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ROTATING COMPRESSIBLE FLOW DUE TO NON-TORSIONALLY OSCILLATING DISK

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A semi infinite expanse of a viscous compressible gas bounded by a disk and initially in a state of rigid rotation is disturbed by small amplitude non-torsional oscillations imposed on the disk. The initial density distribution is a function of the distance from the axis of rotation. This makes the unsteady flow created by the oscillations three dimensional. The Laplace transform technique is used to derive the expressions for velocity and temperature. The diffusive wave motion during the transient time is discussed. The structure of boundary layers formed due to the interactions of compressibility, viscous and Coriolis forces shows several interesting features in comparison with the non-oscillatory case or the case of an incompressible fluid.

1. INTRODUCTION

The theory of rotating fluids has rapidly grown in recent times. The development of the theory, associated with certain physical situations in geophysics and astrophysics, started with the investigations on spin-up, a process of adjustment of fluid from one state of rigid rotation to another state. While incompressible flow theories on spin-up are well developed (Greenspan 1969), the theory of compressible rotating fluids is still developing through the works of Matsuda *et al.* (1975), Iwao Harada (1979) and Venkatasiva Murthy (1981).

The theory of incompressible rotating fluids has also developed through the studies of unsteady motion over a state of rigid rotation, created due to torsional or non-torsional oscillations of the boundary. While the works of Jones (1969), Claire Jacobs (1971), Debnath (1975b) and Venkatasiva Murthy (1979, 1980) deal with motion created by torsional oscillations, the problems of unsteady motion generated by the non-torsional oscillations of a disk in a rigidly rotating incompressible fluid are investigated by Debnath (1974, 1975a) and Soundalgekar (1973). In these theories the structure of the unsteady motion, the growth of multiple boundary layers and the significant effects of rotation and suction at the disk are studied.

The aim of the present paper is to study the effects of compressibility on a rotating fluid subjected to non-torsional oscillations. A viscous compressible gas bounded by an infinite disk and initially in a state of rigid rotation is considered. At some instant, small amplitude non-torsional oscillations are imposed on the disk. The formulation is based on the lines of Iwao Harada (1979) and uses the Laplace transformation technique. For mathematical simplicity the case of unit Prandtl number is considered. The unsteady motion in the boundary layer consisting of diffusive wave systems is discussed. The ultimate boundary layers formed due to the balance of compressibility, viscous and Coriolis forces are determined. The forcing frequency and compressibility are shown to produce several interesting features in the flow in comparison with the non-oscillatory case or the case of an incompressible fluid.

2. Governing Equations

A semi infinite expanse of a compressible gas bounded by a single disk $z' = 0$ is initially in a state of rigid rotation and at time $t' = 0$, small amplitude non-torsional oscillations are imposed on the disk. In a rectangular cartesian coordinate system rotating with an angular velocity Ω, the initial and boundary conditions are

(i) $u' = v' = w' = 0, \rho' = \rho_0', \quad T' = T_0 \; (t' < 0)$

(ii) $u' = \epsilon U' \exp(i\alpha' t'), \quad v' = \epsilon V' \exp(i\alpha' t'), w' = 0,$

$$\rho' = \rho_0', \; T' = T_0 \text{ at } z' = 0 \; (t' > 0),$$

(iii) u', v', w', ρ', T' are bounded as $z' \to \infty$.

Here ϵ is a small non-dimensional parameter, $\bar{q}' = (u', v', w')$ is the velocity vector in the gas, α' is the frequency of oscillation, ρ' is the density and T' is the temperature in the gas. T_0 is the temperature in the undisturbed state. The density ρ_0' in the undisturbed state is obtained by integrating the equation of motion and using the equation of state for a perfect gas.

$$\rho_0' = \rho_c \exp[\gamma M^2 (x'^2 + y'^2)/2 L^2]$$

where ρ_c is the density on the axis of rotation, γ is the ratio of specific heats and M is the Mach number explained later. In the case of an incompressible fluid the density is simply a constant or a linear function of z' if the fluid is linearly stratified (Venkatasiva Murthy 1979), and hence uniform for all x' and y'. For non-torsional oscillations of the disk it will then be appropriate to assume $\frac{\partial}{\partial x'} = \frac{\partial}{\partial y'} = 0$ for all variables except pressure. Also the vertical velocity $w' = 0$ when there is no suction/injection at the disk. However, in the case of a compressible gas, the initial density is a function of x' and y'. The governing equations show that the velocity, temperature and density are coupled. Thus all the variables in the unsteady flow are to be treated as functions of x', y' and z'. If L is any characteristic length, the velocities are non-dimensionalised by ΩL. The characteristic scales for density, temperature and time are chosen as ρ_c, T_0 and Ω^{-1}. The non-dimensional variables are expanded in series of powers of ϵ as follows :

