Indian J. pure appl. Math., 13(8) : 966-972, August 1982

NOTE ON SPACE-TIMES WHICH ADMIT ISOTHERMAL-KRUSKAL TRANSFORMATIONS

M. NAGARAJ AND B. R. SRINIVASA
Department of Mathematics, University of Bangalore, Bangalore 560001

(*Received* 6 *June* 1981)

In this paper it is shown that transformations of the coordinates x^A (A = 1,2,3,4) of a space-time to Isothermal-Kruskal coordinates (u, x^2, x^3, v) will necessarily imply that the functions u and v are independent of x^2 and x^3 and depend only on x^1 and x^4. We have also obtained conditions under which a space-time admits an Isothermal-Kruskal coordinate system. In this paper we have not, however, considered the removal of physical singularities of a space-time by means of such Isothermal-Kruskal coordinate transformations.

1. INTRODUCTION

The well-known 'Schwarzschild exterior solution' in spherical polar curvature coordinates (r, θ, ϕ, t) is

$$ds^2 = (1-2M/r)^{-1}dr^2 - (1-2M/r)dt^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad ...(1.1)$$

Kruskal (1960) transformed the coordinates (r, θ, ϕ, t) to the coordinates (u, θ, ϕ, v) such that (1.1) takes the following form:

$$ds^2 = f^2 (du^2 - dv^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad ...(1.2)$$

Note that the form (1.2) is what Synge (1960) has called 'isothermal' in his book. Further, the singularity at $r = 2M$ in (1.1) is removed in (1.2) if and only if the functions u, v and f are determined essentially uniquely by the following equations (see Kruskal 1960) :

$$u = \{(r/2M) - 1\}^{1/2} \exp (r/4M) \cosh (t/4M) \quad ...(1.3)$$

$$v = \{(r/2M) - 1\}^{1/2} \exp (r/4M) \sinh (t/4M) \quad ...(1.4)$$

$$f^2 = (32M^3/r) \exp (- r/2M). \quad ...(1.5)$$

The coordinate system (u,θ,ϕ,v) such that the singularity in the original metric in the spherically polar coordinate system is removed is called by McVittie (1979) 'Isothermal-Kruskal' or IK, for short.

In this paper we consider any orthogonal coordinate system (x^1,x^2,x^3,x^4) for the space-time so that the line-element is written in the following form:

$$ds^2 = A^2dx^{1^2} - D^2dx^{4^2} + B^2dx^{2^2} + C^2dx^{3^2} \quad ...(1.6)$$

Definition 1.1— We call (u,x^2,x^3,v) an Isothermal-Kruskal coordinate system, and the coordinate transformation (x^1,x^2,x^3,x^4) → (u,x^2,x^3,v) an Isothermal-Kruskal transformation whenever (1) the metric (1 6) is transformed into the form:

$$ds^2 = f^2(du^2 - dv^2) + \bar{B}^2\, dx^{2^2} + \bar{C}^2\, dx^{3^2} \qquad ...(1.7)$$

and (2) any singularity present in the form (1.6) is completely removed in the form (1.7).

In the sequel, we abbreviate 'Isothermal-Kruskal' by IK, as is also done in Synge (1960).

Condition (1) of Definition (1.1) holds only if the functions u and v depend only on x^1 and x^4. This is established in section 2. We study the conditions under which the requirement (2) of Definition (1.1) is satisfied. Section 3 is devoted for this purpose. Finally, we consider some examples in section 4.

2. Equations of Transformation

Consider the transformation: $(x^1,x^2,x^3,x^4) \to (u,x^2,x^3,v)$ where u and v are two smooth functions of x^A $(A = 1,2,3,4)$. Let u_A and v_A denote respectively the partial derivatives of u and v w.r.t. x^A. Comparing the equation (1.6) with (1.7) we obtain the following equations:

$$u_A u_B - v_A v_B = 0 \qquad ...(2.1a)$$

$$f^2(u_B^2 - v_B^2) = P_B, \qquad ...(2.1b)$$

where P_B is given by

$$P_1 = A^2,\ P_2 = B^2 - \bar{B}^2,\ P_3 = C^2 - \bar{C}^2,\ P_4 = -D^2. \qquad ...(2.2)$$

