PERIODIC ORBITS OF COLLISION IN THE RESTRICTED PROBLEM OF THREE BODIES IN A THREE-DIMENSIONAL COORDINATE SYSTEM

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(Communicated by R. S. Varma, F.N.A.)

(Received 29 July 1969; after revision 25 June 1970)

In this paper the equations of motion for the problem are regularized in the neighbourhood of one of the finite masses and the existence of periodic orbits of 3rd kind is studied. Finally it is shown that periodic orbits of collision also exist.

1. Introduction

In this paper we wish to study the three-dimensional generalization of the problem studied by Giacaglia (1967) for the plane case. Since the Hamilton-Jacobi equation for the generating solution takes an unmanagable form for any solution, so we have assumed that the third coordinate ξ_3 of the infinitesimal mass is of the $O(\mu)$. It will be interesting to observe that various equations and results worked out by Giacaglia can be deduced from our results.

In section 2 we have determined the canonical form of the equations of motion and in section 3 these equations are regularized by the generalized Levi-Civita's transformation for three dimension. Equations (20–22) establish the canonical set $(l, L, g, G, h, \bar{H})$ and eqns. (32) form the basis of general perturbation theory for the problem under consideration. Existence of periodic orbits when $\mu \neq 0$ has been shown in section 5. In section 6, we have studied the existence of symmetric or doubly symmetric orbits with respect to ξ_1-plane [ξ_2-plane or ξ_3-plane] or with respect to all the three planes and finally in section 7, we have proved the existence of periodic orbits of collision when $\mu \neq 0$.

2. Equations of Motion

The equations of motion in canonical form of an infinitesimal mass under the gravitational field of two finite and unequal masses and moving in circles are given by

$$\dot{x}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \quad (i = 1, 2, 3) \qquad \ldots \qquad (1)$$

where the Hamiltonian function H and consequently the energy integral is given by

$$H = (\tfrac{1}{2}p_1^2+p_2^2+p_3^2)+(p_1x_2-p_2x_1)-\frac{1-\mu}{r_1}-\frac{\mu}{r_2} = C \quad .. \quad (2)$$

and

$$r_1^2 = (x_1-\mu)^2+x_2^2+x_3^2$$

$$r_2^2 = (x_1-\mu+1)^2+x_2^2+x_3^2$$

$$p_1 = \dot{x}_1-x_2$$

$$p_2 = \dot{x}_2+x_1$$

$$p_3 = \dot{x}_3$$

(x_1, x_2, x_3) are equal to the synodic rectangular dimensionless coordinates of the infinitesimal mass in a uniformly rotating system (mean motion being equal to unity).

3. Regularization of the Solution

We regularize the solution by Levi-Civita's (1906) transformation generated by

$$s = (\mu+\xi_1^2-\xi_2^2)p_1+2\xi_1\xi_2p_2+\xi_3p_3 \quad .. \quad .. \quad .. \quad (3)$$

such that

$$x_i = \frac{\partial s}{\partial p_i}; \quad \pi_i = \frac{\partial s}{\partial \xi_i} \quad (i = 1, 2, 3) \quad .. \quad .. \quad .. \quad (4)$$

where π_i are the momenta associated with the new coordinates ξ_i.
We have from (3) and (4)

$$\pi_1 = \frac{\partial s}{\partial \xi_1} = 2\xi_1p_1+2\xi_2p_2; \quad \pi_2 = \frac{\partial s}{\partial \xi_2} = -2\xi_2p_1+2\xi_1p_2; \quad \pi_3 = \frac{\partial s}{\partial \xi_3} = p_3.$$

From these equations, we have

$$p_1 = \frac{\pi_1\xi_1-\pi_2\xi_2}{2(\xi_1^2+\xi_2^2)}$$

$$p_2 = \frac{\pi_1\xi_2+\pi_2\xi_1}{2(\xi_1^2+\xi_2^2)}$$

$$p_3 = \pi_3.$$

Further

$$x_1 = \mu+\xi_1^2-\xi_2^2$$

$$x_2 = 2\xi_1\xi_2$$

$$x_3 = \xi_3.$$

The Hamiltonian (2) given in terms of these new variables is

$$H = \frac{\pi^2}{8\xi^2}+\frac{\pi_3^2}{2}+\tfrac{1}{2}(\pi_1\xi_2-\pi_2\xi_1)-\frac{\mu}{2}\,\frac{\pi_1\xi_2+\pi_2\xi_1}{\xi^2}-\frac{1-\mu}{r_1}-\frac{\mu}{r_2} = C = \text{const.}$$

where

$$r_1^2 = \xi^4+\xi_3^2$$
$$r_2^2 = 1+\xi^4-2(\xi_1^2-\xi_2^2)+\xi_3^2$$
$$\pi^2 = \pi_1^2+\pi_2^2$$
$$\xi^2 = \xi_1^2+\xi_2^2$$
$$C = C_{(0)}+\mu C_1(\mu)$$
$$= C_0+\mu C_1.$$

Now we introduce a new independent variable τ instead of t defined by

$$dt = r_1\, d\tau \quad (t = 0 \text{ at } \tau = 0) \qquad (5)$$

The equations of motion (1) will be transformed to

$$\frac{d\xi_i}{d\tau} = \frac{\partial K}{\partial \pi_i}; \quad \frac{d\pi_i}{d\tau} = -\frac{\partial K}{\partial \xi_i} \quad (i = 1, 2, 3) \qquad (6)$$

where K is the new Hamiltonian given by

$$K = r_1(H-C)$$
$$= r_1\left[\frac{\pi^2}{8\xi^2}+\tfrac{1}{2}\pi_3^2+\tfrac{1}{2}(\pi_1\xi_2-\pi_2\xi_1)-\frac{\mu}{2}.\frac{\pi_1\xi_2+\pi_2\xi_1}{\xi^2}-\frac{1-\mu}{r_1}-\frac{\mu}{r_2}-C\right]$$

K can be put in the form $K_0+\mu K_1$ where

$$K_0 = \frac{\pi^2 r_1}{8\xi^2}+\tfrac{1}{2}\pi_3^2 r_1+\frac{r_1}{2}(\xi_2\pi_1-\pi_2\xi_1-2C_0)-1 = -\epsilon, \text{ say} \qquad (7)$$

$$K_1 = -\tfrac{1}{2}\left\{\frac{r_1}{\xi^2}(\pi_1\xi_2+\pi_2\xi_1)+\frac{2r_1}{r_2}\right\}-\frac{C-C_0}{\mu}r_1+1. \qquad (8)$$

We have given to K_0 a form which ensures that the orbits analytically continued from the two-body orbits will belong to the $K = 0$ manifold. These are solutions of regularized equation of the restricted problem. Here we have assumed that K_0 is negative (Bhatnagar 1969). Thus the corresponding two-body problem will admit bounded orbits as a solution in rotating co-ordinates. We can easily show that $|\epsilon| < 1$.

