THEOREMS ON BILINEAR GENERATING FUNCTIONS

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Two theorems for obtaining the bilinear generating functions in terms of integrals have been proved in this paper. The theorems have been used to obtain new bilinear generating functions. The results obtained have been further generalized to obtain bilinear generating functions of polynomials of higher variables.

1. INTRODUCTION

Recently, Saran (1970) gave the following theorem for obtaining the bilinear generating function:

If $f_n^\lambda(x) = \mu(n)\, G(x)\, D^n\{g(x)\}$, where $g(x)$ and $G(x)$ are independent of n, $D \equiv \dfrac{d}{dx}$ and

$$F(x, t) = \sum_{m=0}^{\infty} a_m t^m f_m^\lambda(x)$$

then

$$\frac{G(x)F(x-t, ty)}{G(x-t)} = \sum_{r=0}^{\infty} \frac{(-t)^r b_r(y) f_r^\lambda(x)}{\mu(r) r!},$$

where

$$b_r(y) = \sum_{m=0}^{r} (-r)_m \mu(m) a_m y^m.$$

In this paper two theorems for obtaining the bilinear generating functions in terms of integrals are given. These theorems have been used to obtain new bilinear generating functions. The results obtained have been further generalized to obtain bilinear generating functions of polynomials of higher variables.

The theorems are as follows:

Theorem 1—Let

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x) t^n \qquad \ldots \quad \ldots \quad \ldots \qquad (1.1)$$

where $f_n(x)$ is a polynomial of degree n in x, then

$$\frac{1}{\Gamma(b)}\int_0^\infty e^{-p}p^{b-1}\,{}_1F_1(c;\ b;\ yp/y-1)F(x,\,tp)\,dp$$
$$= (1-y)^c \sum_{n=0}^{\infty} (b)_n\,{}_2F_1(-n,\,c;\ b;\ y)f_n(x)t^n \quad \ldots \quad (1.2)$$

provided the integral is convergent.

Theorem 2—Let

$$G(x,\,t) = \sum_{n=0}^{\infty} g_n(x)t^n \quad \ldots \quad (1.3)$$

where $g_n(x)$ is a polynomial of degree n in x, then

$$\frac{1}{\Gamma(a)\Gamma(b)}\int_0^\infty\int_0^\infty e^{-(p+q)}p^{a-1}q^{b-1}\,{}_0F_1(a;\ ypq/y-1)G(x,\,tpq/1-y)\,dp\,dq$$
$$= (1-y)^b \sum_{n=0}^{\infty} (a)_n(b)_n\,{}_2F_1(-n,\,b+n;\ a;\ y)g_n(x)t^n \quad \ldots \quad (1.4)$$

provided the integral is convergent

2. Proof

To prove theorem 1 we put tp for t, multiply both sides by $e^{-p}p^{b-1}$ ${}_1F_1\,(c;\ b;\ yp/y-1)$ and integral w.r.t. p between 0 and ∞. We thus obtain L.H.S. of (1.2). The R.H.S. is equal to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(b)_{m+n}(c)_m}{(b)_m m\,!}\left(\frac{y}{y-1}\right)^m f_n(x)t^n$$
$$= \sum_{n=0}^{\infty} (b)_n f_n(x)t^n\,{}_2F_1(c,\,b+n\,;\,b\,;\,y/y-1).$$

This immediately proves the result, on using the transformation

$${}_2F_1(a,\,b\,;\,c\,;\,x) = (1-x)^{-b}\,{}_2F_1(c-a,\,b\,;\,c\,;\,x/x-1). \quad \ldots \quad (2.1)$$

Similarly if we multiply (1.3) by $e^{-(p+q)}p^{a-1}q^{b-1}\,{}_0F_1\left(a\,;\,\frac{ypq}{y-1}\right)$ with tpq for t and integrate w.r.t. p and q between 0 and ∞, we immediately obtain (1.4) after using (2.1).

