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INSTABILITIES OF INTERNAL GRAVITY WAVES IN STRATIFIED SHEAR FLOWS

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The stability characteristics of internal gravity waves in a three-layered model for a stratified fluid with shear in the presence of a rigid boundary is considered. This configuration is conceived of as simulating atmospheric conditions. The effects of shear in the lowermost layer and the variation of the Brunt-Väisälä frequency in the middle layer of the three-layered model are studied. A jump in the Brunt-Väisälä frequency, at a level different from the shear zone which has a velocity discontinuity, is introduced. Unstable gravity modes, the most unstable of which resembles rather strikingly the disturbances observed in connection with clear-air turbulence, are found. This is brought out by comparision of observed wavelengths with those of the most unstable modes at several heights in the atmosphere.

1. INTRODUCTION

During the past several years observational and theoretical studies, carried out in the atmosphere and oceans, and related studies carried out in the laboratory, have significantly aided progress in understanding clear air turbulence [CAT]. These studies have shown that CAT often results from the hydrodynamic instability of internal waves. Small wave perturbations amplify and may eventually break when the wind shear becomes large enough to overcome the stabilizing influence of the density stratification.

In order to explain the characteristics of internal waves observed in the unstable atmospheric shear zones, Lindzen (1974) studied the stability characteristics of a Helmholtz profile in an infinite Boussinesq fluid with constant Brunt-Väisälä frequency N, and showed that perturbations with horizontal wavenumbers k, such that $k^2 > N^2/2U^2$, are unstable and decay away from the shear zone. Here, $2U$ is the velocity discontinuity at the interface. Subsequently, Lindzen and Rosenthal (1976) studied the stability characteristics of a Helmholtz velocity profile in the presence of a rigid boundary. Sachdev and Satya Narayanan (1981) studied the stability characteristics of a Helmholtz velocity profile in a compressible, stratified fluid with a rigid boundary. They considered the effect of a jump in the Brunt-Väisälä frequency at a level different from the shear zone. Satya Narayanan and Sachdev (1981) extended their previous work by including the variation of the Brunt-Väisälä frequency in the middle layer of a three-layered model. The wavelengths of the most unstable gravity modes for the models considered were in better agreement with the observed ones than those of Lindzen and Rosenthal. Other related works on the stability properties of shear flows in stratified fluids include Jones (1968), Mastrantonio *et al.* (1976), Lalas and Einaudi (1976), Einaudi and Lalas (1976) and Pellacani *et al.* (1978, 1979).

In this paper, we study the stability characteristics of gravity waves in a somewhat more general three-layered model (see Fig. 1) in which the lowermost layer above the ground has a constant shear capped above by a semi-infinite layer of constant velocity with a discontinuity in the velocity at the shear zone. In the middle layer the variation of the Brunt-Väisälä frequency is taken to be exponential in form. There is a jump in the Brunt-Väisälä frequency at a level different from that of the velocity discontinuity. We study the stability properties for the case when the Richardson number is less than 1/4. New unstable modes in the range of low horizontal wavenumbers are found. The agreement of the wavelengths of the most unstable gravity waves at different heights is rather good. While such closeness between the theory and observation is perhaps fortuitous, it nevertheless shows that the distributions of velocity and Brunt-Väisälä frequency in the background are not much different from those assumed here. This is fortified by the fact that the numbers quoted here do not change significantly if the gradient of velocity in the lower layer and the Brunt-Väisälä frequency discontinuity in the middle layer are altered to a small extent. Section 2 deals with the formulation of the problem and the derivation of the dispersion relation. Section 3 gives a discussion of the results while Section 4 presents the conclusions of the present study.

2. Formulation and Dispersion Relation

The velocity and Brunt-Väisälä frequency distributions in our model (Fig. 1) are as follows:

$$U_0 = -U + \alpha z, \quad N = N_0, \quad -D < z < 0 \text{ in layer } A$$

$$U_0 = +U, \quad N = N_0 \exp(sz) = N_1, \; 0 < z < d \text{ in layer } B$$

$$U_0 = +U, \quad N = N_2, \; d < z < \infty \text{ in layer } C.$$

The rigid boundary is at a distance D below the origin, and the jump in the Brunt-Väisälä frequency is situated at a distance d above the origin. The velocity is assumed to have a discontinuity at the origin.