4. Generating Solution

For generating solution, we shall take K_0 for our Hamiltonian function. Since τ is not involved explicitly, so the Hamilton-Jacobi equation may be written as

$$\tfrac{1}{8}\left[\left(\frac{\partial w}{\partial \xi_1}\right)^2+\left(\frac{\partial w}{\partial \xi_2}\right)^2\right]\frac{r_1}{\xi^2}+\tfrac{1}{2}r_1\left(\frac{\partial w}{\partial \xi_3}\right)^2+\frac{r_1}{2}\left\{\xi_2\frac{\partial w}{\partial \xi_1}-\xi_1\frac{\partial w}{\partial \xi_2}-2C_0\right\} = \alpha \qquad (9)$$

where $\alpha = 1-\epsilon$.

We take ξ_3 of the order of μ. Then we have

$$r_1 = \xi^2+O(\mu).$$

Putting

$$\xi_1 = \xi \cos\phi, \quad \xi_2 = \xi \sin\phi$$

eqn. (9) becomes

$$\tfrac{1}{8}\left[\left(\frac{\partial w}{\partial \xi}\right)^2 + \frac{1}{\xi^2}\left(\frac{\partial w}{\partial \phi}\right)^2\right] + \tfrac{1}{8}\xi^2\left(\frac{\partial w}{\partial \xi_3}\right)^2 + \frac{\xi^2}{2}\left(-\frac{\partial w}{\partial \phi} - 2C_0\right) = \alpha. \quad \ldots \quad (10)$$

The solution of eqn. (10) may be written as

$$w = u(\xi) + 2G\phi + \bar{H}\xi_3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

and taking $\xi^2 = z$, we have

$$\left(\frac{du}{dz}\right)^2 = \frac{\bar{H}^2 - 2(G+C_0)}{z^2} f(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$f(z) = \frac{G^2}{2(G+C_0) - \bar{H}^2} - \frac{2\alpha z}{2(G+C_0) - \bar{H}^2} - z^2. \quad \ldots \quad \ldots \quad (13)$$

Suppose

$$G + C_0 < 0.$$

The equation $f(z) = 0$ has two positive roots z_1 and z_2 and $f(z)$ is positive for values of z between z_1 and z_2.
Also

$$z_1 + z_2 = \frac{-2\alpha}{2(G+C_0) - \bar{H}^2} > 0$$

$$z_1 z_2 = \frac{-G^2}{2(G+C_0) - \bar{H}^2} > 0.$$

The solution of eqn. (12) is

$$u(z, G, \alpha) = [\bar{H}^2 - 2(G+C_0)]^{\frac{1}{2}} \int_{z_1}^{z} \frac{\sqrt{f(z)}}{z}\, dz. \quad \ldots \quad \ldots \quad (14)$$

Let us introduce the parameters a, e, l by means of the relation

$$z_1 = a(1-e); \quad z_2 = a(1+e)$$

$$z = z_1 \cos^2 l/2 + z_1 \sin^2 l/2 = a(1 - e\cos l). \quad \ldots \quad \ldots \quad (15)$$

where $0 \leqslant e \leqslant 1$. It may be noted that $z = z_1$ when $l = 0$.

Equations of motion associated with K_0 are

$$\xi'_i = \frac{\partial K_0}{\partial \pi_i} \quad (i = 1, 2, 3)$$

$$\xi'_1 = \frac{\pi_1}{4} + \tfrac{1}{2}\xi^2\xi_2; \quad \xi'_2 = \frac{\pi_2}{4} - \tfrac{1}{2}\xi^2\xi_1; \quad \xi'_3 = \pi_3\xi^2. \quad \ldots \quad \ldots \quad (16)$$

Here $'$ denotes differentiation with respect to τ.

Now

$$\tfrac{1}{4}(\xi_1\pi_1+\xi_2\pi_2) = \xi\xi'.$$

Therefore

$$\frac{dz}{d\tau} = \sqrt{\bar{H}^2-2(G+C_0)}\sqrt{f(z)}.$$

Integrating, we have

$$(\tau-\tau_0)[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}} = \int_{z_1}^{z} \frac{dz}{\sqrt{f(z)}} \qquad (17)$$

where $z = z_1$ at $\tau = \tau_0$.
Introducing L by the relation

$$\alpha = L[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}} > 0, \quad L > 0$$

we have

$$a = \frac{L}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}} > 0$$

$$e = \left[1-\frac{G^2}{L^2}\right]^{\frac{1}{2}} \leqslant 1 \qquad (18)$$

$$\sqrt{f(z)} = ae \sin l$$

$$l = (\tau-\tau_0)[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}. \qquad (19)$$

Now taking L and G for the arbitrary constants instead of α and G, the solution may be given by the relations

$$\frac{\partial w}{\partial L} = \frac{\partial u}{\partial L} = l \qquad (20)$$

$$\frac{\partial w}{\partial G} = 2\phi-f-\frac{\sqrt{L^2-G^2}}{\bar{H}^2-2(G+C_0)}\sin l = g, \text{ say} \qquad (21)$$

where

$$f = \sqrt{1-e^2}\int_0^l \frac{dl}{1-e\cos l},$$

$$\frac{\partial w}{\partial \bar{H}} = \xi_3+\frac{\partial}{\partial H}\left[\int_{z_1}^{z}\left[-G^2+2\alpha z-\{\bar{H}^2-2(G+C_0)\}z^2\right]^{\frac{1}{2}}\frac{dz}{z}\right.$$

$$= \xi_3+\frac{\bar{H}[L^2-G^2]^{\frac{1}{2}}}{\bar{H}^2-2(G+C_0)}\sin l = h, \text{ say} \qquad (22)$$

and for $e = 1$ we have $G = 0, f = 0$. Equations (20), (21) and (22) establish the canonical set $(l, L, g, G, h, \bar{H})$.

