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SLOW STEADY FLOW BETWEEN IMPERMEABLE AND NATURALLY PERMEABLE COAXIALLY TAPERED TUBES

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The flow between two coaxially tapered tubes having the same virtual vertex is considered. The outer tube is impermeable and the inner one is composed of permeable material. The non-slip condition is applied at the outer tube and the slip condition suggested by Jones is applied at the curved surface of the inner tube. The velocity profiles, drop in pressure, flux and skin friction have been determined and discussed graphically.

1. INTRODUCTION

The viscous incompressible flow through tubes and channels have been studied by several authors. Oka (1964) studied the slow, steady motion of a viscous fluid through a tapered tube and examined theoretically the dependence of the distribution of velocity, pressure and flux on the semiangle of the cone. Nishimura and Oka (1965) studied the same problem without neglecting the inertia terms.

In the present investigation we considered the slow steady laminar motion of a viscous incompressible fluid in the region bounded by the two coaxial tapered tubes having the same virtual vertex. The outer tube is impermeable and the inner one is composed of permeable material. We studied the coupled flow by dividing the whole flow field into two regions, (I) free fluid region (between two tubes where Navier-Stoke's equations hold), (II) porous region (where Darcy's equations hold). We applied no-slip condition at the impermeable tube and the slip condition suggested by Jones (1973) at the curved surface of inner permeable tube. The purpose of this paper is to examine the effect of permeability on the distribution of velocity, pressure, flux and coefficient of skin friction.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

We take a spherical coordinate system r, θ and ϕ whose origin is at the common virtual vertex of the two coaxially tapered tubes. A porous material is considered in the region $(0 \leqslant \theta \leqslant \theta_1)$, where a constant pressure gradient is applied at the tapered end $r = 1$ in order to develop the flow. Thus we consider the slow steady laminar motion of a viscous incompressible fluid which has an axial symmetry. We assume that the streamlines are straight lines passing through the tapered end.

The flow in the free fluid region $(\theta_1 \leqslant \theta \leqslant \theta_2)$ is governed by the dimensionless Navier-Stoke's equations. For slow motion, on neglecting the inertia terms the governing equations are reduced to

$$\frac{\partial p}{\partial r} = \left(\nabla^2 v_r - \frac{2v_r}{r^2} \right) \quad \ldots(2.1)$$

$$\frac{\partial p}{\partial \theta} = \left(\frac{2}{r} \frac{\partial v_r}{\partial \theta} \right) \quad \ldots(2.2)$$

and the equation of continuity is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 v_r \right) = 0 \quad \ldots(2.3)$$

where

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right).$$

The flow in the porous region ($0 \leqslant \theta \leqslant \theta_1$) is governed by the dimensionless Darcy's equations:

$$V_r = -K \frac{\partial P}{\partial r} \quad \ldots(2.4)$$

$$0 = -K \frac{\partial P}{r\,\partial \theta} \quad \ldots(2.5)$$

and the equation of continuity is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 V_r \right) = 0 \quad \ldots(2.6)$$

where K is the non-dimensional permeability parameter of the material.

The boundary conditions are

$$\left.\begin{array}{l} \text{at} \quad \theta = \theta_1, \; e_{\theta r} = \dfrac{\alpha}{\sqrt{K}} \left(v_r - V_r \right) \\ \text{where } e_{\theta r} = \dfrac{1}{r} \dfrac{\partial v_r}{\partial \theta}, \\ \text{at} \quad \theta = \theta_2, \; v_r = 0, \end{array}\right\} \quad \ldots(2.7)$$

where α is a constant depending upon the porous material.