3. Applications

(i) Let $f_n(x) = P_n^{(\alpha-n,\,\beta-n)}(x)$, then by Carlitz's result (1963)

$$F(x,\,t) = [1+\tfrac{1}{2}(x+1)t]^\alpha[1+\tfrac{1}{2}(x-1)t]^\beta.$$

Therefore, Theorem 1 gives

$$\sum_{m,\,n,\,p=0}^{\infty} \frac{(b)_{m+n+p}(-\alpha)_m(-\beta)_n(c)_p}{(b)_p\,m\,!\;n\,!\;p\,!}\,[-\tfrac{1}{2}(x+1)t]^m[-\tfrac{1}{2}(x-1)t]^n\left(\frac{y}{y-1}\right)^p$$
$$= (1-y)^c \sum_{n=0}^{\infty} (b)_n\,{}_2F_1(-n,\,c\,;\,b\,;\,y)P_n^{(\alpha-n,\,\beta-n)}(x)t^n.$$

Writing $\frac{t}{k}$ for t, multiplying by $e^k\, k^{-d}$ and evaluating with the help of the relation

$$\frac{1}{2\pi i}\int_C e^k k^{-d}\, dk = \frac{1}{\Gamma(d)} \tag{A}$$

where C is a contour of Hankel type, we obtain (Saran 1954)

$$(1-y)^{-c}F_G\left(b, b, b, c, -\alpha, -\beta; b, d, d; \frac{y}{y-1}, -\tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t\right)$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n}{(d)_n}\, {}_2F_1(-n, c; b; y)\, P_n^{(\alpha-n, \beta-n)}(x)t^n. \quad \ldots \quad \ldots \quad \ldots \tag{3.1}$$

F_G reduces to

$$[1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}(1-y)^c \times F^{(1)}\left(c; -\alpha, -\beta; b; \frac{y(x+1)t}{2+(x+1)t}, \frac{y(x-1)t}{2+(x-1)t}\right),$$

when $d = b$.

This leads to the result obtained by Saran [1970, (3.5)] by a different method.

(ii) Let $f_n(x) = P_n^{(\alpha, \beta)}(x)$, then by Brafman's (1951) result (*see* Rainville 1963, p. 270)

$$F(x, t) = F^{(4)}(1+\beta, 1+\alpha;\ 1+\alpha, 1+\beta;\ \tfrac{1}{2}t(x-1), \tfrac{1}{2}t(x+1)).$$

Then Theorem 1 gives

$$\sum_{m, n, p=0}^{\infty} \frac{(1+\alpha)_{m+n}(b)_{m+n+p}(1+\beta)_{m+n}(c)_p}{(1+\alpha)_m(1+\beta)_n(b)_p m!\ n!\ p!}[\tfrac{1}{2}t(x-1)]^m[\tfrac{1}{2}t(x+1)]^n\left[\frac{y}{y-1}\right]^p$$

$$= (1-y)^c \sum_{n=0}^{\infty} (b)_n\, {}_2F_1(-n, c;\ b;\ y)P_{n(x)}^{(\alpha, \beta)}t^n.$$

Writing $\frac{t}{k}$ for t, multiplying by $e^k\, k^{-1-\alpha}$ and integrating w.r.t. k with the help of (A), we obtain (Saran 1954)

$$(1-y)^{-c}F_E\left(b, b, b, c, 1+\beta, 1+\beta;\ b, 1+\alpha, 1+\beta;\ \frac{y}{y-1}, \tfrac{1}{2}t(x-1), \tfrac{1}{2}t(x+1)\right)$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n}{(1+\alpha)_n}\, {}_2F_1(-n, c;\ b;\ y)P_{n(x)}^{(\alpha, \beta)}t^n. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{3.2}$$