Since $K_0 = \alpha-1$, it follows that

$$K_0 = L[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}-1 > 0$$

and, therefore, for the problem generated by this Hamiltonian (regularized two-body problem in rotating coordinates), we have

$$\frac{dL}{d\tau} = -\frac{\partial K_0}{\partial l} = 0, \quad L = \text{const.} = L_0 \text{ say}$$

$$\frac{dG}{d\tau} = -\frac{\partial K_0}{\partial g} = 0, \quad G = \text{const.} = G_0 \text{ say}$$

$$\frac{d\bar{H}}{d\tau} = -\frac{\partial K_0}{\partial h} = 0, \quad \bar{H} = \text{const.} = H_0 \text{ say}$$

$$\frac{dl}{d\tau} = \frac{\partial K_0}{\partial L} = [\bar{H}^2-2(G+C_0)]^{\frac{1}{2}} = \text{const.} = n_l \quad \therefore l = n_l\tau+l_0$$

$$\frac{dg}{d\tau} = \frac{\partial K_0}{\partial G} = \frac{-L}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}} = \text{const.} = n_g \quad \therefore g = n_g\tau+g_0$$

$$\frac{dh}{d\tau} = \frac{\partial K_0}{\partial \bar{H}} = \frac{L\bar{H}}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}} = \text{const.} = n_h \quad \therefore h = n_h\tau+h_0 \quad .. \quad (23)$$

where l_0, g_0, h_0 are the values of l, g, h respectively at $\tau = 0$.

The angle ϕ is obtained from equation

$$\phi = \tfrac{1}{2}(f+g)+\tfrac{1}{2}\frac{(L^2-G^2)^{\frac{1}{2}}}{\bar{H}^2-2(G+C_0)} \sin l, \text{ when } e \neq 1$$

and

$$\phi = \tfrac{1}{2}g+\frac{L}{2(\bar{H}^2-2C_0)} \sin l, \quad \text{when } e = 1. \quad .. \quad .. \quad .. \quad (24)$$

The variables ξ_i, π_i $(i = 1, 2, 3($ can now be expressed by the canonical elements.

We have

$$\xi_1 = \pm\sqrt{z}\cos\phi$$
$$= \pm[a(1-e\cos l)]^{\frac{1}{2}}\cos\phi$$
$$\xi_2 = \pm\sqrt{z}\sin\phi$$
$$= \pm[a(1-e\cos l)]^{\frac{1}{2}}\sin\phi$$
$$\xi_3 = h-\frac{\bar{H}(L^2-G^2)^{\frac{1}{2}}}{\bar{H}^2-2(G+C_0)}\sin l$$
$$\pi_1 = \cos\phi\frac{\partial w}{\partial\xi}-\frac{\sin\phi}{\xi}\frac{\partial w}{\partial\phi}$$
$$\pi_2 = \sin\phi\frac{\partial w}{\partial\xi}+\frac{\cos\phi}{\xi}\frac{\partial w}{\partial\phi}$$
$$\pi_3 = \frac{\partial w}{\partial\xi_3} = \bar{H}.$$

But

$$\begin{aligned}\frac{\partial w}{\partial \xi} &= \frac{du}{d\xi}\\ &= \frac{du}{dz}\,\frac{dz}{d\xi} = 2\xi\frac{du}{dz}\\ &= 2\xi\left\{\frac{\bar{H}^2-2(G+C_0)}{z^2}f(z)\right\}^{\frac{1}{2}}\\ &= \frac{2eL\sin l}{\pm\sqrt{a(1-e\cos l)}}\end{aligned}$$

and

$$\frac{\partial w}{\partial \phi} = 2G.$$

Therefore

$$\left.\begin{aligned}\pi_1 &= \frac{2eL\sin l\cos\phi-2G\sin\phi}{\pm[a(1-e\cos l)]^{\frac{1}{2}}}\\ \pi_2 &= \frac{2eL\sin l\sin\phi+2G\cos\phi}{\pm[a(1-e\cos l)]^{\frac{1}{2}}}\\ \pi_3 &= \bar{H}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

where ϕ is given by the first of the eqns. (24). When $e = 1$, $(G = 0)$,

$$\left.\begin{aligned}\xi_1 &= \pm\sqrt{2a}\sin l/2\cos\phi\\ \xi_2 &= \pm\sqrt{2a}\sin l/2\sin\phi\\ \xi_3 &= h-\frac{\bar{H}L}{\bar{H}^2-2C_0}\sin l\\ \pi_1 &= \frac{4L}{\pm\sqrt{2a}}\cos l/2\cos\phi\\ \pi_2 &= \frac{4L}{\pm\sqrt{2a}}\cos l/2\sin\phi\\ \pi_3 &= \bar{H}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

where ϕ is given by the second of the eqns. (24).

The original synodic Cartesian coordinates are obtained from equations $(\mu = 0)$,
i.e.

$$\left.\begin{aligned}x_1 &= \xi_1^2-\xi_2^2\\ x_2 &= 2\xi_1\xi_2\\ x_3 &= \xi_3\\ p_1 &= \frac{1}{2z}\{\pi_1\xi_1-\pi_2\xi_2\}\\ p_2 &= \frac{1}{2z}\{\xi_1\pi_2+\xi_2\pi_1\}\\ p_3 &= \pi_3\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

where $z = a(1-e\cos l)$.

