ON THE CAUCHY-POISSON WAVE PROBLEM IN A ROTATING FLUID

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Based on the assumption of linearization, an attempt is made to study an unsteady flow produced by a harmonic pressure distribution of frequency ω (> 0) acting on the horizontal-free surface of an inviscid, incompressible fluid which is rotating with a uniform angular velocity Ω. Two cases (i) $\omega > 2\Omega$ and (ii) $\omega < 2\Omega$ are of particular interest. In accordance with Lamb's classification, it is shown that case (i) corresponds to 'waves of the first class' which are qualitatively similar to the classical surface waves and case (ii), in contrast to the first class waves, corresponds to 'waves of the second class' or inertial waves of frequency 2Ω which are originated entirely due to rotation and have no counterpart in a non-rotating fluid.

With the aid of the joint Laplace and generalised Hankel transformations together with asymptotic methods, the problem is analysed in considerable detail. Both the steady and the transient solutions are obtained explicitly. It is predicted that the solution asymptotically approaches to the steady state as $t \to \infty$. The significant effects of the Coriolis force and the characteristic features of the wave motions are investigated.

1. INTRODUCTION

The existence and importance of wave motions in an incompressible rotating fluid originated solely due to rotation have long been recognized. An early study of such wave phenomena was initiated by Kelvin (1879) and continued by Taylor (1922), Lamb (1932) and many others. The generating mechanism of these waves is often directly attributable to the Coriolis force $2\underset{\sim}{\Omega} \times \underset{\sim}{u}$ involved in the equation of motion.

Görtler (1944, 1957), Stewartson (1952) and Morgan (1951) have made some significant contributions to the wave motions in an inviscid, incompressible rotating fluid. Mention may also be made of the work of Mallick (1957) and Reynolds (1962*a*, *b*). The former author examined the motions originated by a vibrating sphere in a rotating fluid, whereas the latter studied the problem of forced oscillations in a rotating system.

One of the most significant properties of forced oscillations in a rotating fluid is the difference between the motions when the forcing frequency $\omega \gtrless 2\Omega$. More precisely, when $\omega > 2\Omega$, under the assumption of steady motion, the governing flow equation for the disturbance pressure is elliptic in space variables and has resemblance to the Laplace equation. Hence the characteristics of the flow are qualitatively similar to those of the non-rotating fluid. On the other hand, when $\omega < 2\Omega$, the field equation becomes hyperbolic in space coordinates and a fluid in steady rotation can support internal wave motions. In other words, small disturbances in a uniformly rotating fluid can propagate as wave motions through the medium away from the source of disturbance. Nevertheless, in the hyperbolic case, there exist situations in which discontinuities can occur in the flow on the characteristic surfaces. A theoretical and experimental evidence of such cases is available through the work of Oser (1957).

It is also important to point out that a transition from elliptic to hyperbolic equation occurs at the critical frequency $\omega = 2\Omega$ so called the 'resonant frequency'. The steady state solution based on the linearized theory breaks down at the resonant frequency. This is not an unexpected result and is in accordance with the findings of many others. The reason for the singular solution is that the linear approximation is not valid near the resonant frequency. Physically, this situation reveals itself through the wave motions of very large amplitude which cannot be accommodated by the linearized theory. It is thus necessary to include the non-linear and/or the viscous terms in the formulation of the problem in order to achieve a mathematically valid and physically realistic solution.

In this connection, the recent work of Wood (1966), Bains (1967) and Miles (1963) may be mentioned to indicate our interest in this problem. Wood has considered a steady state wave problem in a rigidly rotating fluid due to an oscillating disturbance and observed some anomalous properties of the flow. It is well known that such a boundary value problem with hyperbolic equation is, in general, not well posed because the solution does not depend continuously on the boundary data. In order to overcome this inherent difficulty, Bains (1967) investigated the well-posed initial value problem concerning the forced oscillation of a rigidly rotating fluid of finite extent and resolved the anomalous properties observed by Wood.