Equation (2.1b) leads to

$$u_B^2 = v_B^2 + f^{-2}P_B. \qquad ...(2.3)$$

Since the coordinate transformation is required to be IK, the function f should be different from both zero and infinity. Using eqn. (2.1a) in conjunction with eqn. (2.3), we obtain

$$P_A P_B f^{-2} = -(P_A v_B^2 + P_B v_A^2), \quad \text{where } A \neq B. \qquad ...(2.4)$$

Assume that both P_A and P_B are both different from zero throughout the region under consideration. Then f^{-2} is given by the following expression:

$$f^{-2} = -(v_A^2/P_A) - (v_B^2/P_B,\ A \neq B. \qquad ...(2.5)$$

Thus eqns. (2.4) and (2.5) imply

$$v_1^2/P_1 = v_2^2/P_2 = v_3^2/P_3 = v_4^2/P_4 \qquad ...(2.6)$$

whenever $P_1P_2P_3P_4 \neq 0$. Again, using eqn. (2.4) in (2.3) we obtain:

$$P_A u_B^2 = -P_B v_A^2. \qquad ...(2.7)$$

Thus, it is clear from eqn. (2.7) that both the functions P_A and P_B cannot be of the same sign. The other possibility is that one or both of these two functions vanish identically. We substantiate this further by showing that atleast one of the functions P_A $(A = 1,2,3,4)$ should vanish. For, otherwise eqn. (2.6) holds and using this in (2.3) leads to:

$$u_B^2 = -v_B^2 \; B = 1,2,3,4.$$

This equation cannot hold since the functions u and v are real-valued. Obviously, the functions P_1 and P_4 are everywhere different from zero. Thus, either (or both) of P_2 and P_3 should vanish. Suppose P_2 is different from zero. Put $A = 1$ and $B = 2$ in (2.4), or in (2.5), if P_2 is positive and $A = 4$, $B = 2$ if P_2 is negative. In either case, we are led to a contradiction by eqn. (2.7) since the function f is assumed to be everywhere finite and non-zero. Thus the function P_2 should necessarily vanish. In the same way the function P_3 should vanish identically. Put $A = 2,3$ and $B \neq 2,3$ in (2.4) and make use of the fact that both P_2 and P_3 vanish identically while P_1 and P_4 are both different from zero everywhere. This implies that v_2 and v_3 both vanish identically. Equation (2.3) implies that u_2 and u_3 also vanish identically, so that the smooth real functions u and v are functions of x^1 and x^4 alone. Equations (2.1), (2.3) and (2.5) now take the following form:

$$u_1 u_4 - v_1 v_4 = 0 \quad \text{...(2.8)}$$

$$u_1^2 = v_1^2 + f^{-2}A^2, \; u_4^2 = v_4^2 - f^{-2}D^2 \quad \text{...(2.9)}$$

$$f^{-2} = (D^{-1}v_4)^2 - (A^{-1}v_1)^2. \quad \text{...(2.10)}$$

The line-element (1.7) of the spacetime in IK coordinates takes the following form:

$$ds^2 = f^2(du^2 - dv^2) + B^2 \, dx^{2^2} + C^2 dx^{3^2}. \quad \text{...(2.11)}$$

The next section is devoted to the study of the conditions under which the IK coordinate transformations are admissible for any spacetime.

3. Admissibility of the IK coordinate system

We now consider eqns. (2.8)-(2.11). We first eliminate f^{-2} from (2.9) with the help of (2.10). The following equations are a consequence of (2.8):

$$u_1 = \eta D^{-1}A v_4 \text{ and } u_4 = \eta A^{-1}D v_1 \; (\eta = \pm 1). \quad \text{...(3.1)}$$

We have assumed that both the functions A and D appearing in (3.1) are positive throughout the region under consideration. We further write: $g = D^{-1}A$ in (3.1) so that the equation now takes the form

$$u_1 = \eta g v_4 \text{ and } u_4 = \eta g^{-1} v_1. \quad \text{...(3.2)}$$

The two equations (3.2) are first order partial differential equations for the function u in terms of the function v. The integrability conditions of these equations are as follows:

$$g v_{44} + g_4 v_4 + g^{-2} g_1 v_1 - g^{-1} v_{11} = 0. \quad \text{...(3.3)}$$

This differential equation is defined in the region $\mathcal{D}$ of the spacetime in which both g and g^{-1} assume only positive values.