It can be shown that under the Boussinesq approximation, the basic equations of motion can be reduced to a single differential equation (Lindzen 1974).

$$\left[\left(\frac{D}{Dt}\right)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial z^2}\right) + N^2 \frac{\partial^2}{\partial x^2}\right] w = 0 \qquad \ldots(1)$$

where $$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}.$$

U_0 is the mean horizontal velocity and N is the Brunt-Väisälä frequency. In obtaining (1) we have made use of the fact that d^2U_0/dz^2 is zero everywhere. Assuming sinusoidal forms for the flow characteristics, we can reduce (1) to an equation of the form

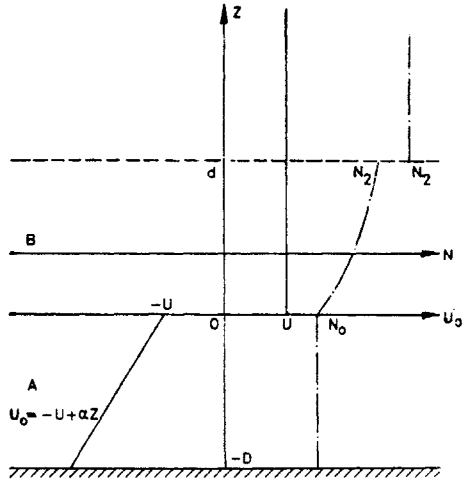

FIG. 1. Three-layered model with a rigid boun… The Brunt-Väisälä frequency jump is at a distance d above the origin.

$$\left[\frac{d^2}{dz^2} + k^2 \left(\frac{N^2}{\Omega^2} - 1 \right) \right] w = 0 \qquad \ldots(2)$$

where Ω is the Doppler-shifted frequency [$\Omega = \omega - kU_0$], k is the horizontal wave-number, and ω is the angular frequency.

The differential equations governing the motion in each of the layers of the three-layered model under consideration are

$$\frac{d^2\hat{w}}{dz^2} + k^2 \left[\frac{N_0^2}{\{\omega - k(\alpha z - U)\}^2} - 1 \right] \hat{w} = 0 \text{ [layer } A] \qquad \ldots(3)$$

$$\frac{d^2\hat{w}}{dz^2} + k^2 \left[\frac{N_0^2 \exp(2sz)}{(\omega - kU)^2} - 1 \right] \hat{w} = 0 \text{ [layer } B] \qquad \ldots(4)$$

$$\frac{d^2\hat{w}}{dz^2} + k^2 \left[\frac{N_2^2}{(\omega - kU)^2} - 1 \right] \hat{w} = \text{ [layer } C]. \qquad \ldots(5)$$

The solution of (3) can be obtained in terms of modified Bessel function I_ν and $I_{-\nu}$ as

$$\hat{w} = A_1 I_\nu(x) + A_2 I_{-\nu}(x) \qquad \ldots(6)$$

where $\nu = ((1/4) - (N_0^2/\alpha^2))^{1/2}$ and $x = (\omega + kU)/\alpha - kz$.

The solution of (4) can be written in terms of the confluent hypergeometric function by suitable transformations (Murphy 1960):

$$\hat{w} = \exp(k' sz + l \exp(sz) - sz/2) \{B_1 F(k', 2k'; x)$$
$$+ B_2 x^{1-2k'} F(1-k', 2-2k'; x)\} \qquad \ldots(7)$$

where $x = -2l \exp(sz)$, $l = ikN_0/s(\omega - kU)$, and $k' = 1/2 + k/s$.
The solution of (5) is

$$\hat{w} = c_1 \exp(i r_e z) \qquad \ldots(8)$$

where $r_e^2 = k^2 \{N_2^2/(\omega - kU)^2 - 1\}$ and $I_m(r_e) > 0$.