3. Method of Solution

(i) *Velocity Distribution*

Equation (2.3) gives $v_r = f(x)/r^2$ ...(3.1)

where $x = \cos\theta$. Eliminating p from (2.1) and (2.2) and using (3.1), we have

$$\left((1 - x^2) \right) \frac{d^2 f}{dx^2} - 2x \frac{df}{dx} + 6f = A_1. \quad \ldots(3.2)$$

The solution of this equation is given by

$$f(x) = A_2 P_2(x) + A_3 Q_2(x) + \frac{1}{6} A_1 \quad \ldots(3.3)$$

where $P_2(x)$ and $Q_2(x)$ are Legendre functions of the first and second kind respectively. A_1, A_2, and A_3 are constants. Thus the radial velocity in the free fluid region is given by

$$v_r = \frac{1}{2r^2}\left\{ A_2(3x^2-1) + A_3\left[\frac{1}{2}(3x^2-1)\log\left(\frac{1+x}{1-x}\right) - 3x\right] + \frac{1}{3}A_1\right\} \quad ...(3.4)$$

From eqns. (2.4) lo (2.6), the velocity in the porous region is given by

$$V_r = C/r_2 \quad ...(3.5)$$

where C is a constant and is given by

$$C = -K\left(\frac{\partial P}{\partial r}\right)_{r=1} \quad ...(3.6)$$

On using the boundary conditions (2.7), we have

$$A_1 = -3\left\{A_2(3x_2^2-1) + A_3\left[\frac{1}{2}(3x_2^2-1)\log\left(\frac{1+x_2}{1-x_2}\right) - 3x_2\right]\right\},$$

$$A_2 = \frac{2C}{3L}\left[3x_1\log\left(\frac{1+x_1}{1-x_1}\right) + \left(\frac{3x_1^2-1}{1-x_1^2}\right) - 3\right],$$

and

$$A_3 = -\frac{4Cx_1}{L}, \quad ...(3.7)$$

where $L = \left\{(x_1^2 - x_2^2)\left[3x_1\log\left(\frac{1+x_1}{1-x_1}\right) + \left(\frac{3x_1^2-1}{1-x_1^2}\right) - 3\right] - x_1(3x_1^2-1)\log\left(\frac{1+x_1}{1-x_1}\right) + x_1(3x_2^2-1)\log\left(\frac{1+x_2}{1-x_2}\right) + 6x_1(x_1-x_2)\right\}$

$x_1 = \cos\theta_1$ and $x_2 = \cos\theta_2$.

(ii) *Pressure Distribution*

Equations (2.1), (2.2) and (3.4) give the pressure in the free fluid region as

$$p = \frac{1}{r^3}\left\{A_2(3x^2-1) + \frac{1}{2}A_3\left[(3x^2-1)\log\left(\frac{1+x}{1-x}\right) - 6x\right] + \frac{1}{3}A_1\right\} + \text{constant.} \quad ...(3.8)$$

Hence the pressure drop in r-direction is given by

$$\left[p(1,\theta) - p(r,\theta)\right] = \left(1-\frac{1}{r^3}\right)\left\{A_2(3x^2-1) + \frac{A_3}{2}\left[(3x^2-1)\log\left(\frac{1+x}{1-x}\right) - 6x\right] + \frac{1}{3}A_1\right\}. \quad ...(3.9)$$

(iii) *Flux of Flow*

Flux of flow per unit time in the free fluid region is given by

$$Q = \int_{\theta_1}^{\theta_2} \int_0^{2\pi} r^2 \, v_r \sin \theta \, d\theta \, d\phi .$$

Putting the value of v_r from (3.4), we get

$$Q = \pi \Big\{ A_2 (x_1^3 - x_2^3 - x_1 + x_2) + \frac{1}{2} A_3 \Big[(x_2 - x_2^3) \log \Big(\frac{1 + x_2}{1 - x_2} \Big) + (x_1^3 - x_1) \log \Big(\frac{1 + x_1}{1 - x_1} \Big) + 2 \Big(x_2^2 - x_1^2 \Big) \Big] + \frac{1}{3} A_1 \Big(x_1 - x_2) \Big) \Big\} . \qquad \ldots(3.10)$$