F_E reduces to

$$(1-tx)^{-b}H_4\left(b, c, 1+\alpha, b;\ \frac{t^2(x^2-1)}{4(1-tx)^2}, \frac{y}{(y-1)(1-tx)}\right)$$

when $\alpha = \beta$, on using the formula $\frac{(b)_{m+n}}{(b)_m(b)_n} = \sum_{s=0}^{\min(m, n)} \frac{(-m)_s(-n)_s}{(b)_s s!}$ where H_4 is defined by Horn (Erdélyi 1953, p. 225) as

$$H_4(\alpha, \beta, \gamma, \delta;\ x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n x^m y^n}{(\gamma)_m(\delta)_n m!\ n!}.$$

This is absolutely convergent if $4r = (s-1)^2$, $|x| < r$, $|y| < s$. We thus obtain

$$(1-y)^{-c}(1-tx)^{-b}H_4\left(b, c, 1+\alpha, b;\ \frac{t^2(x^2-1)}{4(1-tx)^2}, \frac{y}{(y-1)(1-tx)}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(b)_n}{(1+2\alpha)_n}\, {}_2F_1(-n, c;\ b;\ y)C_{n(x)}^{\alpha+\frac{1}{2}}t^n \qquad \ldots \quad \ldots \quad \ldots \ (3.2.1)$$

where C_n^{ν} is a Gegenbaur polynomial defined by

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} C_n^{\nu}(x)t^n,$$

where

$$C_n^{\alpha+\frac{1}{2}}(x) = \frac{(1+2\alpha)_n}{(1+\alpha)_n} P_n^{(\alpha, \alpha)}(x).$$

If $b = 1+2\alpha$, H_4 can be reduced and we obtain Chatterjea's [1969, (3.4)] result in a different way.

The above result (3.2.1) can be further reduced if y is replaced by $\frac{y}{c}$ and c be taken infinitely great. This leads to

$$(1-tx)^{-b}exp(y)H_7\left(b, 1+\alpha, b;\ \frac{t^2(x^2-1)}{4(1-tx)^2}, \frac{-y}{1-tx}\right)$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(1+2\alpha)_n} L_n^{b-1}(y)C_n^{\alpha+\frac{1}{2}}(x) \qquad \ldots \quad \ldots \quad \ldots \ (3.2.1.1)$$

where H_7 is a confluent form of H_4 [Erdélyi 1953, (3.5)]. This again leads to Weisner's result (Rainville 1963, p. 281) if $\frac{1}{2}+\alpha = \nu = \frac{b}{2}$ on using the following result due to Srivastava [1968, (5.4.25.3)]

$$exp\left(x+\frac{y}{2}\right){}_0F_1(\nu+\tfrac{1}{2};\ y^2/16)$$

$$= \left[\frac{4x(x+y)}{(2x+y)^2}\right]H_7\left((2\nu, \nu+\tfrac{1}{2}, 2\nu;\ \frac{y^2}{(2x+y)^2}, \frac{2x(x+y)}{2x+y}\right) \qquad \ldots \ (3.2.2)$$

where x is not equal to zero.

(iii) Let $g_n(x) = P_{n(x)}^{(\alpha, \beta)}$, then Theorem 2 gives

$$\sum_{m, n, p=0}^{\infty} \frac{(b)_{m+n+p}(a)_{m+n+p}(1+\beta)_{n+p}(1+\alpha)_{n+p}}{(a)_m(1+\alpha)_n(1+\beta)_p m!\, n!\, p!} \left(\frac{y}{y-1}\right)^m \left[\frac{1}{2}\ \frac{(x-1)t}{1-y}\right]^n \left[\frac{1}{2}\ \frac{(x+1)t}{1-y}\right]^p$$

$$= (1-y)^b \sum_{n=0}^{\infty} (a)_n(b)_n\, {}_2F_1(-n, b+n;\ a;\ y)P_n^{(\alpha, \beta)}(x)t^n.$$