Applying the matching conditions at the interfaces (Chandrasekhar 1961), and the boundary condition $w = 0$ at $z = -D$ on the rigid boundary, we have the following dispersion relation:

$$(d_{11}d_{22} - d_{12}d_{21})\{d_{33}(d_{44}d_{55} - d_{45}d_{54}) - d_{34}(d_{43}d_{55} - d_{45}d_{53})\}$$
$$+ (d_{12}d_{31} - d_{11}d_{32})\{d_{23}(d_{44}d_{55} - d_{45}d_{54})$$
$$- d_{24}(d_{43}d_{55} - d_{45}d_{53})\} = 0 \qquad \ldots(9)$$

where the d_{ij} 's are given in the appendix.

3. Results and Discussion

The dispersion relation obtained in the previous section is solved numerically for the relevant values of the parameters $\hat{\alpha}$, $\hat{D}$, r, τ, α, and s. The parameters $\hat{\alpha}$ and r, respectively, give the Brunt-Väisälä frequency ratio, $\hat{\alpha} = N_1/N_2$ at $z = d$, and the ratio of the average densities above and below the Brunt-Väisälä frequency jump level. $\hat{D}$ is the non-dimensional parameter, equal to N_0D/U, which characterizes the height of the shear layer from the rigid boundary. τ is the non-dimensional ratio of the distance between the shear zone and the Brunt-Väisälä frequency jump level to the distance between the shear zone and the lower rigid boundary, that is, $\tau = d/D$. The parameter α is the gradient of shear, and s is a measure of the variation of the Brunt-Väisälä frequency in the middle layer of the three-layered model.

The stability properties depend on a large number of parameters so that one is forced to limit the investigation to specific parametric values and study the effects of shear, variation in the Brunt-Väisälä frequency, the presence of the rigid boundary, and the discontinuity in the Brunt-Väisälä frequency at a level different from the shear zone.

It is evident from the studies of Lindzen and Rosenthal (1976) that the curve associated with the Kelvin-Helmholtz instability in the presence of a rigid boundary behaves in a manner similar to the Kelvin-Helmholtz instability in the unbounded case at low horizontal wavenumbers. They showed that at values of k for which only neutral gravity waves existed in the unbounded case (as well as at values of γ where both gravity waves and Kelvin-Helmholtz instabilities coexisted), instabilities appeared when a rigid boundary below was introduced. This was confirmed by Sachdev and Satya Narayanan (1981). These instabilities were associated with the phenomenon of overreflection.

The presence of the Brunt-Väisälä frequency jump at a level different from the shear layer brings in new unstable modes in the range of low horizontal wavenumbers.

The structure of these unstable disturbances is similar to that of neutrally propagating gravity waves. The growth rates of these modes are rather small but significant because they appear in the range of horizontal wavenumbers which are otherwise stable.

Figures 2-4 present $\hat{C}_i$, the normalized imaginary part of the phase speed, for $\hat{\alpha}=0.8$, $\tau=1.0$, $s=0.1$, $\hat{D}=3.24$ and $\alpha=1.0$, 0.5, 1.5, respectively. The maximum growth rate of the most rapidly growing gravity mode at the value kD, where it takes place, is greater than that of the Kelvin-Helmholtz mode. The maximum value of $\hat{C}_i$ for the most unstable gravity mode is greater than the one obtained in an earlier paper by Satya Narayanan and Sachdev (1981). This shows that the presence of shear has a destabilizing effect on the gravity modes at low wavenumbers. Variation

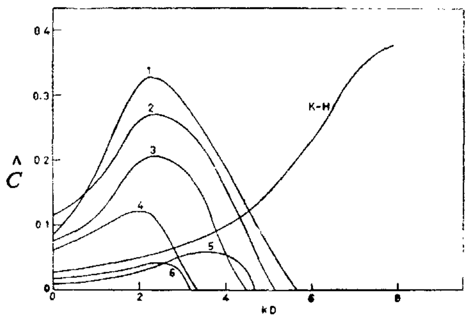

FIG. 2. Properties of unstable modes for $\hat{\alpha}=0.8$, $\tau=1.0$, $\alpha=1.0$, $s=0.1$, and $\hat{D}=3.24$ K—H : Kelvin-Helmholtz mode, 1-6 : Gravity modes.

