ON A NUMERICAL SOLUTION OF AN INTEGRAL EQUATION OF THE FIRST KIND

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(Communicated by J. N. Kapur, F.N.A.)

(*Received 10 August 1970; after revision 25 November 1970*)

An approximate method for the solution of an integral equation of the first kind having a logarithmic singularity has been derived in this paper. Comparison with an analytical solution showed that the approximate solution is sufficiently accurate for use in practical problems.

1. INTRODUCTION

Most boundary value problems of aerodynamics, classical elasticity, potential theory and electrodynamics can conveniently be formulated in terms of integral equations. In such cases, the integral equations involve, in general, singular kernels which hamper their theoretical and numerical treatments. The analytical solution of a boundary integral equation is, in most cases, impossible. An alternative procedure is, therefore, the numerical treatment replacing the integral equation by a system of simultaneous linear algebraic equations and solving them by modern digital computer methods.

Little work has been done so far on integral equations of first kind, but extensive literature exists for those of the second kind. Although several numerical methods were proposed for solving equations of first kind (*see* for example Fox and Goodwin 1953, Young 1954, Phillips 1962, Aleksandrov 1962, Symm 1963), a majority of them deal with only non-singular kernels.

In this paper, we present a numerical method for the solution of the singular integral equation

$$F(x) = \int_0^\infty \left[P(x, t) \log \left|\frac{t-x}{t+x}\right| + Q(x, t)\right] f(t)\, dt \qquad \ldots \qquad \ldots \quad (1.1)$$

where P, Q and F are prescribed well-behaved functions and $f(t)$ is unknown.

Many problems of physical interest reduce to the problem of solving integral equations of type (1.1) (Pearson 1957, Thwaites 1960, Tu and Gazis 1964). The products $P(x, t)f(t)$ and $Q(x, t)f(t)$ in (1.1) will be approximated by the generalized trapezoidal rule in § 2, and a generalized approximation method in § 3. No satisfactory estimation of error has been possible for these numerical methods and, in order to provide an idea of the error involved in the method given in § 2, a numerical example is presented in § 4 and its analytical solution based on the application of Mellin transform is given in § 5. Finally, § 6 contains some computational notes, numerical results and discussion.

2. The Generalized Trapezoidal Rule

Equation (1.1) may be written as

$$F(ih) \doteqdot \sum_{j=0}^{N-1} \int_{jh}^{\overline{j+1}h} \left[P(ih, t) \log \left|\frac{ih-t}{ih+t}\right| + Q(ih, t)\right] f(t)\, dt, \qquad i = 1, 2, \ldots, N \quad .. \quad (2.1)$$

$$= \sum_{j=0}^{N-1} \int_{jh}^{\overline{j+1}h} \left[\left\{\frac{(\overline{j+1}h-t)}{h} . P(ih, jh) f(jh) + \frac{(t-jh)}{h} . P(ih, \overline{j+1}h) f(\overline{j+1}h)\right\} \log \left|\frac{ih-t}{ih+t}\right| + \left\{\frac{(\overline{j+1}h-t)}{h} . Q(ih, jh) f(jh) + \frac{(t-jh)}{h} . Q(ih, \overline{j+1}h) f(\overline{j+1}h)\right\}\right] dt.$$

For the numerical example considered in this paper (*see* § 4), it will be seen that $P(x, 0) = Q(x, 0) = 0$. Setting $t = jh+ph$ in $jh \leqslant t \leqslant \overline{j+1}h$ and simplifying, the above system takes the matrix form $A\underset{\sim}{f} = \underset{\sim}{F}$, where

$$\left.\begin{aligned} a_{ij} &= P_{ij} \int_0^1 \left[p \log \left|\frac{i+1-j-p}{i-1+j+p}\right| + (1-p) \log \left|\frac{i-j-p}{i+j+p}\right|\right] dp + Q_{ij}, \\ & \qquad i = 1, 2, \ldots, N; \quad j = 1, 2, \ldots, N-1 \\ \text{and} \\ a_{iN} &= P_{iN} \int_0^1 p \log \left|\frac{i-N+1-p}{i+N-1+p}\right| dp + \tfrac{1}{2} Q_{iN}. \end{aligned}\right\} \quad .. \quad (2.2)$$

In the above, $P_{ij} = P(ih, jh)$, $Q_{ij} = Q(ih, jh)$, etc.