The sideral Cartesian coordinates are given by

$$\left.\begin{aligned} X_1 &= x_1 \cos t - x_2 \sin t \\ X_2 &= x_1 \sin t + x_2 \cos t \\ X_3 &= x_3 \\ \dot{X}_1 &= p_1 \cos t - p_2 \sin t \\ \dot{X}_2 &= p_1 \sin t + p_2 \cos t \\ \dot{X}_3 &= p_3 \end{aligned}\right\} \qquad \ldots \quad (28)$$

where

$$dt = r_1 \, d\tau \qquad \ldots \quad (29)$$

or

$$t = \int_0^\tau \xi^2 \, d\tau + O(\mu).$$

Therefore

$$t - t_0 = \int_0^\tau z \frac{d\tau}{dz} \, dz$$

$$= \int \frac{a(1 - e \cos l)}{[\bar{H}^2 - 2(G + C_0)]^{\frac{1}{2}}} \, dl$$

$$= \frac{a(l - e \sin l)}{[\bar{H}^2 - 2(G + C_0)]^{\frac{1}{2}}} \qquad \ldots \quad (30)$$

where t_0 is a constant. It is seen that l is the eccentric anomaly of the problem of two-body.

In terms of the canonical variables, the complete Hamiltonian may be written as

$$K = K_0 + \mu K_1$$

$$= L[\bar{H}^2 - 2(G + C_0)]^{\frac{1}{2}} - 1 + \mu \left[-\tfrac{1}{2} \left\{ \frac{r_1}{\xi^2} (\pi_1 \xi_2 + \pi_2 \xi_1) + \frac{2 r_1}{r_2} \right\} - \frac{C - C_0}{\mu} r_1 + 1 \right] \qquad \ldots \quad (31)$$

where

$$r_1^2 = \xi^4 + \xi_3^2$$

$$r_2^2 = 1 + \xi^4 + 2(\xi_1^2 - \xi_2^2) + \xi_3^2$$

$$\xi^2 = \xi_1^2 + \xi_2^2$$

and $\xi_1, \xi_2, \xi_3; \pi_1, \pi_2, \pi_3$ are given by eqns. (25).

The equations of motion for the complete Hamiltonian are

$$\left.\begin{aligned}
\frac{dl}{d\tau} &= \frac{\partial K}{\partial L} = [\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}+\mu\frac{\partial R}{\partial L}\\
\frac{dg}{d\tau} &= \frac{\partial K}{\partial G} = \frac{-L}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}}+\mu\frac{\partial R}{\partial G}\\
\frac{dh}{d\tau} &= \frac{\partial K}{\partial \bar{H}} = \frac{L\bar{H}}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}}+\mu\frac{\partial R}{\partial \bar{H}}\\
\frac{dL}{d\tau} &= -\frac{\partial K}{\partial l} = -\mu\frac{\partial R}{\partial l}\\
\frac{dG}{d\tau} &= -\frac{\partial K}{\partial g} = -\mu\frac{\partial R}{\partial g}\\
\frac{d\bar{H}}{d\tau} &= -\frac{\partial K}{\partial h} = -\mu\frac{\partial R}{\partial h}
\end{aligned}\right\} \quad \ldots \quad (32)$$

These equations form the basis of a general perturbation theory for the problem in question.

5. Existence of Periodic Orbits when $\mu \neq 0$

Here we shall follow the method used by Choudhry (1966). When $\mu = 0$, the eqns. (32) become

$$\left.\begin{aligned}
\frac{dl}{d\tau} &= \frac{\partial K_0}{\partial L} = [\bar{H}^2-2(G+C_0)]^{\frac{1}{2}} = \text{const.}\\
\frac{dg}{d\tau} &= \frac{\partial K_0}{\partial G} = \frac{-L}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}} = \text{const.}\\
\frac{dh}{d\tau} &= \frac{\partial K_0}{\partial \bar{H}} = \frac{L\bar{H}}{[\bar{H}^2-2(G+C_0)]^{\frac{1}{2}}} = \text{const.}\\
\frac{dL}{d\tau} &= -\frac{\partial K_0}{\partial l} = 0\\
\frac{dG}{d\tau} &= -\frac{\partial K_0}{\partial g} = 0\\
\frac{d\bar{H}}{d\tau} &= -\frac{\partial K_0}{\partial h} = 0
\end{aligned}\right\} \quad \ldots \quad (33)$$

Let

$$x_1 = L, \quad x_2 = G, \quad x_3 = \bar{H}$$

$$y_1 = l, \quad y_2 = g, \quad y_3 = h.$$

The eqns. (33) may be written as

$$\frac{dx_i}{d\tau} = 0; \quad \frac{dy_i}{dt} = n_i^{(0)} \text{ say} \quad (i = 1, 2, 3)$$

i.e.

$$x_i^{(0)} = a_i; \quad y_i^{(0)} = n_i^{(0)}\tau+\omega_i.$$

These are the generating solution of the problem of two bodies in a three-dimensional coordinate system. Here a_i and ω_i $(i = 1, 2, 3)$ are arbitrary constants and

$$\left.\begin{aligned} n_1^{(0)} &= \left[-\frac{\partial K_0}{\partial x_1}\right]_{x_1 = a_1} \\ n_2^{(0)} &= \left[-\frac{\partial K_0}{\partial x_2}\right]_{x_2 = a_2} \\ n_3^{(0)} &= \left[-\frac{\partial K_0}{\partial x_3}\right]_{x_3 = a_3} \end{aligned}\right\} . \qquad (34)$$

The generating solution will be periodic with the period τ_0 if

$$y_i(\tau_0) - y_i(0) = n_i^{(0)}\tau_0 = 2K_i\pi$$

and

$$x_i(\tau_0) - x_i(0) = 0.$$

Here K_i $(i = 1, 2, 3)$ are integers, so that $n_i(0)$ are commensurable.

Let the general solution in the neighbourhood of the generating solution be periodic with the period $\tau_0(1+\alpha)$ where α is a negligible quantity of the order μ. Introduce a new independent variable φ by the equation

$$\varphi = \frac{\tau}{1+\alpha}.$$

The period of the general solution will then be $\frac{\tau_0}{1+\alpha}(1+\alpha)$, i.e. τ_0. This period coincides with that of the generating solution.