Again multiplying both sides by $e^{p+q}\, p^{-1-\alpha}\, q^{-1-\beta}$ with $\frac{t}{pq}$ for t and using (A) we obtain (Appell and Kampé de Fériet 1926)

$$(1-y)^{-b}F_c\left[a,\, a,\, a,\, b,\, b,\, b,\, a,\, 1+\alpha,\, 1+\beta;\ \frac{y}{y-1},\, \tfrac{1}{2}t\left(\frac{x-1}{1-y}\right),\, \tfrac{1}{2}t\left(\frac{x+1}{1-y}\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1+\alpha)_n(1+\beta)_n}\, {}_2F_1(-n,\, b+n;\ a;\ y)P_n^{(\alpha,\beta)}(x)t^n. \quad \ldots \quad \ldots \quad (3.3)$$

This reduces to the result obtained by Manocha and Sharma (1967). This can also give the generating function of the product of Bessel polynomials for $P_n^{(\alpha,\beta)}(x)$ and ${}_2F_1\,(-n,\ b+n;\, a;\, y)$ can be reduced to Bessel polynomials as a limiting case (Agarwal 1954).

If y is replaced by $\frac{y}{\beta}$ and $\frac{1}{2}(1-x) = \frac{x}{\beta}$ and then β tends to infinity, the above result transforms into

$$(1-t)^{-1-\rho}\exp\,(y)\Psi_2\left(1+\rho,\, 1+\rho,\, 1+\alpha;\ \frac{-y}{1-t},\, \frac{-xt}{1-t}\right)$$

$$= \sum_{n=0}^{\infty} \frac{n\,!}{(1+\alpha)_n}\, t^n L^{\alpha}{}_n(x) L^{\rho}{}_n(y), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3.1)$$

after taking $b = \beta$, $a = 1+\delta$, where ψ_2 is a confluent case of Appell's function $F^{(4)}$ defined by

$$\Psi_2(\alpha,\, \gamma,\, \gamma';\ x,\, y) = \sum_{m,\, n=0}^{\infty} \frac{(\alpha)_{m+n}x^m y^n}{(\gamma)_m(\gamma')_n m\,!\ n\,!}.$$

This reduces to Hille-Hardy formula (Rainville 1963, p. 212), if we use the following transformation due to Srivastava [1968, (5.4), (27.3), 106)]

$$\Psi_2(\alpha,\, \alpha,\, \alpha;\ x,\, y) = \exp\,(x+y)\,{}_0F_1(\alpha;\ x,\, y).$$

(iv) Again if $g_n(n) = P_n^{\alpha-n,\,\beta-n}(x)$ then Theorem 2 gives

$$(1-y)^{-b}\sum_{m=0}^{\infty}\frac{(b)_m}{m\,!}\left(\frac{y}{y-1}\right)^m F_{2,\,0}^{2,\,1}\left(\begin{matrix} a+m,\, b+m: & -\alpha,\, -\beta; & \tfrac{1}{2}\frac{(x+1)t}{y-1},\, \tfrac{1}{2}\frac{(x+1)t}{y-1} \\ c,\ d & : & \end{matrix}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(d)_n}\, t^n\, {}_2F_1(-n,\, b+n;\ a;\ y)P_n^{(\alpha-n,\,\beta-n)}(x)t^n \quad \ldots \quad \ldots \quad (3.4)$$

after using (A), where $F_{2,\,0}^{2,\,1}$ is a Kampé de Fériet function, defined in a general case, in a modified notation, as

$$F_{r,\,s}^{p,\,q}\left(\begin{matrix}\alpha_p:\ \beta_q,\, \beta'_q;\ x,\, y \\ \gamma_r:\ \delta_s,\, \delta'_s\end{matrix}\right) = \sum_{m,\, n=0}^{\infty}\frac{(\alpha_p)_{m+n}(\beta_q)_m(\beta'_q)_n x^m y^n}{(\gamma_r)_{m+n}(\delta_s)_m(\delta'_s)_n m\,!\ n\,!}.$$

It is convergent if $p+q < r+s+1$. If $p+q = r+s+1$, it is absolutely convergent if $|x|+|y| < \min(1, 2^{s+q+1})$.