3. Generalized Approximation Method

Equation (1.1) may be written as

$$\sum_{j=0}^{N-1} \int_{jh}^{\overline{j+1}h} \left[P(x, t) \log \left|\frac{x-t}{x+t}\right| + Q(x, t)\right] f(t)\, dt = F(x),$$

i.e.

$$\int_0^h \left[P(x, t) \log \left|\frac{x-t}{x+t}\right| + Q(x, t)\right] f(t)\, dt + \sum_{j=1}^{N-2} \int_{jh}^{\overline{j+1}h} \left[P(x, t) \log \left|\frac{x-t}{x+t}\right| + Q(x, t)\right] f(t)\, dt + \int_{\overline{N-1}h}^{Nh} \left[P(x, t) \log \left|\frac{x-t}{x+t}\right| + Q(x, t)\right] f(t)\, dt \doteqdot F(x). \quad .. \quad (3.1)$$

Instead of using the trapezoidal rule throughout, we proceed now as follows: In the interval $(0, h)$, we approximate the products $P(x, t)f(t)$ and $Q(x, t)f(t)$ by a quadratic through $0, h, 2h$; in the interval $(\overline{N-1}h, Nh)$, we approximate the products by a quadratic through $\overline{N-2}h$, $\overline{N-1}h$, Nh; in the remaining intervals $(h, 2h)$, $(2h, 3h)$, ..., $(\overline{N-2}h, \overline{N-1}h)$, we approximate the products by cubics through $\overline{j-1}h$, jh, $\overline{j+1}h$, $\overline{j+2}h$. Finally, as before, we set $x = h, 2h, \ldots, Nh$.

Hence, in the interval $(0, h)$, we get

$$\left.\begin{aligned}
&\int_0^h \left[P(x,t)f(t)\log\left|\frac{x-t}{x+t}\right| + Q(x,t)f(t)\right] dt \\
&\quad \doteqdot P(x,0)\,.\,f_0\,.\int_0^h \frac{(t-h)(t-2h)}{(-h)(-2h)}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + P(x,h)\,.\,f_1\,.\int_0^h \frac{t(t-2h)}{-h^2}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + P(x,2h)\,.\,f_2\,.\int_0^h \frac{t(t-h)}{2h^2}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + Q(x,0)\,.\,f_0\,.\,\frac{5}{12}h + Q(x,h)\,.\,f_1\,.\,\frac{2}{3}h + Q(x,2h)\,.\,f_2\,.\left(-\frac{1}{12}h\right)
\end{aligned}\right\}. \tag{3.2}$$

In the intervals $(jh, \overline{j+1}h)$, $j = 1, 2, \ldots, \overline{N-2}$, we get

$$\left.\begin{aligned}
&\int_{jh}^{\overline{j+1}h} \left[P(x,t)f(t)\log\left|\frac{x-t}{x+t}\right| + Q(x,t)f(t)\right] dt \\
&\quad \doteqdot P(x,\overline{j-1}h)\,.\,f_{j-1}\int_{jh}^{\overline{j+1}h} \frac{(t-jh)(t-\overline{j+1}h)(t-\overline{j+2}h)}{(h)(-h)(-2h)(-3h)}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + P(x,jh)\,.\,f_j\,.\int_{jh}^{\overline{j+1}h} \frac{(t-\overline{j-1}h)(t-\overline{j+1}h)(t-\overline{j+2}h)}{(-h)\,.\,h\,.\,(-2h)}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + P(x,\overline{j+1}h)\,.\,f_{j+1}\,.\int_{jh}^{\overline{j+1}h} \frac{(t-\overline{j-1}h)(t-jh)(t-\overline{j+2}h)}{2h\,.\,h\,.\,(-h)}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + P(x,\overline{j+2}h)\,.\,f_{j+2}\,.\int_{jh}^{\overline{j+1}h} \frac{(t-\overline{j-1}h)(t-jh)(t-\overline{j+1}h)}{3h\,.\,2h\,.\,h}\log\left|\frac{x-t}{x+t}\right| dt \\
&\quad + Q(x,\overline{j-1}h)\,.\,f_{j-1}\,.\left(-\frac{1}{24}h\right) + Q(x,jh)\,.\,f_j\,.\left(\frac{13}{24}h\right) \\
&\quad + Q(x,\overline{j+1}h)\,.\,f_{j+1}\,.\left(\frac{13}{24}h\right) + Q(x,\overline{j+2}h)\,.\,f_{j+2}\,.\left(-\frac{1}{24}h\right)
\end{aligned}\right\}. \tag{3.3}$$