(iv) *Coefficient of Skin Friction*

Coefficient of skin friction at $\theta = \theta_1$ is given by

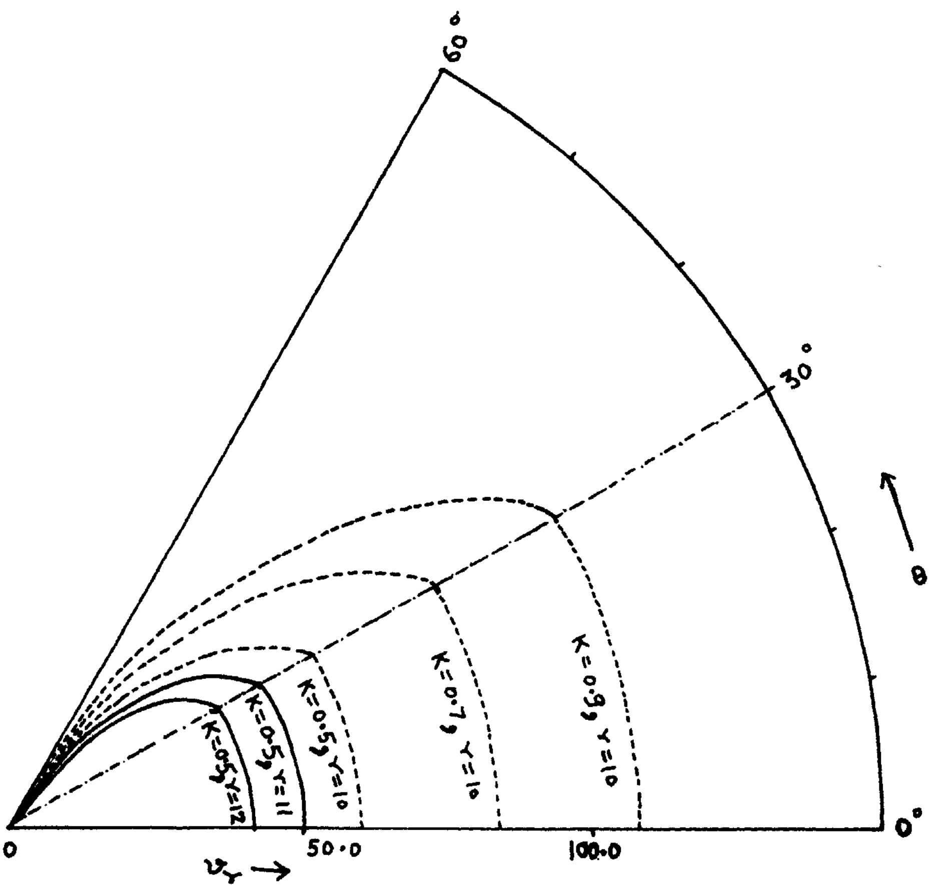

Fig. 1. Radial velocity v_r versus θ.

$$(C_f)_{\theta=\theta_1} = \frac{(1-x_1^2)^{1/2}}{r^3}\left\{3A_2x_1 + \frac{1}{2}A_3\left[3x_1\log\left(\frac{1+x_1}{1-x_1}\right) + \left(\frac{3x_1^2-1}{1-x_1^2}\right) - 3\right]\right\}. \quad \ldots(3.11)$$

Coefficient of skin friction at $\theta = \theta_2$ is given by

$$(C_f)_{\theta=\theta_2} = \frac{(1-x_2^2)^{1/2}}{r^3}\left\{3A_2x_2 + \frac{1}{2}A_3\left[3x_2\log\left(\frac{1+x_2}{1-x_2}\right) + \left(\frac{3x_2^2-1}{1-x_2^2}\right) - 3\right]\right\}. \quad \ldots(3.12)$$

4. Numerical Discussion

The radial velocity profiles are drawn for different values of permeability parameter K and the radial distance r in Fig 1. We find that the radial velocity increases

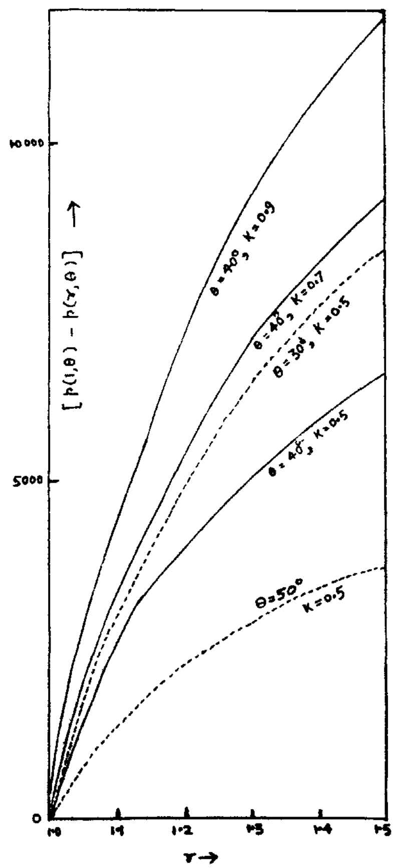

Fig. 2. Drop in radial pressure versus r.