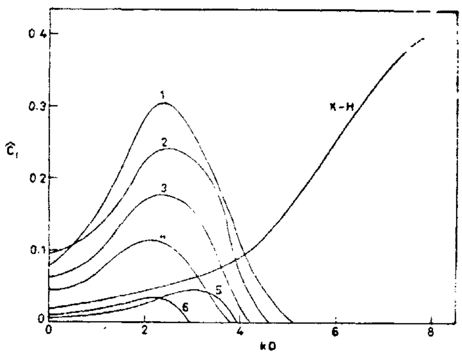

FIG. 3. Properties of unstable modes for $\hat{\alpha}=0.8$, $\tau=1.0$, $\alpha=0.5$, $s=0.1$, and $\hat{D}=3.24$. K—H : Kelvin-Helmholtz mode, 1-6 : gravity modes.

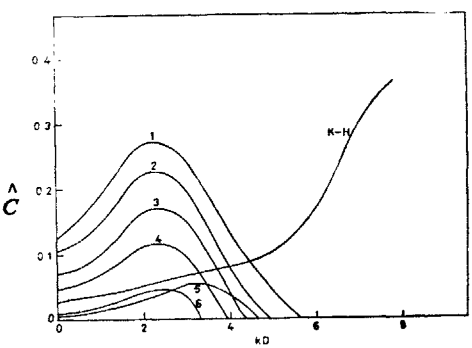

FIG. 4. Properties of unstable modes for $\hat{\alpha}=0.8$, $\tau=1.0$, $\alpha=1.5$, $s=0.1$, and $\hat{D}=3.24$, K—H ; Kelvin-Helmholtz mode, 1-6 : Gravity modes,

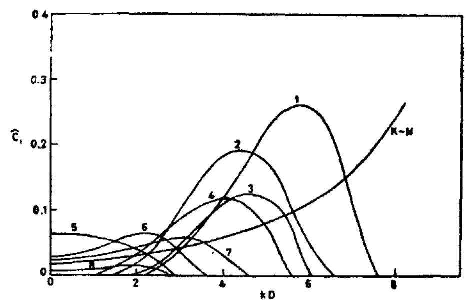

FIG. 5. Properties of unstable modes for $\hat{\alpha}=0.8$, $\tau=1.0$, $\beta=1.25$ and $\hat{D}=7.14$. K—H : Kelvin-Helmholtz mode, 1-8 : gravity modes.

of α does not produce a significant change on the stability characteristics of these modes. The number of unstable modes is the same as in Satya Narayanan and Sachdev (1981). Figure 5 depicts the characteristics of gravity waves for $\hat{\alpha} = 0.8$, $\tau = 1.0$, $\alpha = 1.0$, $s = 0.1$ and $\hat{D} = 7.14$. In this case, the number of unstable modes is more than the ones obtained by Satya Narayanan and Sachdev (1981). As the distance between the velocity discontinuity and the rigid boundary increases, new unstable modes at low horizontal wavenumbers appear. The maximum value of $\hat{C}_i$ for the most unstable mode for $\hat{D} = 7.14$ is less than that for $\hat{D} = 3.24$. Variation of N_1 in the middle layer does not produce significant changes on the growth rates and wavelengths of the unstable modes. As the parameter s in N_1 is varied from 0 to 1, the maximum value of $\hat{C}_i$ tends to decrease which indicates that an increase in Brunt-Väisälä frequency N_1 in the middle layer, has a stabilizing effect on these modes.