4. Generalization

The repeated application of theorems lead to the generalization of the result of § 3. For instance, (3.1) can be generalized as

$$(1-y_r)^{-c_r} \sum_{l_r=0}^{\infty} \frac{(c_r)_{l_r}}{l_r!} \left(\frac{y_r}{y_r-1}\right)^{l_r} F^{r,1}_{r,0}\begin{pmatrix} b_r+l_r: & -\alpha, -\beta; & -\frac{1}{2}(x+1)t, -\frac{1}{2}(x-1)t \\ d_r & : & \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(b_r)_n}{(d_r)_n} t^n {}_2F_1(-n, c_r; b_r; y_r) P_n^{\alpha-n, \beta-n}(x). \quad \ldots \quad (4.1)$$

Similarly (3.2) and (3.3) may be generalized as

$$(1-y_r)^{-c_r} \sum_{l_r=0}^{\infty} \frac{(c_r)_{l_r}}{l_r!} \left(\frac{y_r}{y_r-1}\right)^{l_r} F^{r+1,0}_{s,1}\begin{pmatrix} b_r+l_r, 1+\beta :-; \frac{1}{2}(x-1)t, \frac{1}{2}(x+1)t \\ d_s : 1+\alpha, 1+\beta \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(b_r)_n}{(1+\alpha)_n (d_s)_n} {}_2F_1(-n, c_r; b_r; y_r) P_n^{(\alpha,\beta)}(x) t^n \quad \ldots \quad (4.2)$$

$$(1-y_r)^{-b_r} \sum_{l_r=0}^{\infty} \frac{(b_r)_{l_r}}{l_r!} \left(\frac{y_r}{y_r-1}\right)^{l_r} F^{2r,0}_{s,1}\begin{pmatrix} a_r+l_r, b_r+l_r :-; \frac{t(x-1)}{2(1-y_r)}, \frac{1}{2}\frac{t(x+1)}{(1-y_r)} \\ d_s : 1+\alpha, 1+\beta \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(a_r)_n (b_r)_n}{(1+\alpha)_n (1+\beta)_n (d_s)_n} {}_2F_1(-n, b_r+n; a_r; y_r) P_n^{(\alpha,\beta)}(x) \quad \ldots \quad (4.3)$$

and (3.4) gives

$$(1-y_r)^{-b_r} \sum_{l_r=0}^{\infty} \frac{(b_r)_{l_r}}{l_r!} \left(\frac{y_r}{y_r-1}\right)^{l_r} F^{2r,1}_{2s,0}\begin{pmatrix} a_r+l_r, b_r+l_r :-\alpha, -\beta; -\frac{t}{2}\left(\frac{x+1}{1-y}\right), -\frac{t}{2}\left(\frac{x-1}{1-y}\right) \\ c_s, d_s : \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(a_r)_n (b_r)_n}{(c_s)_n (d_s)_n} t^n {}_2F_1(-n, b_r+n; a_r: y_r) P_n^{(\alpha-n,\beta-n)}(x). \quad \ldots \quad (4.4)$$

Where $(1-y_r)^{-f_r}$ stands for $\prod_{n=1}^{r} (1-y_n)^{-f_n}$ and

$${}_2F_1(-n, a_r; b_r; y_r) = {}_2F_1(-n, a_1; b_1; y_1) \ldots {}_2F_1(-n, a_r; b_r; y_r).$$

Similarly $(A_r)_{l_r}$ denotes $(A_r)_{l_1} \ldots (A_r)_{l_r}$ and $\sum_{l_r=0}^{\infty} = \sum_{l_1=0}^{\infty} \ldots \sum_{l_r=0}^{\infty}$.

