)_{xx} \eta^2 [\eta^4 - 2\eta^2 + 1]$$

$$\theta_{00} = \frac{Re}{4} (1-\eta^2)(3-\eta^2)$$

$$\theta_{10} = -\frac{1}{288} Re^2 s^4 (1/s^4)_x \left[\eta^8 - 8\eta^6 + 18\eta^4 - 16\eta^2 + 5 \right]$$

$$\theta_{20} = Re^3 \left[\sum_{n=1}^{6} \frac{(1-\eta^{2n})}{2n} \left\{ (-1)^{n+1} a_n + \frac{Pr}{144} \frac{b_n}{n} \right\} \right]$$

$$+ \frac{Re}{6} \sum_{n=1}^{3} (-1)^{n+1} c_n (1-\eta^{2n})$$

$$\chi_{11} = \frac{s^3}{1152} \sin\phi\, \eta\, (\eta^6 - 12\eta^4 + 21\eta^2 - 10)$$

$$w_{11} = -\frac{Re\, s}{11520} \cos\phi\eta\, (\eta^2 - 1)\, (\eta^6 - 19\eta^4 + 51\eta^2 - 49)$$

$$\theta_{11} = -\frac{Re\, s^3 \cos\phi\eta}{1382400}\Big[(381 + 1325 Pr) - 15\,(49 + 200 Pr)\eta^2 + 100\,(5 + 26\, Pr)\eta^4 - 25\,(7 + 45\, Pr)\eta^6 + 30\,(1 + 7 Pr)\eta^8 - (1 + 10\, Pr)\eta^{10}\Big]$$

$$\chi_{22} = Re\, s^6 \sin 2\phi \sum_{n=3}^{7} \left[\eta^2 \frac{d_n\, \eta^{2(n-1)} - (n-1)\eta^2 + (n-2)}{(2n-4)\,(2n-2)\,(2n)\,(2n+2)}\right]$$

$$w_{22} = \frac{Re^2 s^4 \eta^2}{20 \times (1152)^2} \Big\{ \cos 2\phi \Big[\sum_{n=1}^{7} A_n d_n\, (1 - \eta^{2n}) - (3.9385 - 4.8333\eta^2 - 14\eta^4 + 24.1\eta^6 - 12.2917\eta^8 + 3.3286\eta^{10} - 0.25\eta^{12} + 0.6079\eta^{14})\Big] + 313.6766 - 676.3333\eta^2 + 635.25\eta^4 - 389.2\eta^6 + 142.7083\eta^8 - 28.3714\eta^{10} + 2.3333\eta^{12} - 0.0635\eta^{14} \Big\}$$

$$\theta_{22} = Pr Re^3 s^6 \eta^2 \Big\{ \frac{1}{(2400) \times (1152)^2} \sum_{n=0}^{8} (B_n \cos 2\phi + D_n)\eta^{2n} - 5 \times 10^{-4} \Big[\sum_{n=1}^{3} (-1)^n c_n\, (1 - \eta^{2n}) \cos 2\phi + \sum_{n=4}^{9} E_n \eta^{2n} - (0.0346 d_7 - 0.0019 d_6 - 0.0061 d_5 - 0.0274 d_4 - 0.2291 d_3) \Big]\Big\}$$

$$- \frac{Re^3 s^6 \eta^2}{20 \times (1152)^2} \Big\{ \cos 2\phi \Big[20 \times (1152)^2 \times \Big[(1 - \eta^2) \times (1.302 d_3 + 0.579 d_4 + 0.306 d_5 + 0.181 d_6) \times 10^{-4} - (1 - \eta^4)\,(4.167 d_3 + 8.333 d_4 + 12.5 d_5 + 16.667 d_6) \times 10^{-2} + (1 - \eta^2)\,(1.667 d_3 + 2.5 d_4 + 3.333 d_5 + 4.167 d_6 + 4.5 d_7) \times 10^{-2} + \text{terms of order } 10^{-5} \text{ and less}\Big] + (0.000\eta^{16} - 0.001\eta^{14} + 0.017\eta^{12} - 0.088\eta^{10} + 0.251\eta^8 - 0.233\eta^6 - 0.151\eta^4 + 0.328\eta^2 - 0.123) \Big] + 0.002\eta^{16}$$