$$(u, v, w) = \epsilon (u_1, v_1, w_1) + \epsilon^2 (u_2, v_1, w_2) + \ldots$$

$$\rho = \rho_0 (1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \ldots)$$

$$T = 1 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \ldots ,$$

where

$$\rho_0 = \exp\left[\tfrac{1}{2}\, \gamma\, M^2 (x^2 + y^2)\right]$$

and the non-dimensional disturbance velocity $\bar{q}$ has components (u, v, w). Substitution of these expansions in the equations of motion written in a rectangular cartesian frame rotating with angular velocity Ω, the linearised non-dimensional equations, on omitting the subscript 1 are

$$\frac{\partial \rho}{\partial t} + \left(\nabla \cdot \bar{q}\right) + \gamma M^2 \left(u x + v y\right) = 0 \qquad \ldots(1)$$

$$\frac{\partial u}{\partial t} - 2 v = - x \theta - \frac{1}{\gamma M^2} \frac{\partial p}{\partial x} + \frac{E}{\rho_0} \left(\nabla^2 u + \frac{1}{3} \frac{\partial}{\partial x} \nabla \cdot \bar{q}\right) \qquad \ldots(2)$$

$$\frac{\partial v}{\partial t} + 2 u = - y \theta - \frac{1}{\gamma M^2} \frac{\partial p}{\partial y} + \frac{E}{\rho_0} \left(\nabla^2 v + \frac{1}{3} \frac{\partial}{\partial y} \nabla \cdot \bar{q}\right) \qquad \ldots(3)$$

$$\frac{\partial w}{\partial t} = - \frac{1}{\gamma M^2} \frac{\partial p}{\partial z} + \frac{E}{\rho_0} \left(\nabla^2 w + \frac{1}{3} \frac{\partial}{\partial z} \nabla \cdot \bar{q}\right) \qquad \ldots(4)$$

$$\frac{\partial \theta}{\partial t} + (\gamma - 1) \nabla \cdot \bar{q} = \frac{\gamma E}{\rho_0 p} \nabla^2 \theta \qquad \ldots(5)$$

$$p = \rho + \theta \qquad \ldots(6)$$

where

$E = \mu/\rho_c\, \Omega\, L^2$ is the Ekman number,

$P = \mu\, C_p/k$ is the Prandtl number,

$\gamma = C_p/C_v$ is the ratio of specific heats, and

$M = \Omega\, L/[\gamma\, R\, T_0/m]^{1/2}$ is the Mach number.

Here μ is the viscosity, k the thermal conductivity, C_p, C_v the specific heats, R the universal gas constant and m the molecular weight of the gas. Making a Boussinesq approximation that the temperature variation gives rise to density variation keeping pressure constant to order E, the new variable $P^* = (E/\rho_0)\, p$ is defined. Further, in view of the well understood incompressible flow theory of rotating fluids we assume that w and z are of order $(E/\rho_0)^{1/2}$ in the boundary layer. The basic equations representing the flow, under the boundary layer approximations for derivatives, are

$$\rho = - \theta \qquad \ldots(7)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \gamma M^2 (u x + v y) = 0 \qquad \ldots(8)$$

$$\frac{\partial u}{\partial t} - 2 v + x \theta = \frac{E}{\rho_0} \frac{\partial^2 u}{\partial z^2} \qquad \ldots(9)$$

$$\frac{\partial v}{\partial t} + 2 u + y \theta = \frac{E}{\rho_0} \frac{\partial^2 y}{\partial z^2} \qquad \ldots(10)$$

$$\frac{\partial w}{\partial t} = - \frac{E}{\rho_0 \gamma M^2} \frac{\partial P^*}{\partial z} + \frac{4}{3} \frac{E}{\rho_0} \frac{\partial^2 w}{\partial z^2} \qquad \ldots(11)$$

$$\frac{\partial\theta}{\partial t}+(\gamma-1)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{\gamma E}{\rho_0 P}\frac{\partial^2\theta}{\partial z^2}. \qquad \ldots(12)$$

The initial conditions are

$$u=v=w=\theta=0\ (t<0).$$

The boundary conditions are

(i) $u=U\exp(i\alpha t),\ v=V\exp(i\alpha t),\ w=0,$

$\theta=0$ on $z=0$, ...(13a)