5. Extension of Theorems to Polynomials of Two and Higher Variables

If $F(x_1, x_2, t_1, t_2) = \sum_{m, n=0}^{\infty} f_{m, n}(x_1, x_2)t_1^m t_2^n$, then analogous theorem to (1.2) can be written as

$$\frac{1}{\Gamma(b_1)\Gamma(b_2)}\int_0^\infty\int_0^\infty e^{-(p_1+p_2)}p_1^{b_1-1}p_2^{b_2-1}\,{}_1F_1(c_1;\ b_1;\ y_1p_1/y_1-1)\,{}_1F_1(c_2;\ b_2;\ y_2p_2/y_2-1)$$
$$F(x_1, x_2, t_1p_1, t_2p_2)dp_1dp_2$$
$$= (1-y_1)^{c_1}(1-y_2)^{c_2}\sum_{m, n=0}^{\infty}(b_1)_m(b_2)_n\,{}_2F_1(-n, c_1;\ b_1;\ y_1)\,{}_2F_1(-m, c_2;\ b_2;\ y_2)$$
$$f_{m, n}(x_1, x_2)t_1^m t_2^n, \qquad (5.1)$$

provided the integrals are convergent.

Similarly the theorem analogous to (1.4) is the following:

If

$$G(x_1, x_2, t_1, t_2) = \sum_{m, n=0}^{\infty} g_{m, n}(x_1, x_2)t_1^m t_2^n,$$

then

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-(p_1+p_2+p_3+p_4)}p_1^{a_1-1}p_2^{a_2-1}p_3^{a_3-1}p_4^{a_4-1}$$
$$\times\,{}_0F_1\left(a_1;\frac{y_1p_1p_3}{y_1-1}\right){}_0F_1\left(a_2;\frac{y_2p_2p_4}{y_2-1}\right)G\left(x_1, x_2\ \frac{t_1p_1p_3}{1-y_1}, \frac{t_2p_2p_4}{1-y_2}\right)dp_1dp_2dp_3dp_4$$
$$= (1-y_1)^{b_1}(1-y_2)^{b_2}\sum_{m, n=0}^{\infty}(a_1)_m(b_1)_m(a_2)_n(b_2)_n\,{}_2F_1(-m, b_1+m;\ a_1;\ y_1)$$
$$.\,{}_2F_1(-n, b_2+n;\ a_2;\ y_2)g_{m,n}(x_1, x_2)t_1^m t_2^n \qquad (5.2)$$

provided the integrals are convergent.

6. Applications

(i) From the generating function

$${}_0F_1(\gamma;\ -xt)\,{}_0F_1(\gamma';\ -y\tau)\,{}_0F_1(1+\alpha-\gamma-\gamma';\ (t+\tau)(1-x-y))$$
$$= \sum_{m\ n=0}^{\infty}\frac{t^m\tau^n}{(1+\alpha-\gamma-\gamma')_{m+n}}\mathcal{F}_{m, n}(\alpha, \gamma, \gamma';\ x, y)$$

where $\mathcal{F}_{m, n}$ is a Jacobi polynomial of two variables (Appell and Kampé de Fériet 1926, p. 100), we get on applying (5.1)

$$\sum_{r, s=0}^{\infty}\frac{(b_1)_r(b_2)_s}{(1+\alpha-\gamma-\gamma')_{r+s}r!\,s!}[t(1-x-y)]^r[\tau(1-x-y)]^s$$
$$.\,\psi_1\left(b_1+r, c_1, b_1, \gamma;\frac{y_1}{y_1-1}, -xt\right)\psi_1\left(b_2+s, c_2, b_2, \gamma';\frac{y_2}{y_2-1}, -y\tau\right)$$

$$= (1-y_1)^{c_1}(1-y_2)^{c_2} \sum_{m,\,n=0}^{\infty} \frac{(b_1)_m(b_2)_n t^m \tau^n}{(1+\alpha-\gamma-\gamma')_{m+n}} \cdot {}_2F_1(-n,\,c_1\,;\,b_1\,;\,y_1)$$

$${}_2F_1(-m,\,c_2\,;\,b_2\,;\,y_2)\mathfrak{F}_{m,\,n}(\alpha,\,\gamma,\,\gamma'\,;\,x,\,y), \qquad \ldots \qquad (6.1)$$

where Ψ is a confluent form of F^2.