The second-order linear partial differential equation (3.3), as it stands, is difficult to integrate. Note that it could be solved if the function g is known. We wish now to show that it could be completely integrated in the two special cases: (i) $g = g(x^1)$, (ii) $g = g(x^4)$.

Case (i)— We assume that g is a function of x^1 alone so that the differential equation (3.3) reduces to:

$$v_{44} + g^{-3}g_1v_1 - g^{-2}v_{11} = 0. \qquad ...(3.4)$$

The variables are separated by putting $v = X(x^1)\,T(x^4)$; we thus obtain two ordinary differential equations for X and T: Let K be some constant. Then

$$\ddot{T} - KT = 0,\; g^{-2}X'' - g^{-3}\,g_1X' - KX = 0. \qquad ...(3.5)$$

A dot denotes differentiation w.r.t. x^4 and a dash denotes differentiation w.r.t. x^1. Let $y = y(x)$ be a solution of

$$(g^{-1}y')' = 0. \qquad ...(3.6)$$

Since the function g is assumed to be neither zero nor infinity at any point of the domain, we see that the first integral: $y'=Lg\,(x^1)$ is neither zero nor infinity anywhere in the domain. Thus the variable x^1 in (3.5) can be changed to y. The equation then becomes

$$L\,\frac{d^2X}{dy^2} - KX = 0.$$

Thus the most general solution v for eqn. (3.4) is given by

$$v = \{P\exp(K^{1/2}y/L) + Q\exp(-K^{1/2}y/L)\}$$
$$\times\{R\exp(K^{1/2}x^4) + S\exp(-K^{1/2}x^4)\} \qquad ...(3.7)$$

where P,Q,R,S are arbitrary constants of integration and the variable y is given by the solution of the equation:

$$y' = Lg(x^1).$$

The Jacobian of the transformation: $(x^1,x^2,x^3,x^4) \rightarrow (u,x^2,x^3,v)$ is given by

$$J = u_1v_4 - u_4v_1.$$

Using (3.2) in this equation, we obtain the value of J as follows:

$$J = \eta\,(gv_4^2 - g^{-1}v_1^2). \qquad ...(3.8)$$

Equations (3.7) and (3.8) now lead to the equation

$$J = 4\eta gK\{PQR^2\exp(2K^{1/2}x^4) + PQS^2\exp(-2K^{1/2}x^4)$$
$$+ P^2RS\exp(2K^{1/2}y/L) + Q^2RS\exp(-2K^{1/2}y/L)\}. \qquad ...(3.9)$$

An identical analysis of the second case, namely when g is a function of x^4, leads to similar results. We now consider that case.

Case (ii) — In this case, g is a function of x^4 alone: $g = g(x^4)$. In this case, therefore, the differential equation reduces to the following:

$$gv_{44} + g_4v_4 - g^{-1}v_{11} = 0. \qquad ...(3.10)$$

Separation of the variables as in case (i) leads to the two ordinary differential equations

$$g^2\,\ddot{T} + gg_4\dot{T} - KT = 0, \quad X'' - KX = 0 \qquad ...(3.11)$$

where $v(x^1,x^4) = X(x^1)T(x^4)$ and K is a constant [not the same as that which appears in eqn. (3.5)]. We introduce a new variable $z = z(x^4)$ as the solution of differential equation.

$$(g\dot{z}) = 0. \qquad \text{...(3.12)}$$

Let L be an arbitrary constant of integration. Then the first integral of eqn. (3.12) is:

$$\dot{z} = Lg^{-1}. \qquad \text{...(3.13)}$$

The change of variable $x^4 \to z$ in (3.11) leads to

$$\frac{d^2T}{dz^2} - (K/L)T = 0, \quad X'' - KX = 0. \qquad \text{...(3.14)}$$

Thus the most general solution of (3.10) is as follows:

$$v(x^1,x^4) = \{P \exp(K^{1/2}x^1) + Q \exp(-K^{1/2}x^1)\} \times \{R \exp(K^{1/2}z/L) + S \exp(-K^{1/2}z/L\}. \qquad \text{...(3.15)}$$