Finally, in $(\overline{N-1}h, Nh)$, we have

$$\left.\begin{aligned}
&\int_{\overline{N-1}h}^{Nh}\left[P(x,t)f(t)\log\left|\frac{x-t}{x+t}\right|+Q(x,t)f(t)\right]dt\\
&\quad\doteqdot P(x,\overline{N-2}h)\,.\,f_{N-2}\,.\int_{\overline{N-1}h}^{Nh}\frac{(t-\overline{N-1}h)(t-Nh)}{(-h)(-2h)}\log\left|\frac{x-t}{x+t}\right|dt\\
&\quad+P(x,\overline{N-1}h)\,.\,f_{N-1}\,.\int_{\overline{N-1}h}^{Nh}\frac{(t-\overline{N-2}h)(t-Nh)}{h(-h)}\log\left|\frac{x-t}{x+t}\right|dt\\
&\quad+P(x,Nh)\,.\,f_N\,.\int_{\overline{N-1}h}^{Nh}\frac{(t-\overline{N-2}h)(t-\overline{N-1}h)}{2h\,.\,h}\log\left|\frac{x-t}{x+t}\right|dt\\
&\quad+Q(x,\overline{N-2}h)\,.\,f_{N-2}\,.\left(-\frac{1}{12}h\right)+Q(x,\overline{N-1}h)\,.\,f_{N-1}\,.\left(\frac{2}{3}h\right)\\
&\quad+Q(x,Nh)\,.\,f_N\,.\left(\frac{5}{12}h\right)
\end{aligned}\right\}. \qquad (3.4)$$

Using $P(x, 0) = Q(x, 0) = 0$, and the relations (3.2), (3.3) and (3.4), and setting $x = ih$, $i = 1, 2, \ldots, N$, eqn. (3.1) reduces, as before, to N equations in N unknowns $f_1, f_2, \ldots, f_N$.

4. Numerical Example

We consider the integral equation

$$F(x) = \int_0^\infty t f(t) \int_0^{2\pi} \frac{d\theta}{(x^2-2xt\cos\theta+t^2)^{\frac{1}{2}}}\,dt \qquad \ldots \quad \ldots \quad (4.1)$$

the solution of which is required in a variety of problems, e.g. the problem of electrified disc (Copson 1947), and a mixed boundary value problem of elasticity (Sneddon 1947).

It is convenient to write (4.1) as

$$F(x) = 4\int_0^\infty \frac{t}{x+t} f(t)\, K\left\{\frac{4xt}{(x+t)^2}\right\}^{\frac{1}{2}} dt \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

where

$$K\left\{\frac{4xt}{(x+t)^2}\right\}^{\frac{1}{2}} = \int_0^{\pi/2} \frac{d\theta}{\left\{1-\dfrac{4xt}{(x+t)^2}\cos^2\theta\right\}^{\frac{1}{2}}}. \qquad \ldots \quad \ldots \quad (4.3)$$

The kernel in (4.2) can be split into the form

$$\left.\begin{aligned} & P(x,t)\log\left|\frac{t-x}{t+x}\right|+Q(x,t) \\ \text{where} \\ & P(x,t) = -\frac{2t}{x+t}\sum_{i=0}^{n} b_i k_1^{2i} \\ & Q(x,t) = \frac{t}{x+t}\sum_{i=0}^{n} a_i k_1^{2i} \\ \text{and} \\ & k_1^2 = \left(\frac{x-t}{x+t}\right)^2 \end{aligned}\right\} \quad \ldots \quad (4.4)$$

In (4.4), the coefficients a_i and b_i are those computed by Cody (1965), and these values (up to those for $n = 6$) are used in the computation. The eqn. (4.2) now becomes

$$\frac{F(x)}{4} = \int_0^\infty \left[P(x,t)\log\left|\frac{t-x}{t+x}\right|+Q(x,t)\right] f(t)\, dt \quad \ldots \quad (4.5)$$

which is of the same type as (1.1), and can be solved by the methods given in §§ 2 and 3.