Miles (1963) has studied the Cauchy-Poisson wave problem in a rigidly rotating fluid of unlimited depth due to an initial surface elevation. He examined the principal features of the wave motions including the effects of the Coriolis force and the curvature of the free surface on the flow.

In recent years, Debnath (1969*a*, *b*) and Debnath and Rosenblat (1969) have made an initial value investigation of wave phenomena in various situations. They suggested many convincing arguments in favour of the initial value approach to the wave problems in general and emphasized that this is the most rigorous way of deriving a unique solution of physical interest without having to resort to the use of a radiation condition or an equivalent device.

This analysis is intended to study the unsteady wave phenomena in an inviscid, incompressible fluid rotating with a uniform angular velocity Ω. The motion is set up by harmonically oscillating pressure distributions acting on the horizontal-free surface of the rotating fluid. The problem is solved by the joint Laplace and generalised Hankel transformations together with asymptotic methods. An asymptotic analysis of the problem related to physically realistic pressure distributions has been carried out. Both the steady and the transient solutions associated with slow and rapid rotation are obtained explicitly. The significant effects of the Coriolis force and the characteristic features of the wave motions are investigated.

2. Statement of the Problem

We consider a linearized axisymmetric wave problem in an inviscid, incompressible, homogeneous, rotating fluid of finite depth h with the paraboloidal-free surface

$$z = z_0(r) = \frac{r^2}{2l}, \quad l = \frac{g}{\Omega^2}. \qquad (2.1)$$

We take the origin of the cylindrical polar coordinates (r, θ, z) on the free surface $z = z_0(r)$ with the z-axis vertically upward. The problem will be considered as an initial value problem under the following circumstances:

(i) In the undisturbed state, the fluid is rotating with a uniform angular velocity Ω about the axis of symmetry $r = 0$.

(ii) The surface tension and the viscosity of the fluid are neglected.

(iii) The motion is set up by an oscillating pressure distribution $p(r, t)$ in the form

$$\left.\begin{aligned} p(r, t) &= Pp(r)e^{i\omega t}H(t), \quad 0 \leq r \leq a \\ &= 0 \qquad\qquad\qquad\quad , \ r > a \end{aligned}\right\} \qquad (2.2)$$

applied at the free surface $z = z_0(r)$, where P is constant, $p(r)$ is an arbitrary function of r, ω is the forcing frequency of oscillations and $H(t)$ is the Heaviside unit function of time t.

The unsteady motion of the flow relative to the rotating axes is governed by the linearized Euler equations and the continuity equation

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial \chi}{\partial r} \qquad (2.3)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0 \qquad (2.4)$$

$$\frac{\partial w}{\partial t} + \frac{\partial \chi}{\partial z} = 0 \qquad (2.5)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \qquad (2.6)$$

where (u, v, w) represent the velocity field and $\chi(r, z; t)$ is the acceleration potential defined by

$$\chi(r, z; t) = \frac{p}{\rho} + g(z - z_0) \qquad \cdots \quad (2.7)$$

where ρ is the density, p the pressure and g is the acceleration due to gravity.

From (2.3)–(2.7), it follows that $\chi(r, z; t)$ satisfies the equation

$$\nabla^2 \chi_{tt} + 4\Omega^2 \chi_{zz} = 0, \; 0 \leq r < \infty, \; -h \leq z \leq z_0(r) \qquad \cdots \quad (2.8)$$

where the axisymmetric Laplacian ∇^2 is

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \qquad \cdots \quad (2.9)$$

The linearized-free surface conditions are

$$\left.\begin{aligned} \chi - g\eta &= \frac{p}{\rho} p(r) e^{i\omega t} \\ \eta_{tt} + u_t \, z_0(r) &= -\chi_z \end{aligned}\right\} \begin{aligned} z &= z_0(r) \\ t &\geq 0 \end{aligned} \qquad \cdots \quad \begin{aligned}(2.10)\\(2.11)\end{aligned}$$

where the disturbed-free surface elevation $\eta(r, t)$ is given by

$$z = z_0(r) + \eta(r, t). \qquad \cdots \quad (2.12)$$

The bottom boundary condition is

$$\frac{\partial \chi}{\partial z} = 0 \text{ at } z = -h. \qquad \cdots \quad (2.13)$$