The equations of motion can be written as

$$\frac{dx_i}{d\varphi} = -(1+\alpha)\frac{\partial K}{\partial y_i}$$

$$\frac{dy_i}{d\varphi} = (1+\alpha)\frac{\partial K}{\partial x_i} \quad (i = 1, 2, 3). \qquad (35)$$

The general solution in the neighbourhood of the generating solution may be taken as

$$x_i = a_i + \beta_i + \xi_i(\varphi)$$

$$y_i = n_i^{(0)}\varphi + \omega_i + \gamma_i + \eta_i(\varphi) \quad (i = 1, 2, 3).$$

Equations (35) can be written in terms of the new variables (ξ_i, η_i) as

$$\frac{d\xi_i}{d\varphi} = -\frac{\partial K'}{\partial \eta_i}; \quad \frac{d\eta_i}{d\varphi} = \frac{\partial K'}{\partial \xi_i} \quad (i = 1, 2, 3) \qquad (36)$$

where

$$K'[\varphi, \xi_i, \eta_i] = (1+\alpha)K[\varphi, a_i+\beta_i+\xi_i, n_i^{(0)}\varphi+\omega_i+\gamma_i+\eta_i(\varphi)]$$

$$-(1+\alpha)K[\varphi, a_i, n_i^{(0)}+\omega_i]+n_1^{(0)}\xi_1+n_2^{(0)}\xi_2+n_3^{(0)}\xi_3$$

$$= (1+\alpha)\left[K(\varphi, a_i, n_i^{(0)}\varphi+\omega_i)+\sum_{i=1}^{3}\left\{\frac{\partial K}{\partial a_i}\xi_i+\frac{\partial K}{\partial \omega_i}\eta_i\right\}\cdots\right]$$

$$-(1+\alpha)[K(\varphi, a_i, n_i^{(0)}\varphi+\omega_i)]+n_1^{(0)}\xi_1+n_2^{(0)}\xi_2+n_3^{(0)}\xi_3.$$

Now in order that the periodic solution may exist the necessary and sufficient conditions are written as

$$x_i(\tau_0)-x_i(0) = \xi_i(\tau_0) = 0 \quad \ldots \quad (37)$$

$$y_i(\tau_0)-y_i(0)-2K_i\pi = \eta_i(\tau_0) = 0 \quad (i = 1, 2, 3). \quad \ldots \quad (38)$$

Restricting our solution only up to the first order infinitesimals, equations of motion (36) may be written as

$$\frac{d\xi_i}{d\varphi} = -\frac{\partial K'}{\partial \eta_i} = -(1+\alpha)\frac{\partial K}{\partial \omega_i} \quad \ldots \quad (39)$$

$$\frac{d\eta_i}{d\varphi} = \frac{\partial K'}{\partial \xi_i} = (1+\alpha)\frac{\partial K}{\partial a_i}+n_i(0) \quad (i = 1, 2, 3). \quad \ldots \quad (40)$$

Expanding $K[\varphi, a_i+\beta_i, \eta_i^{(0)}\varphi+\omega_i+\gamma_i]$ in ascending powers of β_i, γ_i, μ, we find that the eqns. (39) may be written as

$$\frac{d\xi_K}{d\varphi} = -(1+\alpha)\frac{\partial}{\partial \omega_K}[K(\varphi, a_i+\beta_i, n_i^{(0)}\varphi+\omega_i+\gamma_i)]$$

$$= -(1+\alpha)\frac{\partial}{\partial \omega_K}[K_0(\varphi, a_i+\beta_i)+\mu K_1(\varphi, a_i+\beta_i, n_i^{(0)}\varphi+\omega_i+\gamma_i)]$$

$$= -\mu\frac{\partial}{\partial \omega_K}\left[K_1(\varphi, a_i, n_i^{(0)}\varphi+\omega_i+\sum_{i=1}^{3}\left\{\beta_i\frac{\partial K_1}{\partial a_i}+\gamma_i\frac{\partial K_1}{\partial \omega_i}\right\}\right].$$

Integerating and putting the values of ξ_i in (37), we get

$$\frac{\xi_K(\tau_0, \beta_i, \gamma_i, \mu)}{-\mu\tau_0} = \frac{\partial[K_1]}{\partial \omega_K}+\sum_{i=1}^{3}\beta_i\frac{\partial^2[K_1]}{\partial \omega_K\partial a_i}+\sum_{i=1}^{3}\gamma_i\frac{\partial^2[K_1]}{\partial \omega_K\partial \omega_i} = 0 \quad \ldots \quad (41)$$

where

$$[K_1] = \frac{1}{\tau_0}\int_0^{\tau_0} K_1\{\varphi, a_i, n_i^{(0)}\varphi+\omega_i\}\,d\varphi.$$

Equations (40) give

$$\begin{aligned}\frac{d\eta_1}{d\varphi} &= n_1^{(0)}+(1+\alpha)\frac{\partial K}{\partial a_1}\\ &= n_1^{(0)}+(1+\alpha)\frac{\partial}{\partial a_1}[K_0(\varphi, a_i+\beta_i)+\mu K_1(\varphi, a_i+\beta_i, n_i^{(0)}\varphi+\omega_i+\gamma_i)]\\ &= n_1^{(0)}+(1+\alpha)\frac{\partial K_0}{\partial a_1}+\sum_{i=1}^{3}\beta_i\frac{\partial^2 K_0}{\partial a_1\partial a_i}+O(\mu)\\ &= \alpha\frac{\partial K_0}{\partial a_1}+\beta_1\frac{\partial^2 K_0}{\partial a_1^2}+\beta_2\frac{\partial^2 K_0}{\partial a_1\partial a_2}+\beta_3\frac{\partial^2 K_0}{\partial a_1\partial a_3}+O(\mu)\end{aligned}$$

