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STRONG MAGNETOGASDYNAMIC SHOCK WAVES WITH INCREASING ENERGY

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The total energy of the flow between the shock and the point of explosion is not, in general, constant for flows driven out by an expanding surface. Under the assumption that the flow is caused by an expanding cylindrical piston, this paper investigates the similarity flows existing behind strong shock waves in the presence of a transverse magnetic field when the total energy varies according to a prescribed law.

1. INTRODUCTION

Summers (1975) has constructed a spherically symmetric model which incorporates a self-consistent magnetic field and describes a blast wave in the solar wind caused by a solar flare. The same model may be adopted to include a 'driven' wave produced by a flare energy release that is time dependent. Rosenau and Frankenthal (1976) have also suggested that the total energy content of the flow may be expressed as time dependent. Similarity flows with energy increasing with time were studied by Jones (1953), Lees and Kubota (1957) and Rogers (1957). If energy increases with time, the solutions correspond to a blast wave produced by intense, prolonged solar flare activity when the wave is driven by fresh erupting solar plasma for some time and its energy tends to increase as it propagates from the sun (Summers 1975).

The assumption that the total energy of the flow between the shock and the point of explosion is constant, does not hold good for the flows driven out by an expanding surface since the pressure exerted by it on the gas increases the total energy between the shock and the expanding surface. Assuming that the flow is caused by an expanding cylindrical piston, it is proposed, in the present paper, to investigate the similarity flows existing behind strong shock waves in the presence of a transverse magnetic field when the total energy E is allowed to vary according to the law,

$$E = E_c\, t^s \qquad \ldots(1.1)$$

where E_c and s are constants. Only the positive values of s have been considered, for, when $s = 0$ the solution corresponds to a blast wave. The density and magnetic field of the undisturbed gas ahead of the advancing shock have been taken to be non-uniform. The flow is headed by a strong shock front and has an expanding surface as an inner boundary which is a contact discontinuity because there is no particle flow across it. The solutions are valid in the region between the contact discontinuity and the shock front. As discussed by Rogers (1958), cavity formation is theoretically feasible, though in Nature, the formation and maintenance of a cavity would be greatly inhibited by energy losses at the shock front and also by gravitation.

2. Fundamental Equations, Boundary Conditions and Similarity Transformations

The equations describing the one dimensional unsteady flow of a perfect gas are,

$$\frac{D\sigma}{Dt} + \sigma\frac{\partial u}{\partial r} + \frac{\sigma u}{r} = 0 \qquad \text{...(2.1)}$$

$$\frac{Du}{Dt} + \frac{1}{\sigma}\left[\frac{\partial p}{\partial r} + \frac{1}{2}\frac{\partial h^2}{\partial r} + \frac{h^2}{r}\right] = 0 \qquad \text{...(2.2)}$$

$$\frac{Dh}{Dt} + h\frac{\partial u}{\partial r} + \frac{hu}{r} = 0 \qquad \text{...(2.3)}$$

$$\frac{D}{Dt}(p\sigma^{-\gamma}) = 0 \qquad \text{...(2.4)}$$

where u, p, σ and h are respectively the velocity, pressure, density and transverse magnetic field tangential to the flow. The Rankine-Hugoniot boundary conditions in the case of a strong shock moving with velocity V take the form,

$$\left.\begin{aligned} u_1 &= \frac{2V}{\gamma+1}, \quad h_1 = h_0\frac{\gamma+1}{\gamma-1} \\ p_2 &= \frac{2\sigma_0 V^2}{\gamma+1} \text{ and } \sigma_1 = \sigma_0 = \frac{\gamma+1}{\gamma-1} \end{aligned}\right\} \qquad \text{...(2.5)}$$

where $u_1\ h_1$, p_1, σ_1 are the values of the variables immediately behind the shock front and σ_0 and h_0 are the values of the mass density and magnetic field immediately ahead of the shock. Initial conditions provided by the initial spatial distribution of the state variables are,

$$\sigma_0 = \sigma_c r^{-w} \qquad \text{...(2.6)}$$

$$u = p = T = 0 \qquad \text{...(2.7)}$$

$$h_0 = h_c r^{-m} \qquad \text{...(2.8)}$$

where σ_c and h_c are constants and $0 \leqslant w < 2$, $0 \leqslant m \leqslant 1$. At $t = 0$ a cylindrical piston begins to expand and propel the gas radially outward, creating a shock which propagates outward. The piston motion is assumed to obey a power law time dependence, r_p is proportional to t^N where r_p defines the position of the piston. This implies a boundary condition on the gas speed at the piston which suffices to determine the problem. Since we are concerned with self-similar motions, we postulate that the shock speed obeys the law,