(ii) u, v, w, θ are bounded as $z\to\infty$. ...(13b)

equations (7)–(12) are Laplace transformed and combined to give a single differential equation for $\bar{v}$) (transforms of v), as

$$\left[L_z^2\ L_z'+(\gamma-1)M^2(x^2+y^2)L_z+4L_z'\right]\bar{v}=0 \qquad \ldots(14)$$

where

$$L_z=s-\frac{E}{\rho_0}\frac{\partial^2}{\partial z^2}\ \text{and}\ L_z'=s-\frac{E}{\rho_0 P}\frac{\partial^2}{\partial z^2}.$$

The Laplace transformed boundary conditions are

(i) $\bar{u}=U/(s-i\alpha),\ \bar{v}=V/(s-i\alpha),$

$\bar{w}=0,\ \bar{\theta}=0$ at $z=0$ ...(15a)

(ii) $\bar{u}, \bar{v}, \bar{w}, \bar{\theta}$ are boudded as $z\to\infty$. ...(15b)

3. The Solution

Equation (14) admits solutions of the form $\exp(\delta z)$ where δ satisfies a sixth order algebraic equation and it is not possible to find the roots in general and to find the exact inversions of transformed variables. Hence attention is confined to the case $P=1$ in this paper and this will not prevent from studying the compressibility effects on the flow. In the non-oscillatory case $\alpha=0$, the assumption $P=1$ is unnecessary for a steady problem and the steady solution valid for all values of P can be given. When $P=1$, the solution for the transformed variables is

$$(\bar{u},\bar{v},\bar{\theta})=(s-i\alpha)^{-1}\sum_{i=1}^{3}(U_i,V_i,\theta_i)\exp(-h_i z),$$

$$\bar{w}=(s-i\alpha)^{-1}G(s,\sigma^2)$$

where

$$(s-i\alpha)G(s,\sigma^2)=\sum_{i=1}^{3}\left\{-\frac{1}{2}(U_i x+V_i y)\gamma M^2 z\exp(-h_i z)\right.$$

$$+\left[\frac{\partial U_i}{\partial x}+\frac{\partial V_i}{\partial y}-\frac{1}{2}\gamma M^2(U_i x+V_i y)\right]h_i^{-1}$$

$$\left.[\exp(-h_i z)-1]+\frac{E}{\rho_0}\theta_i h_i[1-\exp(-h_i z)]\right\}$$

(equation continued on p. 977)

$$+ \sum_{i=2}^{3} \frac{2(-1)^{(2i+1)/2}\,\theta_i\,\sigma^2}{(\gamma-1)\,h_i}\,[1-\exp(-h_i z)],$$

$$h_1 = (s\,\rho_0/E)^{1/2},\quad h_{2,3} = [(s \pm 2\,i\,\sigma^2)\,\rho_0/E]^{1/2},$$

$$\sigma^4 = 1 + \frac{1}{4}(\gamma-1)\,P\,M^2\,(x^2+y^2),\quad U_1 = -\frac{y}{x}\,V_1,$$

$$U_{2,3} = \frac{i\,\sigma^2\,x \mp y}{i\,\sigma^2\,y \pm x}\,V_{2,3},\quad \theta_1 = \frac{2}{x}\,V_1,$$

$$\theta_{2,3} = \frac{\mp 2(\sigma^4-1)}{i\,\sigma^2\,y \pm x}\,V_{2,3},\quad V_1 = V - V_2 - V_3,$$

$$V_{2,3} = \tfrac{1}{4}\,[\mp 2\,(U+y\,V/x)\,(i\,\sigma^2\,x+y)+2\,V\,(x^2+y^2)/x]\,[x \pm i\,\sigma^2\,y]\,\sigma^{-4}\,(x^2+y^2)^{-1}$$

The Laplace inversion gives the following solutions for u, v, w and θ satisfying the required boundary conditions (13).

$$(u, v, \theta) = (U_1, V_1, \theta_1)\,f^+(\alpha, 0) + (U_2, V_2, \theta_2)\,f^+(\alpha, \sigma^2) + (U_3, V_3, \theta_3)\,f^+(\alpha, -\sigma^2) \qquad \ldots(16)$$