5. Application of Mellin Transform

From (4.2), we obtain

$$\begin{aligned} \int_0^\infty F(x)\,.\,x^{u-1}\,dx &= 4\int_0^\infty \frac{t}{x+t} f(t)\,.\int_0^\infty x^{u-1} K\left\{\frac{4xt}{(x+t)^2}\right\}^{\frac{1}{2}} dt\,dx \\ &= 4\int_0^\infty t f(t)\,.\int_0^\infty \frac{x^{u-1}}{x+t} K\left\{\frac{4xt}{(x+t)^2}\right\}^{\frac{1}{2}} dt\,dx \\ &= 4\int_0^\infty t^u f(t)\,.\int_0^\infty \frac{x^{u-1}}{1+x} K\left\{\frac{4x}{(1+x)^2}\right\}^{\frac{1}{2}} dx\,dt. \quad \ldots \quad (5.1) \end{aligned}$$

Using the relations given in Bateman (1953, 1954), the inner integral of the right side of (5.1) can be simplified, and finally eqn. (5.1) becomes

$$\int_0^\infty F(x)\,x^{u-1}\,dx = \frac{2\pi^3}{\sin \pi u}\cdot\frac{1}{[\Gamma(\frac{1}{2}+\frac{1}{2}u)]^2\,[\Gamma(1-\frac{1}{2}u)]^2}\cdot\int_0^\infty t^u f(t)\,dt.$$

On interchanging x and t, this gives

$$\int_0^\infty x^u f(x)\,dx = \frac{\sin \pi u}{2\pi^3}\,.\,[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}u)\,.\,\Gamma(1-\tfrac{1}{2}u)]^2\,.\int_0^\infty F(t)\,t^{u-1}\,dt.$$

Writing the left-hand side as $\int_0^\infty x^{u-1}.\,x\,.f(x)\,dx$ and using the definition of the inverse Mellin transform, we obtain

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-u-1}\,.\,\frac{\sin \pi u}{2\pi^3}\,.\left[\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(\frac{2-u}{2}\right)\right]^2.\int_0^\infty t^{u-1}F(t)\,dt\,du,$$
$$0 < c < 1.$$

in which the order of integration cannot be changed.
But rewriting this double integral in the form

$$f(x) = \frac{d^2}{dx^2}\frac{1}{2\pi^3}\int_0^\infty F(t)\cdot\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{x}\right)^{u-1}\cdot\frac{\sin\pi u}{u(u-1)}\cdot\left[\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(1-\frac{u}{2}\right)\right]^2 du\,dt \quad \ldots \quad (5.2)$$

we can evaluate it in both cases when $\frac{t}{x} < 1$ and $\frac{t}{x} > 1$.

Now, consider the inner integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{t}{x}\right)^{u-1}\cdot\frac{\sin\pi u}{u(u-1)}\left[\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(1-\frac{u}{2}\right)\right]^2 du, \quad 0 < c < 1.$$

When $\frac{t}{x} = z < 1$, the integral can be evaluated by taking a semicircle on the right as the contour, and its value is

$$-4\pi\sum_{n=1}^{\infty} z^{2n-1}\cdot\left[\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n)}\right]^2\cdot\frac{1}{2n(2n-1)} \quad \ldots \quad \ldots \quad (5.3)$$

the negative sign being taken because the contour is encircled in a clockwise direction.

When $\frac{t}{x} = z > 1$, the integral is evaluated by taking a semicircle on the left as the contour, and its value is

$$-4\pi\sum_{n=1}^{\infty} z^{-2n}\cdot\left[\frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n)}\right]^2\cdot\frac{1}{2n(2n-1)}. \quad \ldots \quad \ldots \quad (5.4)$$

Hence

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{z^{u-1}\sin\pi u}{u(u-1)}\cdot\left[\Gamma\left(\frac{u+1}{2}\right)\Gamma\left(1-\frac{u}{2}\right)\right]^2 du$$

$$= \begin{cases} -4\pi\sum_{n=1}^{\infty} z^{2n-1}\cdot\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right]^2\cdot\frac{1}{2n(2n-1)}, & z < 1 \\ -4\pi\sum_{n=1}^{\infty} z^{-2n}\cdot\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right]^2\cdot\frac{1}{2n(2n-1)}, & z > 1. \end{cases} \quad \ldots \quad (5.5)$$