The initial conditions of the problem are

$$\left.\begin{aligned} & z < z_0 \\ \chi = \chi_t = 0, \; & t = 0, \\ & z = z_0(r) \\ \eta = \eta_t = 0, \; & t = 0. \end{aligned}\right\} \qquad \cdots \quad (2.14)$$

Further, we assume that $\chi(r, z; t)$, $\eta(r, t)$ and $p(r)$ possess the Hankel transformation in the generalised sense (Lighthill 1958) with respect to r.

3. Solution of the Problem (Planar Approximation)

The case of planar approximation corresponds to the horizontal-free surface of the rotating fluid, that is, $z_0(r) \equiv 0$. In terms of the dimensionless parameter $\alpha = \Omega^2 l/g$ introduced by Miles (1963), this corresponds to $\alpha \doteq \infty$. It deserves attention to distinguish the effect of the free surface curvature on the flow.

It is convenient to introduce the Laplace transform $\tilde{\chi}$ of χ with respect to t by the integral (Sen 1969)

$$\tilde{\chi} = \tilde{\chi}(r, z; s) = \int_0^\infty e^{-st} \chi(r, z; t) dt \qquad \cdots \quad (3.1)$$

and then the generalised Hankel transform $\bar{\bar{\chi}}$ of $\bar{\chi}$ with respect to r by

$$\bar{\bar{\chi}} = \bar{\bar{\chi}}(k, z;\ s) = \int_0^\infty rJ_0(kr)\bar{\chi}(r, z;\ s)dr, \qquad (3.2)$$

where $J_0(kr)$ is the Bessel function of the first kind and order zero.

The joint Laplace and Hankel transforms enable us to obtain the solution of the transformed equation

$$\bar{\bar{\chi}}_{zz} = \frac{s^2k^2}{\lambda^2}\bar{\bar{\chi}},\ \lambda^2 = s^2+4\Omega^2 \qquad (3.3)$$

subject to the transformed boundary and free surface conditions in the form

$$\bar{\bar{\chi}}(k, z;\ s) = \frac{P\bar{p}(k)}{\rho(s-i\omega)} \frac{\lambda \cosh \frac{sk}{\lambda}(z+h)}{\left(\lambda s \cosh \frac{khs}{\lambda} + gk \sinh \frac{khs}{\lambda}\right)}. \qquad (3.4)$$

In the case of a very deep fluid, $h \to \infty$, solution (3.4) assumes the form

$$\bar{\bar{\chi}}(k, z;\ s) = \frac{P}{\rho} \frac{\bar{p}(k)}{(s-i\omega)} \frac{s\lambda}{(s\lambda+gk)} \exp\left(\frac{skz}{\lambda}\right). \qquad (3.5)$$

The corresponding expressions for $\bar{\bar{\eta}}(k, s)$ in finite and infinite depth are

$$\bar{\bar{\eta}}(k, s) = -\frac{P}{\rho} \frac{\bar{p}(k)}{(s-i\omega)} \frac{k \sinh \frac{khs}{\lambda}}{\left(\lambda s \cosh \frac{khs}{\lambda} + gk \sinh \frac{khs}{\lambda}\right)} \qquad (3.6)$$

$$\bar{\bar{\eta}}(k, s) = -\frac{P}{\rho} \frac{\bar{p}(k)}{(s-i\omega)} \frac{k}{(s\lambda+gk)}. \qquad (3.7)$$

It has already been stated that two cases (i) $\omega > 2\Omega$ (elliptic) and (ii) $\omega < 2\Omega$ (hyperbolic) are of particular interest.