Here P,Q,R,S represent arbitrary constants of integration (not the same as appear in eqn. (3.7)). The Jacobian of the transformation: $(x^1,x^2,x^3,x^4) \to (u,x^2,x^3,v)$ can be calculated in the same way as in case (i); but we shall not do it here, as we shall not require this particular case in our further discussion.

4. Maximal Extendibility of the Spacetime

Let (M,g) be an Einstein-Riemann manifold of dimension 4. We assume that M is smooth (i.e., C^∞). Let g be given in the coordinate chart (x^1,x^2,x^3,x^4) by (1,6). Further, let the C^∞ function defined by the symbol g as in (3.2) be neither 0 or ∞ anywhere in the coordinate neighbourhood. In such a case, there are no abnormal properties of the spacetime when referred to the coordinate system (x^1,x^2,x^3,x^4) or to the IK coordinate system (u,x^2,x^3,v). However, the situation is different if the function g assumes the value 0 or becomes infinite over a subset of the coordinate neighbourhood. For example, consider the Schwartzschild spacetime referred to polar coordinates as in eqn. (1.1). The function g in this case is given by

$$g = \left(1 - \frac{2M}{r}\right)^{-1}$$

which certainly becomes infinite on the surface $r = 2M$. In such cases, the question is: what further conditions should be satisfied in order that the 'singular behaviour' in g is resolved in the IK coordinate system? The main reason for seeking this question is the fact that the function g appears in an essential form in the Jacobian of the IK coordinate transformation.

In this paper, we consider only the situation resulting from the Case (i) discussed in section 3. (Of course, an identical analysis could be made for the situation determined by Case (ii) of section 3 also. But we shall not make it in this paper.) The funtion g is then a function of x^1 alone. Let us write that in the following form:

$$g = (x^1 - x_0^1)^{\overrightarrow{m}} g(x_1) \qquad \text{...(4.1)}$$

where m is a real number and the function $\bar{g}$ is smooth and does not have a zero in an open neighbourhood of x^1_0; we shall also assume that it remains bounded in this neighbourhood. There are two cases to be considered here. These arise as a consequence of the requirement that the Jacobian J given by (3.9) should always remain non-zero and finite. Substitute (4.1) into (3.9). Then J remains finite and non-zero if and only if either (i) $P = 0$, $Q \neq 0$; or (ii) $P \neq 0$, $Q = 0$. Then J takes the following form:

$$J = 4\eta K\bar{g}Q^2RS \exp\left\{m \log (x^1 - x^1_0) - 2K^{1/2}y/L\right\} (P = 0,\ Q \neq 0) \quad \text{...(4.2a)}$$

$$J = 4\eta K\bar{g}P^2RS \exp\left\{m \log (x^1 - x^1_0) + 2K^{1/2}y/L\right\} (P \neq 0, Q = 0). \quad \text{...(4.2b)}$$

Note that both the constants R and S are different from zero. The function $y = y(x^1)$ is the solution of the differential equation: $\frac{d}{dx^1}y = Lg$, where L is a constant. The requirement that the Jacobian J should remain bounded and non-zero in an open neighbourhood of x^1_0 requires that y should be of the following form:

$$y = \begin{cases} (Lm/2K^{1/2}) \log (x^1 - x^1_0) + F(x^1) & \text{...(4.2a)} \\ -(Lm/2K^{1/2}) \log (x^1 - x^1_0) + F(x^1) & \text{...(4.2b)} \end{cases}$$

where the (smooth) function $F(x^1)$ remains bounded in an open neighbourhood of the point x^1_0. Substituting $F(x^1) = (x^1 - x^1_0)^n\bar{F}(x^1)$, where n is a non-negative integer and $\bar{F}(x^1)$ is a smooth finite function which is nowhere zero in an open neighbourhood of x^1_0, into the right hand side of the equation for y above, we obtain

$$y = (x^1 - x^1_0)^n \bar{F}(x^1) \pm (Lm/K^{1/2}) \log (x^1 - x^1_0)^{1/2}. \quad \text{...(4.3)}$$