$$\begin{aligned} w = {} & F(\alpha, 0)\,[f^-(\alpha, 0) - 2\,erf\,(i\,\alpha\,t)^{1/2}] \\ & + \{F(\alpha, \pm\sigma^2)[f^-(\alpha, \pm\sigma^2) - 2\,erf\,(i\,(\alpha \pm 2\,\sigma^2)\,t)^{1/2}]\} \\ & + F_1(\alpha, 0)\,f^+(\alpha, 0) + \{F_1(\alpha, \pm\sigma^2)\,f^+(\alpha, \pm\sigma^2)\} \\ & + [\theta_1 + \theta_2 \exp(-2\,i\,\sigma^2\,t) + \theta_3 \exp(2\,i\,\sigma^2\,t)]\,[1-\exp(-\rho_0\,z^2/4\,Et)] \\ & [E/\rho_0\,\pi\,t]^{1/2} + 2\,\{F_2(\alpha, \pm\sigma^2)\exp(\mp 2i\,\sigma^2\,t)\}\,[E\,t/\rho_0\,\pi]^{1/2} \\ & [\exp(-\rho_0\,z^2/4\,E\,t) - 1] + \Big\{F_2(\alpha, \pm\sigma^2)\Big(-\frac{z}{2}\,f^+(\alpha, \pm\sigma^2) \\ & - \frac{1}{2}(E/\rho_0)^{1/2}\,f^-(\alpha, \pm\sigma^2)\,[i\,(\alpha \pm 2\,\sigma^2)]^{-1/2} + (E/\rho_0)^{1/2}\,[i\,(\alpha \pm 2\,\sigma^2)]^{-1/2} \\ & erf\,[i\,(\alpha \pm 2\sigma^2)\,t]^{-1/2}\Big)\Big\} \end{aligned} \qquad \ldots(17)$$

where each expression in the brackets { } takes upper sign once and lower sign once and the sum should be taken.

$$\begin{aligned} F(\alpha, \sigma^2) = {} & \Big[\frac{\partial U_2}{\partial x} + \frac{\partial V_2}{\partial y} - \frac{1}{2}\,\gamma\,M^2\,(U_2\,x + V_2\,y) - i\,\theta_2\,(\alpha + 2\,\sigma^2) \\ & - \frac{2\,i\,\theta_2\,\sigma^2}{(\gamma-1)}\Big]\exp(i\,\alpha\,t) \Big/ 2\,[i\,(\alpha + 2\sigma^2)\rho_0/E]^{1/2}, \end{aligned}$$

$$F_1(\alpha, \sigma^2) = -\frac{1}{4}\,(U_2\,x + V_2\,y)\,\gamma\,M^2\,z\exp(i\,\alpha\,t),$$

$$F_2(\alpha, \sigma^2) = \gamma\,M^2\,(U_2\,x + V_2\,y)/4\,(\alpha + 2\,\sigma^2)\,\sigma^2,$$

$$f^{\pm}(\alpha, \sigma^2) = \exp\{-z\,[i\,(\alpha + 2\,\sigma^2)\,\rho_0/E]^{1/2}\}\,erfc\,\Big\{\frac{z}{2}\,\Big(\frac{\rho_0}{E\,t}\Big)^{1/2}$$

(*equation continued on p.* 978)

$$- [i(\alpha+2\sigma^2)t]^{1/2}\Big\} \pm \exp\{z[i(\alpha+2\sigma^2)\rho_0/E]^{1/2}\}$$

$$erfc\left\{\frac{z}{2}\left(\frac{\rho_0}{E t}\right)^{1/2} + [i(\alpha+2\sigma^2)t]^{1/2}\right\}.$$

The steady state solution in the limit $t \to \infty$ is given by

$$(u, v, \theta) = \left\{\sum_{i=1}^{3} (U_i, V_i, \theta_i) \exp(-g_i z)\right\} \exp(i\alpha t), \qquad \ldots(18)$$

$$w = G(i\alpha, \sigma^2) \exp(i\alpha t) \qquad \ldots(19)$$

where $g_1 = [i\alpha\rho_0/E]^{1/2}$, $g_{2,3} = [i(\alpha \pm 2\sigma^2)\rho_0/E]^{1/2}$.

4. Discussion of the Results

(i) *Wave Motion in the Boundary Layer*

The transient motion is represented by the solution in equations (16) and (17). The function *erfc* appearing in f^+ and f^- with a complex argument can be conveniently expanded using a result due to Strand (1965) in a series of functions suitable to explore the wave character. Development of the expressions f^+ and f^- on the lines of Venkatasiva Murthy (1980) shows the existence of wave motion in the boundary layer. The difussing boundary layer interacts with the inertial oscillations, compressibility and Coriolis force in producing three different wave systems travelling away from the disk and represented by the functions

$$f^{\pm}(\alpha, 0), f^{\pm}(\alpha, \sigma^2) f^{\pm}(\alpha - \sigma^2).$$

These waves diffuse with effective diffusivity μ/ρ_c travelling away from the disk with velocities

$$[2\alpha E/\rho_0]^{1/2}, [2 \mid \alpha + 2\sigma^2 \mid E/\rho_0]^{1/2}, [2 \mid \alpha - 2\sigma^2 \mid E/\rho_0]^{1/2}.$$