The inversion theorems for the Laplace and Hankel transformations enable us to obtain an integral representation for the surface elevation $\eta(r, t)$ related to the first case in the form

$$\eta(r, t) = -\frac{P}{2\pi i\rho}\int_0^\infty k^2\bar{p}(k)J_0(kr)dk \int_{c-i\infty}^{c+i\infty} \frac{e^{st}\,ds}{(s-i\omega)\left(\lambda s \coth \frac{khs}{\lambda} + gk\right)} \qquad (3.8)$$

$$\eta(r, t) = -\frac{P}{2\pi i\rho}\int_0^\infty k^2\bar{p}(k)J_0(kr)dk \int_{c-i\infty}^{c+i\infty} \frac{e^{st}\,ds}{(s-i\omega)(s\lambda+gk)}. \qquad (3.9)$$

Remark: It is, however, interesting to point out that in the limit $\Omega \to 0$, the integrals (3.8) and (3.9) are in excellent agreement with those obtained by Debnath (1969*a*, *b*) in connection with the non-rotating fluid flow.

In order to facilitate the study of the problem in physical situations and to investigate effects of slow rotation on the flow, it is necessary to evaluate integral (3.9) asymptotically for large times t. Before we embark on the actual asymptotic analysis, it should be pointed out at least some essential points about the complex s-integral involved in (3.9). The s-integral has two branch points at $s = \pm 2i\Omega$ in addition to the polar singularities of the integrand of (3.9).

In the elliptic case, it is reasonable to approximate $s\lambda + gk$ by $s^2+2\Omega^2+gk$. This is a good approximation of the slowly rotating case and also shows a perfect agreement with Miles (1963) analysis.

The convolution theorem for the Laplace transformation together with the above results yields

$$\eta(r, t) = \frac{P}{\rho}\int_0^\infty \frac{k^2\bar{p}(k)J_0(kr)(i\omega \sin \beta t+\beta \cos \beta t-\beta e^{i\omega t})dk}{\sqrt{gk+2\Omega^2}\,(gk-\omega^2+2\Omega^2)} \quad .. \quad (3.10)$$

where

$$\beta^2 \equiv gk+2\Omega^2. \quad .. \quad .. \quad .. \quad .. \quad (3.11)$$

An expression similar to (3.10) can be found for $\eta(r, t)$ in a slowly rotating fluid of finite depth.

It may be mentioned that the solution (3.10) is convergent for all values of k in the domain of integration $(0, \infty)$, provided the function $\bar{p}(k)$ is well behaved in $(0, \infty)$. We shall mention such suitable functions $p(r)$ to discover the characteristic features of the flow.

Integral (3.10) is very much similar to that obtained by the author (Debnath 1969*a*, *b*) in a non-rotating wave problem. Hence a similar asymptotic analysis can be carried out to derive an explicit form for $\eta(r, t)$. In order to avoid duplication of similar procedure, it may probably be fair to refer to our earlier work stated above. It may be of some help to the readers to have some acquaintance with that work. However, in order to make the present paper self-contained to some extent, the necessary results without proof can be recalled from those papers. This makes it possible to outline the asymptotic evaluation of (3.10).

It is convenient to rewrite (3.10) in the form

$$\eta(r, t) = I_{tr}-I_{st} \quad .. \quad .. \quad .. \quad .. \quad (3.12)$$

where the steady state and the transient wave integrals are

$$I_{st} = \frac{Pe^{i\omega t}}{\rho}\int_0^\infty \frac{k^2\bar{p}(k)J_0(kr)dk}{gk-(\omega^2-2\Omega^2)} \quad .. \quad .. \quad .. \quad .. \quad (3.13)$$

$$I_{tr} = \frac{P}{\rho}\int_0^\infty \frac{k^2\bar{p}(k)J_0(kr)(i\omega \sin \beta t+\beta \cos \beta t)dk}{\sqrt{gk+2\Omega^2}\,(gk-\omega^2+2\Omega^2)}. \quad .. \quad .. \quad (3.14)$$