In order to have a quantitative idea of the stability characteristics of the gravity waves for the present configuration, we give in Table I, a comparative study of the results of Lindzen and Rosenthal (1976), Sachdev and Satya Narayanan (1981), and Satya Narayanan and Sachdev (1981), and the present work for parameters relevant to the experimental observations by Hook *et al.* (1973), Hardy *et al.* (1973), Ottersten *et al.* (1973), and Reed and Hardy (1972). In Table I, we have chosen $\tau = 0.25$ for

TABLE I

Stability characteristics of the most unstable gravity waves for the present model

Case	D km	U ms^{-1}	N_o, 10^{-2}s	Observed wave length, km.	λ_i, km L & R	S & S*	S & S	Present
1	.12	1	2.7	.35	.309	.32	.364	.344
2	2.5	7	2.0	2.7	3.04	2.91	2.86	2.76
3	11	10	2.0	6	4.41	4.9	5.1	5.58
4	9	28	1.3	15.20	17.9	17.2	16.5	15.42

Case	C_r/U L&R	S&S*	S & S	Present	C_i/U L & R	S & S*	S & S	Present	$(kC_i)^{-1}$, s L & R	S & S*	S & S	Present
1	−.15	.124	.114	.261	.257	.278	.31	.321	192	183	187	171
2	−.104	.178	.162	.126	.178	.236	.254	.258	387	281	256	243
3	.026	.052	.084	.0735	.109	.118	.1385	.142	647	660	586	625
4	.108	.117	.134	.226	.273	.228	.2416	.258	373	429	388	340

Here λ_i, wavelength; L & R — Lindzen and Rosenthal (1976); S & S* — Sachdev and Satya Narayanan (1981); S & S — Satya Narayanan and Sachdev (1981).

the cases 3 and 4. This corresponds to the sharp change in the Brunt-Väisälä frequency between 11-14 km, a value close to the actual distribution of Brunt-Väisälä frequency as quoted by Gossard and Hooke (1975).

4. Conclusions

The results obtained for the present configuration agree well with the experimental observations. As noted in the introduction, such close agreement of theoretical and observed values should be considered more as a coincidence; nevertheless we believe that the configuration we have considered is probably not far from the actual conditions prevailing at the heights where these instabilities occur. The insensitivity of the results to small changes in the shear in the lower layer and the distribution of the Brunt-Väisälä frequency in the middle layer, confirms the above conclusion.

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References

Chandrasekhar, S. (1961). Hydrodynamic and Hydromagnetic Stability. Oxford University Press, Oxford.

Einaudi, F., and Lalas, D. P. (1976). The effect of boundaries on the stability of inviscid stratified shear flows. *J. appl. Mech.*, **98**, 243–48.

Gossard, E. E., and Hooke, W. H. (1975). Waves in the Atmosphere. Elsevier, Amsterdam.

Hardy, K. N., Reed, R. J., and Mather, G. K. (1973). Observation of Kelvin-Helmholtz billows and their mesoscale environment by radar, instrumented aircraft, and a dense radiosonde network. *Q. Jl. Roy. Meteorol. Soc.*, 99, 279–93.

Hooke, W. H., Hall (Jr), F. F., and Gossard, E. E. (1973). Observed generation of an atmospheric gravity wave by shear instability in the mean flow of the planetary boundary layer. *Boundary Layer Meteorol.*, **5**, 29–41.

Jones, W. L. (1968). Reflection and stability of waves in stably stratified fluids with shear flow : A numerical study. *J. Fluid Mech.*, **34**, 609–24.

Lalas, D. P., and Einaudi, F. (1976). On the characteristics of gravity waves generated by atmospheric shear layers. *J. Atmos. Sci.*, **33**, 1248–59.

Lindzen, R. S. (1974). Stability of a Helmholtz velocity profile in a continuously stratified, infinite Boussinesq fluid-applications to clear air turbulence. *J. Atmos. Sci.*, **31**, 1507–14.

Lindzen, R. S., and Rosenthal, A. J. (1976). On the instability of Helmholtz velocity profiles in stably stratified fluids when a lower boundary is present. *J. Geophys. Res.*, **81**, 1561–71.

Mastrantonio, G., Einaudi, F., Fua, D., and Lalas, D. P. (1976). Generation of gravity waves by jet streams in the atmosphere. *J. Atmos Sci.*, 33, 1730–38.

Murphy, G. M. (1960). Ordinary Differential Equations and Their Solutions. D. Van Nostrand.