$$V = V_c t^{N-1} \qquad \text{...(2.9)}$$

where V_c is a constant which depends on the speed of the piston. As is often the case in problems of this nature, it is more convenient to solve for the piston motion in terms of the shock motion rather than vice-versa. We would therefore adopt this point of view forthwith and instead of the piston speed take V_c as a known parameter of the problem.

We define a non-dimensional variable η by

$$\eta = \frac{r}{R} = r\, t^{-\delta} \left[\frac{\sigma_c}{\alpha\, E_c} \right]^{1/(4-w)} \qquad \ldots(2.10)$$

$$\delta = \frac{2+s}{4-w} = \frac{2}{2m+2-w} = N \qquad \ldots(2.11)$$

and $R(t)$ is the shock radius having the dimensions of length; α is some constant depending only on the parameters γ, s, w and m as shown below.

The total energy between the piston and the shock surface is given by

$$E = E_c\, t^s = 2\pi \int_{r_p}^{R} \left(\tfrac{1}{2}\sigma u^2 + \frac{p}{\gamma - 1} + \frac{h^2}{2} \right) r\, dr. \qquad \ldots(2.12)$$

Let us make use of the following similarity transformations:

$$\left.\begin{aligned} u &= \frac{r}{t}\, U(\eta), \quad \sigma = \frac{\sigma_c}{r^w}\, G(\eta) \\ p &= \frac{\sigma_c}{r^{w-2} t^2}\, P^2(\eta), \quad h = \frac{\sigma_c^{\frac{1}{2}}}{r^{(w-2)/2}\, t}\, H(\eta)\,. \end{aligned}\right\} \qquad \ldots(2.13)$$

The variable η assumes the values η_p and 1 at the piston surface and at the shock respectively. This enables us to write the piston radius $r_p = R\, \eta_p$.

As a consequence of (2.13), we have from (2.12),

$$R = \left(\frac{E_c}{2\pi J \sigma_c} \right)^{1/(4-w)} t^{(2+s)/(4-w)} \qquad \ldots(2.14)$$

Also, from (2.11), R is given by

$$R = \left(\frac{\alpha E_c}{\sigma_c} \right)^{1/(4-w)} t^{(2+s)/(4-w)} \qquad \ldots(2.15)$$

so that from (2.14) and (2.15), we get,

$$\alpha = \frac{1}{2\pi\, J}$$

where J and consequently α are functions of γ, s, w and m as mentioned earlier.

From (2.10) the shock radius $R = V_c\, t^N N^{-1}$. Hence from (2.14),

$$\frac{dR}{dt} = \left(\frac{2+s}{4-w} \right) \left(\frac{E_c}{2\pi\, J\, \sigma_c} \right)^{1/(4-w)} t^{(s-w-2)/(4+w)} = V_c\, t^{N-1} \qquad \ldots(2.16)$$

so that η can also be written as

$$\eta = \left(\frac{N}{V_e}\right) r\, t^{-N}.$$

Also (2.16) gives

$$V_c = \left(\frac{2+s}{4-w}\right)\left(\frac{E_c}{2\pi\, J\, \sigma_c}\right)^{1/(4-w)}$$

and

$$N = \frac{2+s}{4-w} = \delta$$

so that

$$\left(\frac{\alpha E_c}{\sigma_c}\right)^{1/(4-w)} V_e\, N^{-1}\,.$$

It is also convenient to define a dimensionless acoustic speed and Alfvén speed in terms of the variables Z and X respectively, where

$$\frac{\gamma p}{\sigma} = \frac{\gamma P}{R}\left(\frac{r}{t}\right)^2 = Z\left(\frac{r}{t}\right)^2 \qquad \ldots(2.17)$$

$$\frac{h^2}{\sigma} = \frac{H^2}{G}\left(\frac{r}{t}\right)^2 = X\left(\frac{r}{t}\right)^2. \qquad \ldots(2.18)$$