It would be fair to indicate that the nature of both of the ultimate wave system and of the asymptotic approach to it is determined by the singularities of (3.13) and (3.14). The remainder of this section is therefore concerned with locating and characterizing these singularities. The simple pole of (3.13) and (3.14) is given by

$$k \equiv k_0 = \frac{(\omega^2 - 2\Omega^2)}{g}. \quad \cdots \quad (3.15)$$

Thus the dominant contribution to I_{st} as $r \to \infty$ comes from the pole at $k = k_0$. To obtain the contribution for large r, it is convenient to replace $J_0(kr)$ by the pair of Hankel functions $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$. Making reference to the author's earlier work, it turns out that

$$I_{st} \sim -\frac{Pi}{\rho g}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \bar{p}(k_0)k_0^2\, e^{i\left(\omega t - rk_0 + \frac{\pi}{4}\right)} + 0\left(\frac{1}{r}\right). \quad \cdots \quad (3.16)$$

The transient integral I_{tr} can also be evaluated asymptotically for large t by using the author's earlier technique. This can be done by employing the stationary phase and saddle point methods. It readily turns out that the asymptotic expansion of I_{tr} for large t has the form

$$I_{tr} \sim \frac{P}{\rho}\frac{k_1^2\bar{p}(k_1)}{\{4rt\,|\,f''_-(k_1)\,|\,\}^{\frac{1}{2}}\beta(k_1)}\left[\frac{\exp i\{rk_1 - t\beta(k_1)\}}{\beta(k_1)+\omega} + \frac{\exp i\{t\beta(k_1) - rk_1\}}{\beta(k_1)-\omega}\right] + 0\left(\frac{1}{t^2}\right) \quad \cdots \quad (3.17)$$

where the stationary and saddle points related to integral (3.14) are given by

$$k \equiv k_1 = \left(\frac{gt^2}{4r^2} - \frac{2\Omega^2}{g}\right) \quad \cdots \quad (3.18)$$

and

$$f_-(k) = \beta(k) - \frac{rk}{t}. \quad \cdots \quad (3.19)$$

It still remains to calculate the contribution to I_{tr} from its polar singularities which are the same as those of I_{st}. By a precisely similar argument advanced before for the evaluation of I_{st}, it can readily be shown that

$$I_{tr}\text{ (polar)} \sim \frac{Pi}{\rho g}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \bar{p}(k_0)\, k_0^2\, e^{i\left(\omega t - k_0 r + \frac{\pi}{4}\right)}. \quad \cdots \quad (3\ 20)$$

Thus the asymptotic representation for $\eta(r, t)$ has the form

$$\eta(r, t) = \eta_{st}(r, t) + \eta_{tr}(r, t), \quad \cdots \quad (3.21)$$

where η_{st}, η_{tr} represent the steady state and the transient components of $\eta(r, t)$ respectively.

It may be of interest to point out that the steady state solution is essentially made up of the contributions to I_{st} and I_{tr} from their simple pole at $k = k_0$. On the other hand, the transient solution η_{tr} comes from the stationary and saddle points at $k = k_1$.

Thus the final form of the solution for $\eta_{st}(r, t)$ is

$$\eta_{st}(r, t) \sim \frac{iP}{\rho g}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} k_0^2\bar{p}(k_0)\, e^{i\left(\omega t - r_0 k + \frac{\pi}{4}\right)}. \quad \cdots \quad (3.22)$$

The transient solution $\eta_{tr}(r, t)$ is exactly identical with (3.17). Hence an asymptotic solution for $\eta(r, t)$ is obtained explicitly.

Remarks: The asymptotic solution for $\eta(r, t)$ in a non-rotating fluid (Debnath 1969*a*, *b*) can be readily recovered as a limit $\Omega \to 0$ from (3.21).