and eqns. (38) give

$$\eta_1(\tau_0, \beta_i, \gamma_i, \mu) = \alpha\tau_0\frac{\partial K_0}{\partial a_1}+\beta_1\tau_0\frac{\partial^2 K_0}{\partial a_1^2}+\beta_2\tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_2}+\beta_3\tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_3}+O(\mu) = 0 \quad (42)$$

$$\eta_2(\tau_0, \beta_i, \gamma_i, \mu) = \alpha\tau_0\frac{\partial K_0}{\partial a_2}+\beta_1\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_1}+\beta_2\tau_0\frac{\partial^2 K_0}{\partial a_2^2}+\beta_3\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_3}+O(\mu) = 0 \quad (43)$$

$$\eta_3(\tau_0, \beta_i, \gamma_i, \mu) = \alpha\tau_0\frac{\partial K_0}{\partial a_3}+\beta_1\tau_0\frac{\partial^2 K_0}{\partial a_3\partial a_1}+\beta_2\tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_3}+\beta_3\tau_0\frac{\partial^2 K_0}{\partial a_3^2}+O(\mu) = 0. \quad (44)$$

Following Choudhry (1966) we may regard β_1, β_2, β_3, γ_2, γ_3 as unknowns reducing to zero with μ if the following conditions are satisfied, i.e.

$$\frac{\partial[K_1]}{\partial\omega_i} = 0 \quad (i = 1, 2, 3) \quad \ldots \quad (45)$$

$$\frac{\partial[K_1]}{\partial a_i} = 0 \quad (i = 1, 2, 3) \quad \ldots \quad (46)$$

$$\frac{\partial(\xi_2, \xi_3, \eta_1, \eta_2, \eta_3)}{\partial(\gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3)} \neq 0, \quad \text{when } \mu = \beta_i = \gamma_i = 0. \quad \ldots \quad (47)$$

Jacobian on left-hand side of (47) can be written as

$$\begin{vmatrix} \frac{\partial^2[K_1]}{\partial\omega_2^2} & \frac{\partial^2[K_1]}{\partial\omega_3\partial\omega_2} & 0 & 0 & 0\\ \frac{\partial^2[K_1]}{\partial\omega_2\partial\omega_3} & \frac{\partial^2[K_1]}{\partial\omega_3^2} & 0 & 0 & 0\\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_1} & \frac{\partial^2[K_1]}{\partial\omega_3\partial a_1} & \tau_0\frac{\partial^2 K_0}{\partial a_1^2} & \tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_1} & \tau_0\frac{\partial^2 K_0}{\partial a_3\partial a_1}\\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_2} & \frac{\partial^2[K_1]}{\partial\omega_3\partial a_2} & \tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_2} & \tau_0\frac{\partial^2 K_0}{\partial a_2^2} & \tau_0\frac{\partial^2 K_0}{\partial a_3\partial a_2}\\ \frac{\partial^2[K_1]}{\partial\omega_2\partial a_3} & \frac{\partial^2[K_1]}{\partial\omega_3\partial a_3} & \tau_0\frac{\partial^2 K_0}{\partial a_1\partial a_3} & \tau_0\frac{\partial^2 K_0}{\partial a_2\partial a_3} & \tau_0\frac{\partial^2 K_0}{\partial a_3^2} \end{vmatrix}.$$

It may be noted that

$$\frac{\partial^2[K_1]}{\partial a_j\partial\omega_i} = 0 \text{ for } \frac{\partial[K_1]}{\partial\omega_i} = 0 \quad (i = 1, 2, 3;\ j = 1, 2, 3).$$

The condition (47) becomes
i.e. if

$$\begin{vmatrix} \dfrac{\partial^2[K_1]}{\partial\omega_2^2} & \dfrac{\partial^2[K_1]}{\partial\omega_3\partial\omega_2} & 0 & 0 & 0 \\ \dfrac{\partial^2[K_1]}{\partial\omega_2\partial\omega_3} & \dfrac{\partial^2[K_1]}{\partial\omega_3^2} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial^2 K_0}{\partial a_1^2} & \dfrac{\partial^2 K_0}{\partial a_2\partial a_1} & \dfrac{\partial^2 K_0}{\partial a_3\partial a_1} \\ 0 & 0 & \dfrac{\partial^2 K_0}{\partial a_1\partial a_2} & \dfrac{\partial^2 K_0}{\partial a_2^2} & \dfrac{\partial^2 K_0}{\partial a_3\partial a_2} \\ 0 & 0 & \dfrac{\partial^2 K_0}{\partial a_1\partial a_3} & \dfrac{\partial^2 K_0}{\partial a_2\partial a_3} & \dfrac{\partial^2 K_0}{\partial a_3^2} \end{vmatrix} \neq 0. \quad .. \quad .. \quad (48)$$

We know

$$K_0 = L[\overline{H}^2-2(G+C_0)]^{\frac{1}{2}}-1$$
$$= x_1[x_3^2-2(x_2+C_0)]^{\frac{1}{2}}-1$$
$$= a_1[a_3^2-2(a_2+C_0)]^{\frac{1}{2}}-1.$$

By actual verification, we can show that

$$\begin{vmatrix} \dfrac{\partial^2 K_0}{\partial a_1^2} & \dfrac{\partial^2 K_0}{\partial a_2\partial a_1} & \dfrac{\partial^2 K_0}{\partial a_3\partial a_1} \\ \dfrac{\partial^2 K_0}{\partial a_1\partial a_2} & \dfrac{\partial^2 K_0}{\partial a_2^2} & \dfrac{\partial^2 K_0}{\partial a_3\partial a_2} \\ \dfrac{\partial^2 K_0}{\partial a_1\partial a_3} & \dfrac{\partial^2 K_0}{\partial a_2\partial a_3} & \dfrac{\partial^2 K_0}{\partial a_3^2} \end{vmatrix} = \frac{-a_1}{[a_3^2-2(a_2+C_0)]^{\frac{3}{2}}} \neq 0.$$