The positive or the negative sign in (4.3) is chosen according as eqn. (4.2a) or (4.2b) is valid. The value of m can now be determined by substituting (4.3) into the differential equation giving y: $y' = Lg = L(x^1 - x^1_0)^m\bar{g}(x^1)$. In fact, it is easily seen to be -1. This in turn enables us to determine the function $\bar{g}(x^1)$:

$$\bar{g}(x^1) = \pm \frac{m}{2K^{1/2}} + n(x^1 - x^1_0)^n\bar{F}(x^1) + (x^1 - x^1_0)^{n+1}\bar{F}'(x^1):$$

where n is a non-negative integer and $m = -1$. Thus we obtain:

Theorem 4.1— The IK coordinate transformation $(x^1,x^2,x^3,x^4) \rightarrow (u,x^2,x^3,v)$ enables us to extend the spacetime:

$$ds^2 = A^2dx^{1^2} - D^2dx^{1^2} + B^2dx^{2^2} + C^2dx^{3^2}$$

to a (coordinate) singularity-free IK spacetime

$$ds^2 = f^2(du^2 - dv^2) + B^2dx^{2^2} + C^2dx^3$$

beyond $x^1 = x^1_0$, at which point the function $g = D^{-1}A$ which is assumed to be a function of x^1 alone, is of the form:

$$g(x^1) = (x^1 - x_0^1)^{-1} \{\mp (4K)^{-1/2} + n(x^1 - x_0^1)^n \bar{F}(x^1) + (x^1 - x_0^1)^{n+1} \bar{F}'(x^1)\} \quad \ldots(4.4)$$

where n is a non-negative integer and the smooth function $\bar{F}$ is finite and non-zero in an open neighbourhood of the point x_0^1. The sign to be chosen in (4.4) corresponds to the choice of sign in eqn. (4.3).

When the conditions of the Theorem 4.1 are satisfied, the functions u and v appearing in the IK coordinate transformation are given by

$$\left.\begin{aligned} u &= \eta \int (g\, v_4\, dx^1 + g^{-1} v_1\, dx^4) \\ v &= \exp(\pm K^{1/2} y/L) \{R \exp(K^{1/2} x^4) + S \exp(-K^{1/2} x^4)\} \end{aligned}\right\} \quad \ldots(4.5)$$

where the choice of sign is determined by that in equation (4.3).

The function f^2 appearing in the IK form of the line element given in Theorem 4.1 is now easily determined with the help of eqn. (2.10) and (4.5) as follows:

$$f^2 = A^2 (g^2 v_4^2 - v_1^2)^{-1} = -\frac{D^2}{4KRS} \exp(\mp 2K^{1/2} y/L). \quad \ldots(4.6)$$

The procedure discussed here can be extended to the case when the function $g(x^1)$ or its reciprocal has more than one zero. Such a situation does exist for meaningful physical solutions, e.g. the Reissner-Nördstrom solution. We shall not discuss that here. However, we give the formulae for v and J in the case of the Schwarzschild spacetime corresponding to a mass M situated at the point $r = 0$. The function $g = D^{-1}A$ is $(1 - 2M/r)^{-1}$. The function v is given by

$$v = (r - 2M)^{1/2} \exp(r/4M) \{R \exp(x^4/4M) + S \exp(-x^4/4M)\},$$

$$f^2 = 32M^3 r^{-1} \exp(-r/2M),$$

$$J = (P^2 RS/4M^2) \exp(h/2M - 1).$$

References

Kruskal, M. D. (1960). Maximal Extension of Schwarzschildmetric. *Phys. Rev.*, 119, 1743.

McVittie, G. C. (1979). Isothermal-Kruskal Coordinates and Einsten equations. *Commun. Soc. Edinb.*, 215–16.

Synge, J.L. (1960), Relativity-The General Theory. North-Holland Publishers.