4. Pressure Distributions of Physical Interest

To investigate the principal features of the wave motions, it may be of interest to mention some simple form of $p(r)$:

$$\text{(a)}\ p(r) = \frac{\delta(r)}{r}, \qquad \text{(b)}\ p(r) = 1, \qquad \text{(c)}\ p(r) = e^{-r^2/r_0^2},$$

$$\text{(d)}\ p(r) = (a^2-r^2)^n, \quad \text{(e)}\ p(r) = e^{-r^2}L_n(r^2), \quad \text{(f)}\ p(r) = J_0(\lambda r),$$

$$\text{(g)}\ p(r) = r^{n-1}\,e^{-mr^2},$$

where $m > 0$, $n > -1$, $L_n(x)$ is the Laguerre function of degree n and $\delta(r)$ is the Dirac function of distribution.

Making reference to Erdélyi (1954), Sneddon (1951) and Lavoine (1959, 1963), the Hankel transformation $\bar{p}(k)$ related to (a)–(g) are respectively

$$1,\ \frac{a}{k}J_1(ak),\ 2m^2e^{-k^2m^2}\left(m^2 = \frac{r_0^2}{4}\right),\ 2^n\,\Gamma(n+1)a^{2(n+1)}(ak)^{-(n+1)}J_{n+1}(ak),$$

$$(n\,!)^{-1}2^{-(2n+1)}k^{2n}\exp\left(-\frac{k^2}{4}\right),\ \int_0^a xJ_0(\lambda x)J_0(kx)dx,$$

$$m^{-n/2}k^{-1}\,\Gamma\left(\frac{n+1}{2}\right)\exp\left(-\frac{k^2}{8m}\right)M_{\frac{n}{2},0}\left(\frac{k^2}{4m}\right);$$

where $J_n(x)$ is the Bessel function of the first kind and order n, $M_{a,b}(x)$ is the Whittaker function (Whittaker and Watson 1920) and the above integral is a standard one (Watson 1922).

Using the expressions for $\bar{p}(k)$, the asymptotic solution for $\eta(r, t)$ can be obtained in each individual case.

5. Discussions of the Wave Motions

The most significant feature of the present initial value investigation is that a unique solution of physical interest is achieved without the use of a radiation condition or an equivalent device.

A careful examination reveals that the transient component of (3.21) does tend to zero in the limit $t \to \infty$, for fixed r in all special cases (b)–(g) of section 4. This clearly confirms the ultimate approach to the steady state in the limit. The steady state solution represents the progressive circular waves propagating in the medium with the phase velocity $\frac{\omega g}{(\omega^2-2\Omega^2)}$ and the group velocity $\frac{\omega g}{2(\omega^2-2\Omega^2)}$. These results indicate that the phase and the group velocities are reduced by a small amount due to rotation. The amplitude of the wave trains is also modified by rotation, and this modification is essentially small.

The most striking conclusion of the analysis is that there is a phase shift in both the steady and the transient solutions by an amount $\dfrac{2\Omega^2 r}{g}$. This may be regarded as the significant effect of the Coriolis force on the wave motions.

It may also be worth noting that the solution for $\eta(r, t)$ related to case (a) follows from that of (b) in the limit $P \to \infty$, $a \to 0$ provided Pa^2 is a finite constant.

Finally, the present analysis is in perfect agreement with that of the non-rotating fluid considered by Debnath (1969*a*, *b*).

6. Asymptotic Development in Rapid Rotation

In this case, $2\Omega \gg \omega$ and the asymptotic representation of $\eta(r, t)$ as $\Omega t \to \infty$ is of most interest.

As indicated before that the poles of (3.9) are at $s = i\omega$, $\pm i\alpha$, where α^2 is given by

$$\alpha^2 = 2\Omega^2 + \sqrt{g^2k^2 + 4\Omega^4}, \; \alpha > 2\Omega. \quad \ldots \quad \ldots \quad \ldots \quad (6.1)$$

It is evident that the poles at $\pm i\alpha$ lie on the imaginary axis outside the branch cuts at $s = \pm 2i\Omega$. Hence the path of integration can be replaced by a path round the infinite semicircle $R(s) < 0$ and round the branch cuts. We anticipate that the contributions to (3.9) from the branch points correspond to the second class (or inertial) waves in contrast to the surface (or the first class) waves associated with the poles as shown before in connection with the slowly rotating case. In this case, the residue contribution from the poles are asymptotically insignificant as $\Omega t \to \infty$.