Ottersten, H., Hardy, K. R., and Little, C. G. (1973). Radar and sodar probing of waves and turbulence in statically stable clear air layers. *Boundary Layer Meteorol.*, 4, 47–89.

Pellacani, C., Tebaldi, C., and Tosi, E. (1978). Shear instabilities in the atmosphere in the presence of a jump in the Brunt-Väisälä frequency. *J. Atmos. Sci.*, **35**, 1633–43.

Pellacani, C., Tebaldi, C., and Tosi, E. (1979). Instabilities induced by over-reflection in stratified fluids in horizontal sheared motion. *Phys. Fluids*, **22**, 599–602.

Reed, R. J., and Hardy, K. R. (1972). A case study of persistent, intense clear air turbulence in an upper level frontal zone. *J. appl. Meteorol.*, **11**, 541–49.

Sachdev, P. L., and Satya Narayanan, A. (1981). Instabilities of a compressible stratified fluid in horizontal sheared motion. *Phys. Fluids*, 24, 1421–24.

Satya Narayanan, A., and Sachdev, P. L. (1981). Instabilities induced by variation of Brunt-Väisälä frequency in compressible stratified shear flows. *Phys. Fluids* (revised version submitted for publication).

APPENDIX

$$d_{11} = I_\nu(x_1),\ d_{12} = I_{-\nu}(x_1),\ d_{21} = \Omega_+^{-1} I_\nu(x_2),\ d_{22} = \Omega_+^{-1} I_{-\nu}(x_2),$$

$$d_{23} = -\Omega_-^{-1} F(k', 2k'; -2l),\ d_{24} = -\Omega_-^{-1}(-2l)^{1-2k'} F(k', 2-2k'; -2l),$$

$$d_{31} = \frac{-k\,\Omega_+}{2}(I_{\nu+1}(x_2) + I_{\nu-1}(x_2)) + \alpha k I_\nu(x_2),$$

$$d_{32} = \frac{k\,\Omega_+}{2}(I_{-\nu+1}(x_2) + I_{-\nu-1}(x_2)) + \alpha k I_{-\nu}(x_2),$$

$$d_{33} = \Omega_- \exp(l)\, C_1(-2l)^{1-2k'} F(I-k', 2-2k'; -2l)/k + \Omega_- \exp(l)$$
$$[(1-2k')(-2l)^{-2k'} F(1-k', 2-2k'; -2l)$$
$$+ (-2l)^{1-2k'} F(2-k', 3-2k'; -2l)/2],$$

$$d_{43} = \exp(C_2)\, F(k', 2k'; C_3),\ d_{44} = \exp(C_2)\, C_3^{1-2k'} F(1-k', 2-2k'; C_3),$$
$$d_{45} = -\exp(ir_e d),$$
$$d_{53} = \frac{\Omega_-}{k}\exp(C_2)\left(k's + ls\exp(sd) - \frac{1}{2}s\right) F(k', 2k'; C_3) - \frac{1}{2}\Omega_-\exp(C_2)$$
$$F(1+k', 1+2k'; C_3),$$
$$d_{54} = \frac{\Omega_-}{k}\exp(C_2)\left(ks + ls\exp(sd) - \frac{1}{2}s\right) C_3^{1-2k'} F(1+k', 1+2k'; +C_3)$$
$$-\Omega_-\exp(C_2)\,[(1-2k')C_3^{-2k'} F(1-k', 2-2k'; C_3)$$
$$+\frac{1}{2}C_3^{1-2k'} F(2-k', 3-2k'; C_3)],$$

$$d_{55} = \frac{r\,\Omega_-}{k}(ir_e)\exp(ir_e d) - \frac{g(1-r)}{k\,\Omega_-}\exp(ir_e d).$$

Here $\nu = \left(\frac{1}{4} - (N_e^2/\alpha^2)\right)^{1/2}$, $C_1 = Ck's + ls - \frac{1}{2}s$, $C_2 = k'sd + l\exp(sd) - sd/2$ and $C_3 = -2l\exp(sd)$.