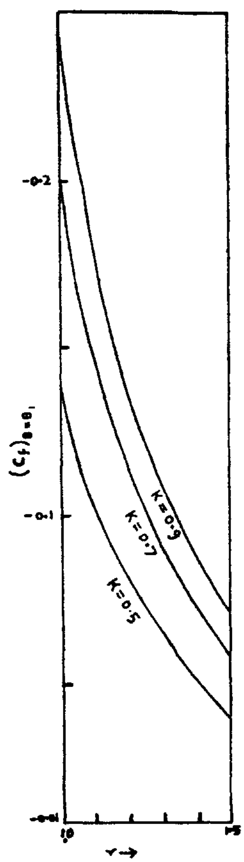

Fig. 3. Coefficient of skin friction versus r at $\theta=\theta_1$.

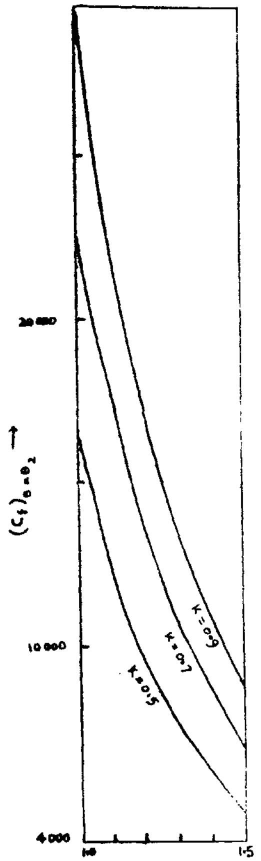

Fig. 4. Coefficient of skin friction versus r at $\theta=\theta_2$.

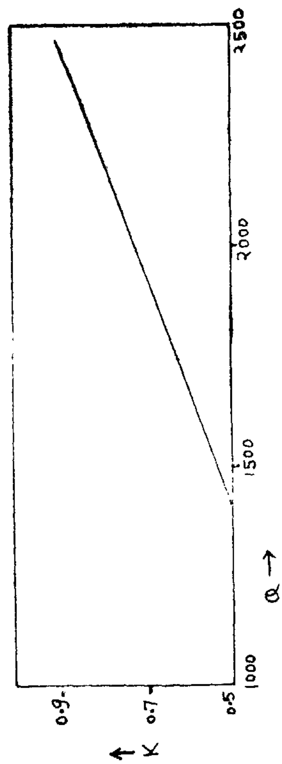

Fig. 5. Flux versus k.

with the increase of K, while it decreases with the increase of r. For a particular value of K and r, the velocity in the porous region is constant and it decreases in the free fluid region with the increase of θ and becomes zero at the outer impermeable tube. In Fig. 2, the drop in radial pressure has been drawn against r for different values of K and θ. It is noted that the drop in radial pressure increases with the increase of K and r while it decreases with the increase of θ. Figures 3 and 4 show the variation of the coefficient of skin friction at both the bouudaries. The coefficient of skin friction increases in magnitude with the increase of K, while it decreases with the increase of radial distance r at both the boundaries. Figure 5 depicts the variation of flux in the free fluid region. We find that the flux increases with the increase of permeability and is independent of the radial distance. [All the numerical analysis is done, keeping the initial pressure gradient $\left(-\frac{\partial P}{\partial r}\right)_{r=1} = 12000$].

The results may find applications in engineering, polymer industry etc.

References

Jones, I. P. (1973). Low Reynolds number flow past a porous spherical shell. *Proc. Camb. phil. Soc.*, **73**, 231.

Nishimura, J., and Oka, S. (1965). The steady flow of a viscous fluid through a tapered tube. *J. Phys. Soc. Japan*, **20**, 449.

Oka, S. (1964). The steady slow motion of a viscous fluid through a tapered tube. *J. Phys. Soc. Japan*, **19**, 1481.