(*equation continued on p.* 953)

$$- 0.0093\eta^{14} + 0.1473\eta^{12} - 1.0193\eta^{10} + 4.0542\eta^{8} - 10.5875\eta^{6} + 21.1354\eta^{4} - 26.1397\eta^{2} + 12.4182\Big\}$$

$$\pi = -\frac{Re\, s^2 s_x \sin\phi}{11520 \times 18}\eta \Big\{ s^2\Big[0.7074\eta^{12} + 14.175\eta^{10} + 45.9378\eta^{8} - 34.3746\eta^{6} - 74.9997\eta^{4} - 5.7474\eta^{2} + 54.3015\Big] + s\Big[8.8336\eta^{11} - 128.3696\eta^{9} + 46.3024\eta^{7} - 70.704\eta^{5} + 543.78\eta^{2} - 399.8424\Big] + 288\,(1 - \eta^2)\Big\}$$

$$w_{21} = \cos 2\phi \Big\{ s^2 s_x \eta\,(0.0045\eta^6 - 0.0324\eta^4 + 0.0783\eta^2 - 0.0504) + Re\, s_x \eta\,(5.333\eta^4 - 15.9999\eta^2 + 10.6666) + Re^2 s_x \Big[0.2857\eta^{13} - 0.7005\eta^{11} + 36.0025\eta^{9} - 85.0056\eta^{7} + 99.3396\eta^{5} - 49.003\eta^{3} - 0.9187\Big] + Re^2 s s_x \eta \Big[s\,(0.0002 - 0.0002\eta^2 + 0.0006\eta^4 + 0.0001\eta^6) - (0.0011 - 0.0019\eta^2 + 0.0008\eta^4)\Big]\Big\}$$

$$\theta_{21} = \eta \cos\phi \Big\{ Pr\, Re^3 s^6 s_x \Big[0.0001\eta^{14} - 0.0001\eta^{12} + 0.001\eta^{10} - 0.001\eta^{8} + 0.002\eta^{6} - 0.002\eta^{4} + 0.0001\eta^{2} - 0.001 \Big] - Pr\, Re^2 s^2 s_x\,(0.0001\eta^6 - 0.0002\eta^4 + 0.0004\eta^2 - 0.0002) + Re^2 s_x\,(0.0137\eta^{14} - 0.0042\eta^{12} + 0.3\eta^{10} - 1.0626\eta^{8} + 2.1807\eta^{6} - 2.7085\eta^{4} + 1.2185\eta^{2} + 0.0624) + Re s^2 s_x\,(0.0001\eta^8 - 0.0007\eta^6 + 0.003\eta^4 - 0.0025) + \text{terms of order } 10^{-7} \text{ and less} \Big\}$$

where a_n, b_n, c_n, d_n, A_n, B_n, C_n, D_n and E_n appearing in these solutions are recorded for convenience in the appendix

The dimensionless heat flux is given by

$$Nu = -\int_0^x \frac{Pr}{s}\,(\partial\theta/\partial\eta)_{\eta=1}$$

and can be calculated from the solution for the temperature distribution. The pressure term upto the first order of approximation is given by

$$p_{00} + p_{10} + Ra p_{11}$$

$$= -16 \Big\{ \int_0^x \frac{dx}{\epsilon s^4} + \frac{Re}{4s^4} + \frac{11}{135}\,\epsilon\, Re^2 \int_0^x s_x^2 \frac{dx}{s^4}$$

(*equation continued on p.* 954)

$$+ \frac{11}{180}\epsilon\, Re^2 \frac{s_x}{s^5} + \int_0^x \left[5s_x^2 - ss_{xx} \right] \frac{dx}{s^4} + \frac{2}{3}\epsilon$$

$$+ \epsilon \frac{s_x}{s^3}\left[\frac{1}{2} + \eta^2\right] + \frac{Ra}{788\, Re}\,(2\eta^5 - 12\eta^3 + 29\eta) \int_0^x s\, dx \Big\} .$$

4. Discussion

To understand the combined effect of the non-uniformity of the tube cross-section and the secondary convection we have considered flow in taper tubes i.e. (i) $s(x) = 1 - \epsilon x$ and flow in constricted tubes i.e., (ii) $s(x) = 1 + \epsilon \sin 2\pi x$ and (iii) $s(x) = 1 - \epsilon \exp(-xv^2/2)$. Numerical computations are carried out and the results are depicted graphically.