The effective decay lengths of these three wave systems are of order

$$[E/\alpha\rho_0]^{1/2}, [E/ \mid \alpha + 2\sigma^2 \mid \rho_0]^{1/2}, [E/ \mid \alpha - 2\sigma^2 \mid \rho_0]^{1/2}$$

so that the waves damp out at times of order

$$\alpha^{-1}, \mid \alpha + 2\sigma^2 \mid^{-1}, \mid \alpha - 2\sigma^2 \mid^{-1}.$$

When $\sigma = 1$ (incompressible fluid) and $\alpha = 0$, the time is simply the viscous diffusion time of order Ω^{-1} in dimension. The waves represented by $f^{\pm}(\alpha, 0)$ are produced by the oscillation ($\alpha \neq 0$) of the disk and are absent in the non-oscillatory case $\alpha = 0$, when the disk moves with uniform velocities U and V in x and y directions. These waves are also absent in the case of an incompressible fluid even if $\alpha \neq 0$, as can be noted from the fact that $V_2 = V_3 = \frac{1}{2} V$ and $V_1 = 0$ when $\sigma = 1$. When $\sigma > 1$, $\alpha \neq 0$, the three wave systems travel with a velocity of the same order $[2\alpha E/\rho_0]^{1/2}$ when the frequency α of oscillations is large ($\alpha \gg 2\sigma^2$). When $\alpha \ll 2\sigma^2$ the velocities of the two wave systems represented by $f(\alpha, \pm\sigma^2)$ are approximately equal and of order $[4\sigma^2 E/\rho_0]^{1/2}$ while the velocity of the wave system represented by

$f^{\pm}(\alpha, 0)$ is $[2\alpha E/\rho_0]^{1/2}$ and the later wave velocity is smaller than that of the former two systems.

The third and fourth terms in equation (17) show that the fluid region diffuses parabolically with diffusivity $2E/\rho_0$ and the diffusing boundary layer supports harmonic oscillations. The period of inertial oscillations decreases with the increase in the Mach number.

(ii) *Boundary Layer Thickness*

When the transient motion decays, the steady state solution in the limit $t \to \infty$ is given by equations (18) and (19), showing the existence of three boundary layers of thickness order

$$[E/\alpha\,\rho_0]^{1/2},\ [E/\mid \alpha \pm 2\sigma^2 \mid \rho_0]^{1/2}$$

formed due to the balance of compressibility, Coriolis and viscous forces. The first mentioned layer arises purely in the presence of the forcing frequency α. When $\alpha = 0$, we have only two boundary layers. Thus to satisfy the boundary conditions, there arises a new boundary layer in addition to those observed in the non-oscillatory case. It should also be noted that for an incompressible fluid ($\sigma = 1$), $V_1 = 0$ and the first mentioned layer is absent even when $\alpha \neq 0$. When $\alpha \neq 0$, $\sigma \neq 1$, for low frequency oscillations $\alpha \ll 2\sigma^2$, the layer of order $[E/\alpha\rho_0)^{1/2}$. is the thickest of the three and hence is the effective depth of penetration. However for high frequency oscillations $\alpha \gg 2\sigma^2$, all the three layers penetrate to depths of same order $(E/\alpha\rho_0)^{1/2}$ Thus it is for intermediate frequency oscillations that the compressibility effects dominate in determining the depth of penetration. In the case of an incompressible fluid the layers of thickness order $(E/\mid \alpha \pm 2 \mid$ ——$)^{1/2}$ are the classical Stokes layers. In the non-oscillatory case of an incompressible fluid an Ekman layer order $E^{1/2}$ is formed on the disk.

It can easily be verified from solution (16)–(19) that the results of an incompressible fluid are obtained by putting $\sigma = 1$. All the variables except the pressure are functions of z only and the vertical velocity $w = 0$. When $\sigma \neq 1$, the vertical velocity w exists to compensate the variations in horizontal velocities with the radial distance from the axis of rotation. The Coriolis force and the compressibility manifest together in producing a non-zero vertical velocity in the boundary layer.

When the exponential terms in z in the steady solution decay, the conditions at the edge of the boundary layer are $u = v = \theta = 0$, $w \neq 0$. In the case of an incompressible fluid, U_i, V_i and θ_i are all constants and hence $w = 0$. The compressibility effects of the fluid requires a vertical velocity from outside to maintain the horizontal velocities in the boundary layer.

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