Further we have

$$K_1 = -\tfrac{1}{2}\left[\frac{r_1}{\xi^2}(\pi_1\xi_2+\pi_2\xi_1)+\frac{2r_1}{r_2}\right]-C_1 r_1+1$$

$$r_1^2 = \xi^4+\xi_3^2; \quad r_2^2 = 1+\xi^4+2(\xi_1^2-\xi_2^2)+\xi_3^2$$

$$\xi^2 = \xi_1^2+\xi_2^2; \quad z = \xi^2 = a(1-e\cos l);$$

$$\phi = \tfrac{1}{2}(f+g)+\tfrac{1}{2}\frac{\sqrt{L^2-G^2}}{\pi^2-2(G+C_0)}\sin l$$

$$f = \sqrt{1-e^2}\int_0^l \frac{dl}{1-e\cos l}$$

$$\xi_3 = h-\frac{\overline{H}(L^2-G^2)^{\frac{1}{2}}}{\overline{H}^2-2(G+C_0)}\sin l$$

$$L = x_1; \quad G = x_2; \quad \overline{H} = x_3$$

$$l = y_1; \quad g = y_2; \quad h = y_3$$

$$x_i = a_i \text{ (approx.)}; \quad y_i = n_i^{(0)}+\omega_i \text{ (approx.)}.$$

Taking only zero order terms, we have

$$z = \xi^2 = a$$

$$r_1^2 = a^2+\xi_3^2; \qquad r_2^2 = 1+a^2+2a\cos 2\phi+\xi_3^2$$

$$\pi_1 = \frac{-2G\sin\phi}{\sqrt{a}}; \quad \pi_2 = \frac{2G\cos\phi}{\sqrt{a}}.$$

In canonical elements, we have

$$[K_1] = -\frac{r_1 G}{a}\cos 2\phi - \frac{r_1}{r_2} - C_1 r_1 + 1$$

where

$$\phi = \tfrac{1}{2}[y_1+y_2]+\tfrac{1}{2}\frac{\sqrt{x_1^2-x_2^2}}{x_3^2-2(x_2+C_0)}\sin y_1$$

and

$$\xi_3 = y_3 - \frac{x_3(x_1^2-x_2^2)}{x_3^2-2(x_2+C_0)}\sin y_1$$

$$x_i = a_i$$

$$y_i = n_i^{(0)}+\omega_i.$$

Thus

$$\frac{\partial[K_1]}{\partial\omega_2} = r_1\sin 2\phi\left[\frac{x_2}{a}-\frac{a^2}{r_2^3}\right] = 0 \quad \text{gives}$$

either

$$2\phi = 0 \text{ or } x_2 = \frac{a^2}{r_2^3}.$$

Now

$$\frac{\partial^2[K_1]}{\partial\omega_2^2} = r_1\cos 2\phi\left[\frac{x_2}{a}-\frac{a^2}{r_2^3}\right] - \frac{3r_1a^4\sin^2 2\phi}{r_2^5}.$$

When, either

$$2\phi = 0,\ \pi \text{ or } x_2 = \frac{a^2}{r_2^3},$$

$$\frac{\partial^2[K_1]}{\partial\omega_2^2} \neq 0; \quad \frac{\partial^2[K_1]}{\partial\omega_3\partial\omega_2} = \sin 2\phi\frac{\partial}{\partial\omega_3}\left\{r_1\left(\frac{x_2}{a}-\frac{a^2}{r_2^3}\right)\right\}$$

$$= 0.$$

Now if $\frac{\partial^2[K_1]}{\partial\omega_3^2} \neq 0$, then the determinant (48) will not be equal to zero. Now

$$\frac{\partial[K_1]}{\partial\omega_3} = \frac{-x_2\cos 2\phi}{a}\cdot\frac{\xi_3}{r_1}+\frac{r_1}{r_2^2}\frac{\xi_3}{r_2}-C_1\frac{\xi_3}{r_1}$$

$$= \xi_3 B, \text{ say.}$$

When

$$\frac{\partial[K_1]}{\partial\omega_3} = 0, \text{ either } \xi_3 = 0 \text{ or } B = 0.$$

Therefore

$$\frac{\partial^2[K_1]}{\partial\omega_3^2} = B+\xi_3\frac{\partial B}{\partial\omega_3} \neq 0, \quad \text{when either } \xi_3 = 0 \text{ or } B = 0.$$

Hence, we find that the determinant (48) is not equal to zero.

6. Existence of Symmetric or Doubly Symmetric Periodic Orbits

Periodicity for $\mu \neq 0$ will be established if we can show that periodic orbits exist for $\mu = 0$, since the function involved are holomorphic in this neighbourhood.

Here we may also observe that the equations of motion (6) in the form

$$\xi''_i = f_i(\xi, \xi'), \quad i = 1, 2, 3$$

do not change under the following transformations:

$$\text{(i)} \quad \xi_1 \to -\xi_1; \quad \xi_2 \to \xi_2; \quad \xi_3 \to \xi_3; \quad \tau \to -\tau$$

$$\text{(ii)} \quad \xi_2 \to -\xi_2; \quad \xi_3 \to \xi_3; \quad \xi_1 \to \xi_1; \quad \tau \to -\tau$$

$$\text{(iii)} \quad \xi_3 \to -\xi_3; \quad \xi_1 \to \xi_1; \quad \xi_2 \to \xi_2; \quad \tau \to -\tau$$

which shows that symmetric orbits may exist for any value of μ.