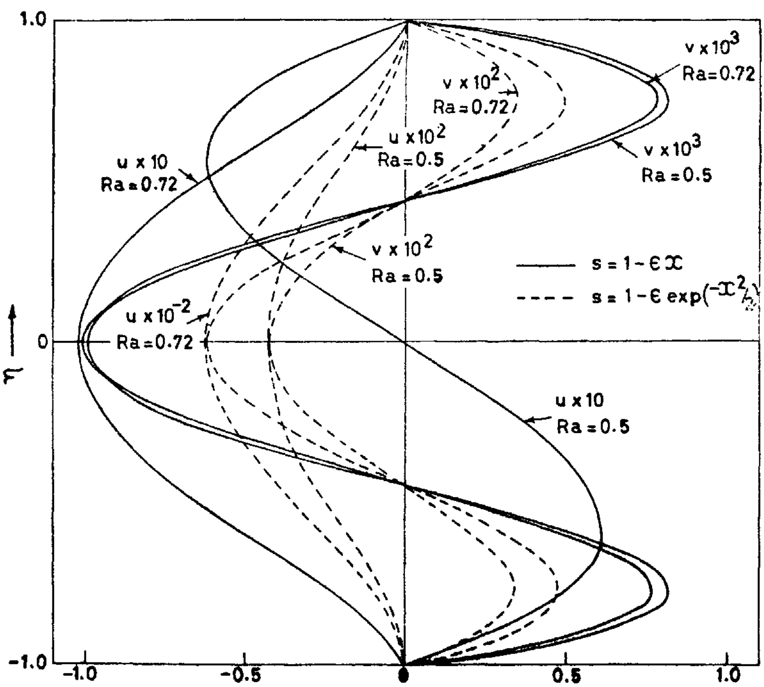

FIG. 1. The non-dimensional radial and tangential velocity components taken upto second order approximation in planes normal to the pipe axis. The radial component u is taken vertically through the pipe axis. The transverse component v is a horizontal profile through the axis.

The numerical results show that the stream function as such decreases monotonically as ϵ increases from zero to one. Consequently the radial and transverse flows at each cross-section is in the negative direction. But it can be seen that near the wall, where the effect of the tube geometry is dominant, these become positive. This feature corresponds to the trapping of the fluid observed by Manton (1975). Due to the presence of free convection, however small, there is a significant heating of the fluid only near the wall. This, in turn, enhances the secondary motion resulting in the increase of the velocity there. But this phenomenon is gradually attenuated by sufficiently strong convective force. Figure 1 shows the radial and transverse velocity components to be similar to that of Morton (1959). Thus the magnitude variations in the velocity components can be attributed mainly to the tube geometry.

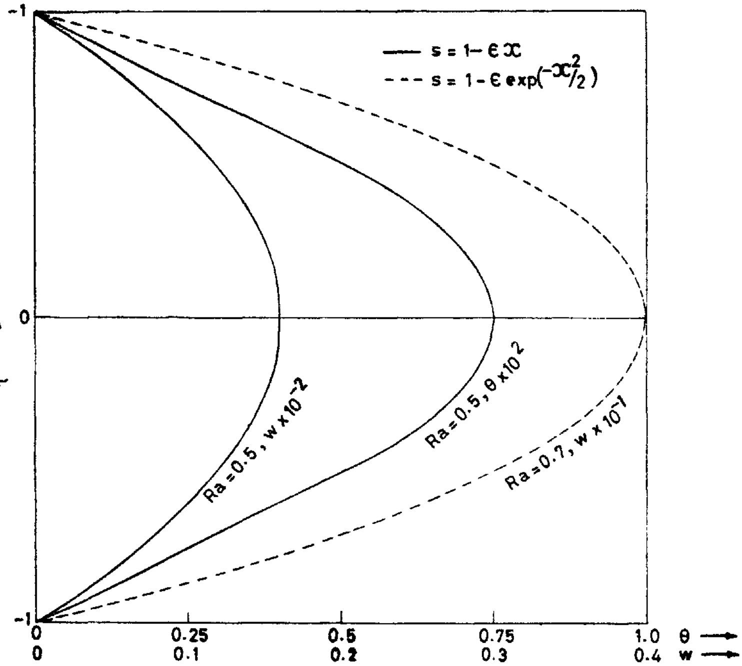

FIG. 2. The non-dimensional axial velocity component and temperature distribution for two different geometries.