To compute the s-integral of (3.9) round the branch cuts, it is convenient to transfer the origin to the branch point $s = 2i\Omega$ by making transformation $s = z + 2i\Omega$. Thus the s-integral becomes

$$\frac{\exp(2i\Omega t)}{2\pi i} \int_{Br_2} \frac{e^{zt} dz}{(2i\Omega + z - i\omega)\{(2i\Omega + z)\sqrt{z^2 + 4i\Omega z} + gk\}} \quad \ldots \quad (6.2)$$

where the notation Br_2 is used in the sense of McLachlan (1955).

Making use of the binomial expansion for small $|z|$ with reasonable approximation to the integrand of (6.2), integral (6.2) takes the form

$$\frac{\exp(2i\Omega t)}{gk(2i\Omega - i\omega)} \frac{1}{2\pi i} \int_{Br_2} e^{zt} \left(1 + \frac{iz}{2\Omega - \omega} - \frac{4i\Omega}{gk} \sqrt{i\Omega z}\right) dz. \quad \ldots \quad (6.3)$$

A similar integral can be obtained from the branch cut at $s = -2i\Omega$ by replacing i by $-i$. Finally, adding them together, it readily turns out that the s-integral becomes

$$-\frac{8\Omega^{\frac{3}{2}}\cos\left(2\Omega t+\frac{\pi}{4}\right)}{g^2k^2(2\Omega-\omega)}\frac{1}{2\pi i}\int_{Br_2}\sqrt{z}\,e^{zt}dz,$$

which is, by McLachlan (1955),

$$\sim-\frac{8\Omega^{\frac{3}{2}}\cos\left(2\Omega t+\frac{\pi}{4}\right)}{g^2k^2\sqrt{\pi}\,t^{\frac{3}{2}}(2\Omega-\omega)r}. \qquad \ldots \quad \ldots \quad \ldots \quad (6.4)$$

Substituting (6.4) into (3.9), $\eta(r, t)$ assumes the form

$$\eta_2(r, t)\sim\frac{8\underline{P}\Omega^{\frac{3}{2}}\cos\left(2\Omega t+\frac{\pi}{4}\right)}{g^2\rho\sqrt{\pi}\,t^{\frac{3}{2}}(2\Omega-\omega)r}\int_0^\infty\bar{p}(k)J_0(kr)dk. \qquad \ldots \quad \ldots \quad (6.5)$$

This represents the second class wave solution in a very rapidly rotating flow for an arbitrary $p(r)$. However, it will be sufficient for the determination of the characteristic features of the flow to take the simple form of $p(r)$ stated in case (a) of section 4. In such a case, the asymptotic solution of physical interest has the form

$$\eta_2(r, t)\sim\frac{8\underset{\sim}{P}\Omega^{\frac{3}{2}}\cos\left(2\Omega t+\frac{\pi}{4}\right)}{\rho g^2\sqrt{\pi}\,t^{\frac{3}{2}}\,(2\Omega-\omega)r}. \qquad \ldots \quad \ldots \quad \ldots \quad (6.6)$$

A solution similar to (6.6) associated with other particular cases of $p(r)$ can be written down.

It remains to calculate the contributions of (3.9) from its poles at $s=i\omega$ and $\pm i\alpha$. To obtain these contributions, it would be convenient to replace $J_0(kr)$ by the pair of Hankel functions $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$ and then an identical procedure as advanced by Debnath (1969*a*, *b*) can be adopted without difficulty. This leads us to conclude that the residue-contributions from the poles are exponentially small for large Ωt and r.

7. Conclusions

This analysis concerning the rapidly rotating basic flow reveals that solution (6.6) has entirely different properties in contrast to (3.21) and represents inertial waves (or the second class waves according to Lamb (1932)) of frequency 2Ω propagating into the fluid. These waves, in contrast to the surface waves, are originated essentially due to strong rotation and has no counterpart in a slowly rotating or a non-rotating fluid.

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