It may be noted that
when

$$\xi_2 = 0, \quad \xi_3 = 0, \quad \pi_2 = 0$$

we have

$$\xi'_1 = \frac{\partial K_0}{\partial \pi_1} = \frac{\pi_1}{4} + \tfrac{1}{2}\xi^2\xi_2 = 0,$$

when

$$\xi_3 = 0, \quad \xi_1 = 0, \quad \pi_2 = 0,$$

we have

$$\xi'_2 = \frac{\partial K_0}{\partial \pi_2} = \frac{\pi_2}{4} - \tfrac{1}{2}\xi^2\xi_1 = 0$$

and when

$$\xi_1 = 0, \quad \xi_2 = 0, \quad \pi_3 = 0,$$

we have

$$\xi'_3 = \frac{\partial K_0}{\partial \pi_3} = \pi_3\xi^2 = 0.$$

Thus we may conclude the following:

(a) If at $\tau = 0$, $\xi_2(0) = 0$, $\xi_3(0) = 0$ and $\pi_1(0) = 0$

$[\xi_3(0) = 0$, $\xi_1(0) = 0$ and $\pi_2(0) = 0$; $\xi_1(0) = 0$, $\xi_2(0) = 0$ and $\pi_3(0) = 0]$

and at $\tau = \tau_1$, $\xi_2(\tau_1) = 0$, $\xi_3(\tau_1) = 0$ and $\pi_1(\tau_1) = 0$

$[\xi_3(\tau_1) = 0$, $\xi_1(\tau_1) = 0$ and $\pi_2(\tau_1) = 0$; $\xi_1(\tau_1) = 0$, $\xi_2(\tau_1) = 0$ and $\pi_3(\tau_1) = 0]$

then the orbit is periodic and the period is $2\tau_1$.

(b) If at $\tau = 0$, $\xi_2(0) = 0$, $\xi_3(0) = 0$, $\pi_1(0) = 0$

and at $\tau = \tau_1$, $\xi_3(\tau_1) = 0$, $\xi_1(\tau_1) = 0$, $\pi_2(\tau_1) = 0$

$\xi_1(\tau_1) = 0$, $\xi_2(\tau_1) = 0$, $\pi_3(\tau_1) = 0$

then the orbit is periodic and the period is $4\tau_1$.

Case (a) corresponds to orbits symmetric with respect to ξ_1-plane [ξ_2-plane; ξ_3-plane] and case (b) corresponds to orbits symmetric with respect to all the three planes.

7. Periodic Orbits of Collision when $\mu \neq 0$

Levi-Civita (1904) established a relation which is satisfied along a collision orbit in the restricted problem. His main result, in that paper, may be stated briefly as follows. He proved that the invariant relation for collision orbits can be analytically continued from the one that corresponds to the problem of two bodies. (In our paper it is given by $G = 0$ (Bhatnagar 1969)). This means that when $\mu \neq 0$, the condition $G = 0$ will change into

$$G+\mu F(l, L, g, G, h, \bar{H}, \mu) = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (49)$$

for μ sufficiently small.

Levi-Civita (1904) has also shown, in particular, such relation is a uniform integral of the differential equations of motion along any collision orbit. The integral is a power series in terms of the distance from the origin. And the series is convergent but the radius of convergence is generally small.

When $\mu \neq 0$ generally the initial conditions generating the above two-body orbit will not conserve the periodicity and collision. We have already shown that periodicity is conserved by continuation. Now we will show that the characteristic of collision is also conserved by continuation.

For showing the validity of the continuation, orbits corresponding to the case when $e = 1$ (i.e. $G = 0$) are considered. The motion in $\xi_1\xi_2$ plane, in our case, is exactly the same as is found out by Giacaglia (1967), and as we have taken ξ_3 of the $O(\mu)$ therefore, when $e = 1$, the orbit starts as an ejection from the origin and returns to it after time $T/4$.

Consider now a periodic orbit ($\mu \neq 0$) starting at the origin as an ejection orbit. Levi-Civita (1904) found out the condition for collision as

$$\dot{\theta}+1 = \rho' f(\rho', \theta), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

where

$$\tan\theta = \frac{x_2}{x_1-\mu}; \quad \rho' = \sqrt{r_1}$$

$$= \frac{2\xi_1\xi_2}{\xi_1^2-\xi_2^2}$$

$$= \tan 2\phi.$$

Therefore,

$$\theta = 2\phi.$$

The condition (50) becomes, in our case, as

$$2\dot{\phi}+1 = r_1^{\frac{1}{2}} f(r_1^{\frac{1}{2}}, 2\phi)$$

or

$$2\frac{d\phi}{d\tau}\frac{d\tau}{dt}+1 = r_1^{\frac{1}{2}}f(r_1^{\frac{1}{2}}, 2\phi)$$

or

$$2\frac{d\phi}{d\tau}+r_1 = r_1^{\frac{3}{2}}f(r_1^{\frac{1}{2}}, 2\phi). \quad \ldots \quad (51)$$

But

$$\tan\phi = \frac{\xi_2}{\xi_1}$$

therefore

$$\frac{d\phi}{d\tau} = \frac{1}{4(\xi_1^2+\xi_2^2)}(\xi_1\pi_2-\xi_2\pi_1)-\tfrac{1}{2}\xi^2$$

$$= \frac{G}{2\xi^2}-\frac{\xi^2}{2}$$

where

$$r_1^2 = \xi^4+\xi_3^2; \quad \xi^2 = \xi_1^2+\xi_2^2.$$

Condition (51) becomes

$$\frac{G}{\xi^2}-\xi^2+r_1 = r_1^{\frac{3}{2}}f(r_1^{\frac{1}{2}}, 2\phi)$$

or

$$G = \xi^4-r_1\xi^2+\xi^2 r_1^{\frac{3}{2}}f(r_1^{\frac{1}{2}}, 2\phi). \quad \ldots \quad (52)$$

The condition (52) corresponds to (49). At $\tau = 0$, eqn. (52) is satisfied since both $G(0) = 0$ and $\xi(0) = 0$.

Following Bhatnagar (1969), the orbit chosen will certainly satisfy eqn. (52) since it starts at the origin. If the ejection angle for $\mu \neq 0$ case is almost the same as for the $\mu = 0$ case, then for sufficiently small value of μ, the two orbits will remain as close as we wish in a finite length of time (Poincaré 1905). This means that after a time $T/4$ has ellapsed the $\mu \neq 0$ orbit will re-enter an arbitrary small neighbourhood of $r_1 = 0$. Since eqn. (52) is satisfied along the entire orbit, the infinitesimal body will approach the origin with characteristic of a collision orbit (Levi-Civita 1904). Hence the proof of the existence of such periodic orbit in the collision is fully established.

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