Figure 2 depicts the axial velocity component and the temperature distribution. The temperature fields' dependence on the tube geometry is insignificant as it changes in magnitude only with respect to the convective force. Whereas the magnitude of the axial velocity shows a marked dependence both on the tube geometry and on the convection parameter. Figure 3 depicts the deviation of the stream function from that of a rigid tube (Morton 1959).

Figure 4 shows the variation of the Nusselt number with respect to the tube geometry and for different Rayleigh numbers. The heat flux at the tube wall prominantly fluctuates from its mean. But the amplitude of oscillation decreases as '*Ra*' increases and it almost coincides with its mean for sufficiently large *Ra*. This is to

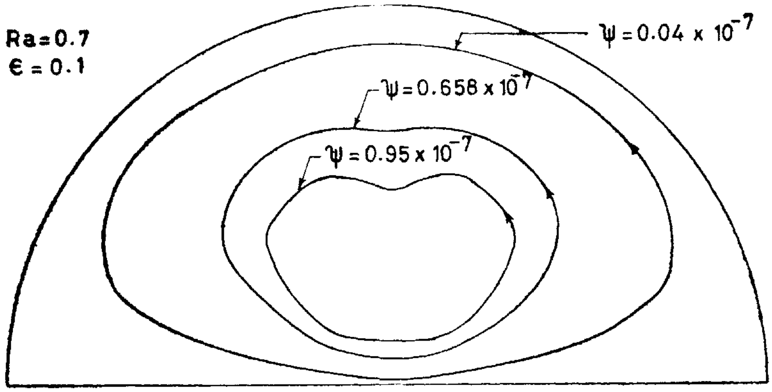

FIG. 3. The stream lines corresponding upto the second order approximation for flow in planes normal to the pipe axis.

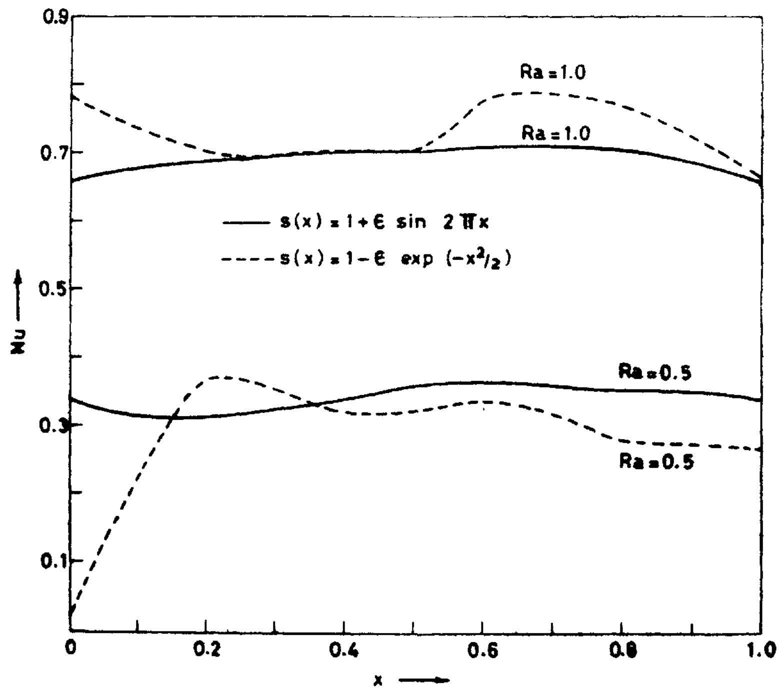

FIG. 4. The Nusselt Number as a function of the axial coordinate x.

be expected, as in the fully developed region, the secondary flow gradually increases. This minimises the temperature fluctuations in the flow field by means of convection. Hence after sufficiently long time and with strong convection, the heat flux becomes steady irrespective of the tube geometry, whereas in the analysis of Chow and Soda (1973) it remains oscillatory throughout. But the inherent restriction on the value of the Reynolds number in our analysis prevents us from making any further quantitative comparison with their analysis.

It is interesting to note that, as emphasised by Casal and Gill (1962), we also find the pressure distribution to be non-uniform along the axial direction and that the flow and temperature distributions also depend on the axial distance. Further the values of Ra and Re taken as 60 and 50 respectively by Morton (1959) seem to be questionable in view of the perturbation scheme. Moreover an increase in Ra would imply greater temperature difference in the flow field under which circumstances, the other physical properties of the fluid may not remain constant. Hence, our restriction of the numerical calculations to small Ra is more appropriate. Such low Rayleigh numbers are considered by Schneider (1981) for a horizontal plate and by McComas and Eckert (1966).