With these new variables, the governing equations and associated boundary conditions for a gas become,

$$\eta \frac{dU}{d\eta} + \frac{\eta}{G}\frac{dG}{d\eta}(U-\delta) + 2 - w = 0 \qquad \ldots(2.19)$$

$$2\gamma\eta\,(U-\delta)\frac{dU}{d\eta} + 2\eta\frac{dZ}{d\eta} + (2Z+\gamma X)\frac{\eta}{G}\frac{dG}{d\eta} + \gamma\eta\frac{dX}{d\eta} + 2\gamma U(U-1) + (2-w)(2Z+\gamma X) + 2\gamma X = 0 \qquad \ldots(2.20)$$

$$2X\eta\frac{dU}{d\eta} + (U-\delta)\,\eta\frac{dX}{d\eta} + (U-\delta)\,X\frac{\eta}{G}\frac{dG}{d\eta} - (6-w)\,XU - 2X = 0 \qquad \ldots(2.21)$$

$$(U-\delta)\,\eta\frac{dZ}{d\eta} - Z(\gamma-1)(U-\delta)\frac{\eta}{G}\frac{dG}{d\eta} + [w(\gamma-1)+2]\,ZU - 2Z = 0 \qquad \ldots(2.22)$$

which can be re-arranged to yield

$$U' = \frac{(\delta-U)\,U\,(U-1) - Z\,(a-2U) - X\,(b-U)}{\eta\,[(U-\delta)^2 - (X+Z)]} \qquad \ldots(2.23)$$

$$G' = \frac{G\,[\eta U' + (2-w)\,U]}{\eta\,(\delta-U)} \qquad \ldots(2.24)$$

$$Z' = \frac{Z\,[(\gamma-1)\,\eta U' + 2\gamma U - 2]}{\eta\,(\delta-U)} \qquad \ldots(2.25)$$

$$X' = \frac{X[\eta U' + 2(2U-1)]}{\eta(\delta - U)} \quad \ldots(2.26)$$

where $\gamma a = 2 + \delta(w-2)$, $2b = 2 + \delta(w-4)$ and a dash (′) denotes differentiation with respect to η. The regions of integration subject to the boundary conditions

$$U = \frac{2\delta}{\gamma+1}, \quad G = \frac{\gamma+1}{\gamma-1},$$

$$Z = \frac{2\gamma(\gamma-1)\delta^2}{(\gamma+1)^2}, \quad X = M_A^{-2}\delta^2 \frac{\gamma+1}{\gamma-1}$$

are,

$$\eta > 0, X + Z - (U-\delta)^2 > 0, \quad \frac{2\delta}{\gamma+1} \leqslant U \leqslant \delta.$$

where M_A is the Alfvén Mach number.

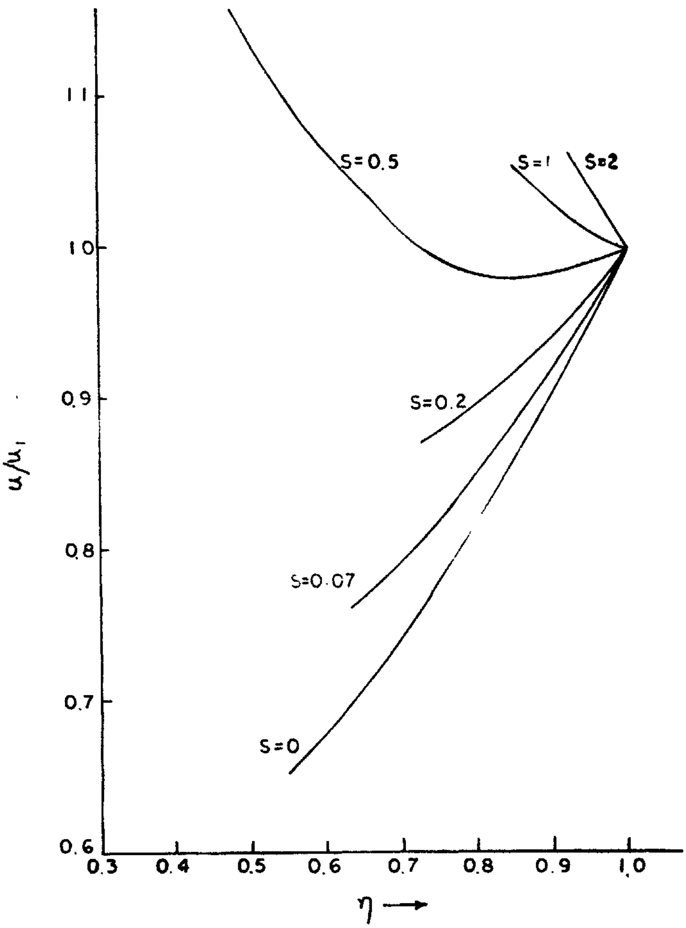

Fig. 1. Velocity Distribution.