(ii) If (5.2) is applied, then

$$\sum_{r,\,s=0}^{\infty} \frac{(a_1)_r(b_1)_r(a_2)_s(b_2)_s}{(1+\alpha-\gamma-\gamma')_{r+s} r!\; s!} [t(1-x-y)]^r [\tau(1-x-y)]^s$$

$$\cdot F_4\left(a_1+r,\, b_1+r;\; a_1,\, \gamma;\; y_1/y_1-1,\, \frac{-xt}{1-y_1}\right) F_4\left(a_2+s,\, b_2+s;\; a_2,\, \gamma';\; \frac{y_2}{y_2-1},\, \frac{-y\tau}{1-y_2}\right)$$

$$= (1-y_1)^{b_1}(1-y_2)^{b_2} \sum_{m,\,n=0}^{\infty} \frac{(a_1)_m(b_1)_m(a_2)_n(b_2)_n}{(1+\alpha-\gamma-\gamma')_{m+n}}$$

$$\cdot\, {}_2F_1(-m,\, b_1+m;\; a_1;\; y_1)\, {}_2F_1(-n,\, b_2+n;\; a_2;\; y_2)\mathfrak{F}_{m,\,n}(\alpha,\,\gamma,\,\gamma_3';\; x,\, y)t^m\tau^n. \qquad (6.2)$$

7. Remarks

The theorems (5.1) and (5.2) suggest that it is possible to extend it further to polynomials of any finite degree.

The results of § 4 indicate that even ${}_2F_1(-n,\, c;\, b,\, y)$ and ${}_2F_1(-n,\, c+n;\, b\,;\, y)$ can be generalized to ${}_pF_q$. For instance, writing (3.2) as

$$(1-y)^{-c} \sum_{l=0}^{\infty} \frac{(c)_l}{l!} \left(\frac{y}{y-1}\right)^l F^{1,\,1}_{1,\,0}\left(\begin{matrix} b+l\colon & -\alpha,\, -\beta;\; -\tfrac{1}{2}(x+1)t,\, -\tfrac{1}{2}(x-1)t \\ d \quad \colon & - \end{matrix}\right)$$

$$= \sum_{l=0}^{\infty} \frac{(b)_l}{(d)_l} {}_2F_1(-l,\, c;\; b;\; y) P_l^{(\alpha-l,\, \beta-l)}(x)t^l$$

we can have

$$(1-y)^{-c_p} \sum_{r=0}^{\infty} \frac{(c_1)_r \ldots (c_p)_r}{r!} \left(\frac{y}{y-1}\right)^r$$

$$\cdot F^{p,\,1}_{p,\,0}\left(\begin{matrix} b_1+r,\, \ldots,\, b_p+r\colon & -\alpha,\, -\beta;\; -\tfrac{1}{2}(x+1)t,\, -\tfrac{1}{2}(x-1)t \\ d_1,\, \ldots,\, d_p & \end{matrix}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(b_1)_n \ldots (b_p)_n}{(d_1)_n \ldots (d_p)_n} {}_2F_1(-n,\, c_p;\; b_p;\; y) P_n^{(\alpha-n,\, \beta-n)}(x)t. \qquad \ldots \qquad (7.1)$$

Similarly we may write three more results corresponding to (4.2), (4.3) and (4.4).

From Theorems 1 and 2 it is possible to obtain the polynomials generated by hypergeometric functions of three variables, defined and studied by Saran (1954) by comparing the integrals given by the said functions in Laplacian form (Saran 1957).

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