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Appendix

The constants appearing in the solutions of the governing equations are given below :

$$a_1 = \frac{1}{900}\left(\frac{19}{24}P_1 + 13Q_1\right) \qquad b_1 = -(1.75\,P_1 - Q_1)$$

$$a_2 = \frac{1}{720}\left(\frac{11}{3}P_1 + 29Q_1\right) \qquad b_2 = (13.75\,P_1 - 7Q_1)$$

$$a_3 = \frac{1}{24}\left(\frac{2}{9}P_1 + Q_1\right) \qquad b_3 = -(35.5\,P_1 + 16Q_1)$$

$$a_4 = \frac{1}{144}(P_1 + 3Q_1) \qquad b_4 = (34.5\,P_1 + 10Q_1)$$

$$a_5 = \frac{1}{480}\left(P_1 + \frac{8}{3}Q_1\right) \qquad b_5 = -(10.75\,P_1 - 5Q_1)$$

$$a_6 = \frac{1}{1800}\left(\frac{1}{3}P_1 + Q_1\right) \qquad b_6 = -\left(\frac{1}{8}P_1 + Q_1\right)$$

where

$$P_1 = 16(s_x/s)^2 \text{ and } Q_1 = 1.25P_1 - 4(s_{xx}/s)$$

$$c_1 = 2c_3 = 5s_x^2 - ss_{xx},\ c_2 = 7.25s_x^2 - 1.75ss_{xx}$$

$$d_1 = d_2 = 1,\ d_3 = 3\left[Q_2(49 + 200Pr) - 54P_2\right]$$

$$d_4 = 15Q_2(7 + 45Pr) - 444P_2,\ d_5 = -\left[40Q_2(5 + 26Pr) + 87P_2\right]$$

$$d_6 = -\left[24Q_2(1 + 7Pr) + 66P_2\right],\ d_7 = Q_2(1 + 10Pr) + 9P_2$$

where

$$P_2 = \frac{1}{331526},\ Q_2 = \frac{1}{276480}$$

$$A_1 = 92224.5d_3 + 36889.8d_4 + 18444.9d_5 + 10541.3d_6 + 6587.83d_7$$

$$A_2 = 69175.3d_3 + 20752.6d_4 + 9224.5d_5 + 4942.6d_6 + 2964.2d_7$$

$$A_3 = 18455.97,\ A_4 = 2296.62,\ A_5 = 525.73,\ A_6 = 166.02\ A_7 = 55.34.$$

$$B_0 = -(74.2773 + 26.7653Pr),\ B_1 = 108.5 - 362.5Pr,$$

$$B_2 = 53.5 + 1262.5Pr,\ B_3 = -(192.9 + 1612.5Pr),$$

$$B_4 = 147.5 + 1034.375Pr,\ B_5 = -(50.5866 + 351.9627Pr),$$

$$B_6 = 8.375 + 61.25Pr,\ B_7 = 0.5236 + 4.522Pr,\ B_8 = 0.0125 + 0.125Pr$$

$$C_1 = 3.45d_3 + 0.69d_4 + 0.23d_5 + 0.2d_6 + 0.12d_7$$

$$C_2 = 1.63d_3 + 5.2d_4 + 0.24d_5 + 0.13d_6 + 0.08d_7$$

$$C_3 = 0.35d_3 + 0.11d_4 + 0.05d_5 + 0.02d_6 + 0.01d_7$$

$$D_0 = 2483\,9118 + 8606.632Pr,\ D_1 = -(5117 + 19275Pr),$$

$$D_2 = 4688.8125 + 19668.25Pr,\ D_3 = 2860.1332 + 13756.666Pr,$$

$$D_4 = 1115.625 + 6242.1857Pr,\ D_5 = -(349.1429 + 1766\,5713Pr)$$

$$D_6 = 40.5417 + 281.0937Pr,\ D_7 = -(2.6712 + 20.988Pr),$$

$$D_8 = 0.0563 + 0.563Pr,\ E_4 = -0.3333,$$

$$E_5 = 0.0208(5d_3 - 2d_4),\ E_6 = 0.0048(3d_4 - 2d_5),$$

$$E_7 = 0.0005(7d_5 - 6d_6),\ E_8 = 0.0011(d_6 - d_7),\ E_9 = 0.0357.$$