From (2.11) and (2.13), the relationships between the similarity solution and the physical variables are,

$$\frac{U}{U_1} = \frac{(\gamma+1)\, U\eta}{2\delta} \quad \ldots(2.27)$$

$$\frac{\sigma}{\sigma_1} = \left(\frac{\gamma-1}{\gamma+1}\right) G\eta^{-w} \quad \ldots(2.28)$$

$$\frac{p}{p_1} = \frac{(\gamma+1)\, Z\, G\eta^{2-w}}{2\gamma\delta^2} \quad \ldots(2.29)$$

$$\frac{h}{h_1} = \frac{M_A\,(\gamma-1)\,(GX)^{1/2}}{\delta\,(\gamma+1)}\, \eta^{(2-w)/2} \quad \ldots(2.30)$$

$$\frac{T}{T_1} = \frac{(\gamma+1)^2\, Z\eta^2}{2\gamma\,(\gamma-1)\,\delta^2} \quad \ldots(2.31)$$

where the suffix 1 stands for values just behind the shock front.

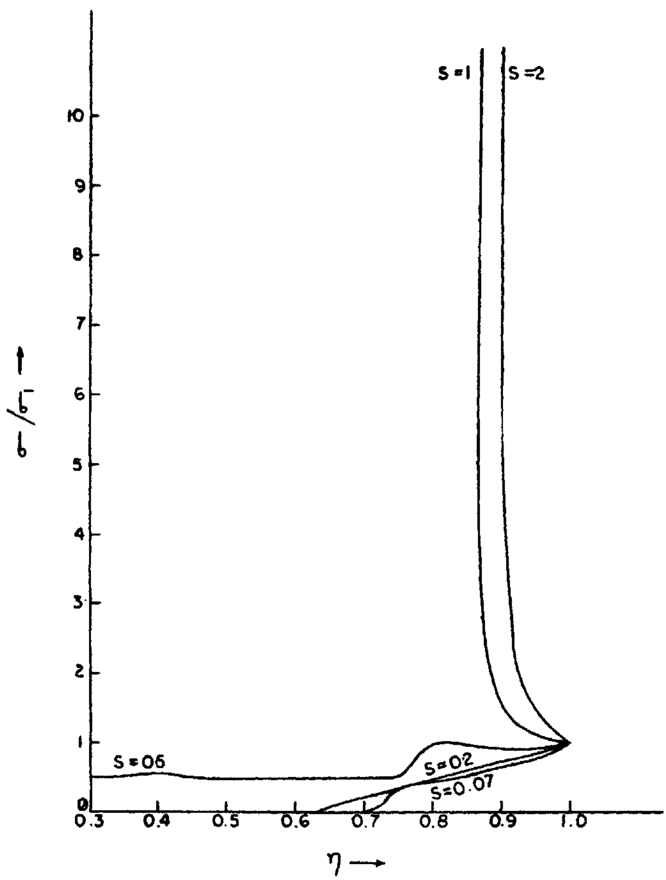

Fig. 2. Density Distribution.

3. Numerical Computation and Discussion

In order to have a clear picture of the behaviour of the flow and field variables, numerical computations have been carried out on digital computer employing Runge-Kutta method and corresponding graphs have been plotted. The following values for constants have been used.

$$w = 1, \quad \gamma = 1.4, \quad M_A = 10 \text{ and } s = 0, 0.07, 0.2, 0.5, 1, 2.$$

Figures 1-4 show the dependence of U/U_1, σ_1/σ, ρ/ρ_1 and H/H_1 respectively with η. It may be observed that as s increases from 0 to 2, all these ratios have an increasing tendency. At $s = 0.5$, however, they indicate a typical behaviour.

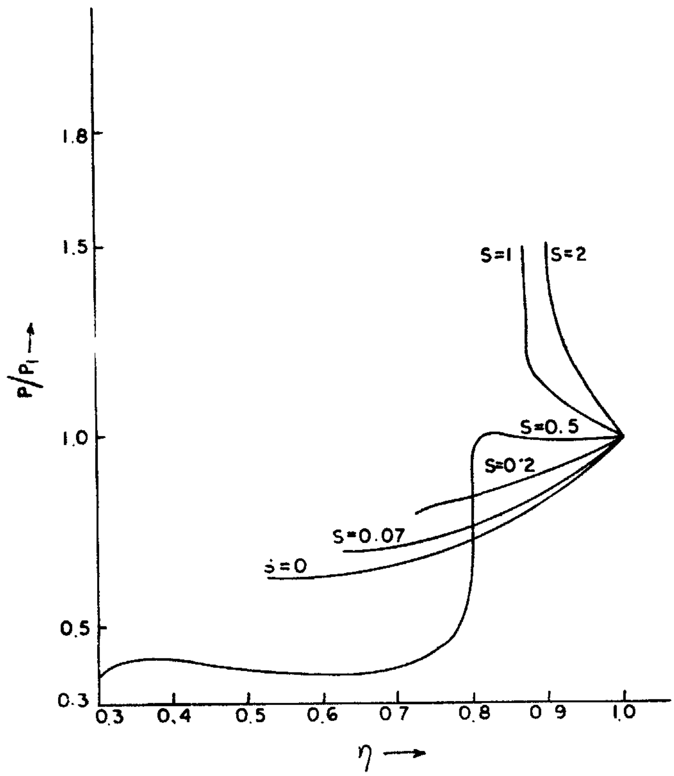

Fig. 3. Pressure Distribution.

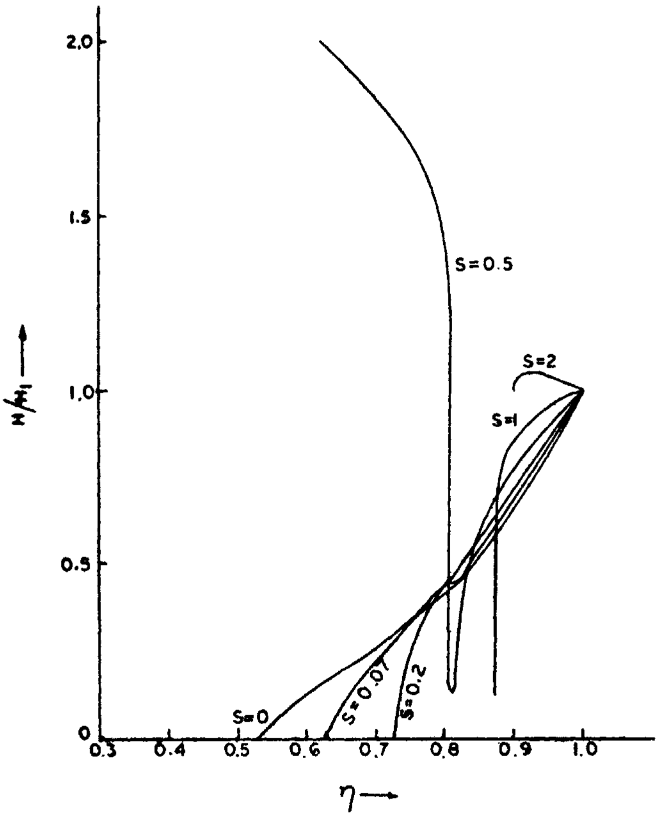

Fig. 4. Magnetic field distribution.

References

Jones, C.W. (1953). On reducible non-linear differential equations occurring in mechanics. *Proc. R. Soc.*, A 217, 327.

Lees, L., and Kubota, T. (1957). *J. Aeronaut. Sci.*, **24**, 195.

Rogers, M.H. (1957). Analytic solutions for the blast wave problem with an atmosphere of varying density. *Ap. J.*, **125**, 478.

———(1958). Similarity flows behind strong shock waves. *Q. Jl Mech. appl. Math.*, **11**, 411.

Rosenau, P., and Frankenthal, S. (1976). Equatorial propagation of axi-symmetric magnetohydrodynamic shocks. *Phys. Fluids*, **19**, (12), 1889.

Summers, D. (1975). An idealised model of a magnetohydrodynamic spherical blast wave applied to a flare produced shock in the solar wind. *Astr. Ap.*, **45**, 151.