Now

$$\sum_{n=1}^{\infty} z^{2n}\cdot\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right]^2\cdot\frac{1}{2n(2n-1)}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty} z^{2n}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(n-\frac{1}{2})}{\Gamma(n)\,\Gamma(n+1)}$$

$$= -\tfrac{1}{2}z\,E'(z) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.6)$$

since

$$E(z) = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; z^2) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n (\frac{1}{2})_n}{n!\,(1)_n} z^{2n}$$

$$= -\frac{1}{4} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,\Gamma(n-\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+1)} z^{2n}$$

using

$$\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$$

and

$$E'(z) = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})\,\Gamma(n-\frac{1}{2})}{\Gamma(n)\,\Gamma(n+1)} z^{2n-1}$$

from which follows the result (5.6) above.

Hence (5.5), where $z = \dfrac{t}{x}$, now becomes

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{t}{x}\right)^{u-1} \sin \pi u}{u(u-1)} \left[\Gamma\left(\frac{u+1}{2}\right) \Gamma\left(1-\frac{u}{2}\right)\right]^2 du$$

$$= \begin{cases} 2\pi\, E'\left(\dfrac{t}{x}\right), & \text{for } t < x \\ 2\pi \dfrac{x}{t}\, E'\left(\dfrac{x}{t}\right), & \text{for } t > x. \end{cases}$$

Therefore, (5.2) gives

$$f(x) = \frac{1}{\pi^2} \frac{d^2}{dx^2} \left[\int_0^x F(t)\, E'\left(\frac{t}{x}\right) dt + \int_x^\infty \frac{x}{t} F(t)\, E'\left(\frac{x}{t}\right) dt\right].$$

By using the formula for integration by parts, the above equation can be put into the more convenient form

$$f(x) = \frac{1}{\pi^2} \frac{d^2}{dx^2} \left[2x\, F(x) - \int_0^x x\, E\left(\frac{t}{x}\right) F'(t)\, dt + \int_x^\infty E\left(\frac{x}{t}\right) \frac{d}{dt}\Big[t\, F(t)\Big]\, dt\right] \tag{5.7}$$

which is the solution of (4.1).

6. Computational Considerations

The method of § 2 was programmed in 'Atlas Autocode' for the KDF 9 computer at Edinburgh University and those of §§ 3 and 5 in Fortran IV for the IBM-7044 computer at I.I.T., Kanpur.

To allow maximum storage space in each case, the programmes were subdivided as follows:

(i) Calculation of integrals of the form

$$I_n = \int_0^1 t^n \log \left|\frac{x-t}{x+t}\right| dt.$$

This was accomplished by the formula

$$I_n = \frac{1}{n+1}\left[\log\left|\frac{1-x}{1+x}\right| - x^{n+1}\{\log|1-x| + (-1)^n \log|1+x|\}\right.$$
$$\left. + x^{n+1}\{1+(-1)^n\}\log|x|\right] - \frac{2}{n+1}\left[\frac{x}{n} + \frac{x^3}{n-2} + \frac{x^5}{n-4} + \dots\right].$$

(ii) Calculation of P_{ij}, Q_{ij}, and setting up of linear equations.

(iii) Solution of equations, which was carried out by a standard sub-routine, which factorizes the matrix A into the product LU and then with back substitution solves the equations and stores the values of $f(t)$.

For the Mellin transform method of § 5, the derivatives required on the right side of eqn. (5.7) are computed by using a sub-routine 'DIFFL' which is based on the evaluation of the coefficients (Giammo 1962) for the n-ordinates in the n—*pt* finite difference expression for the Kth derivative evaluated at a point, say xj.

Table I illustrates the effect of the number of equations on the stability of the solution for the methods of §§ 2 and 3 with $F(x) = \exp(-x)$. N is the number of equations used in the evaluation of the unknown function $f(x)$; the next column is the number of decimal places (D_1) unaffected by increasing N in the generalized trapezoidal rule; and the last column is the number of decimal places (D_2) in $f(x)$ unaffected by increasing N in the generalized approximation method.

TABLE I

N	D_1	D_2
10	4	4
20	5	5
30	5	6
40	5	6

TABLE II

X	Generalized trapezoidal rule	Mellin transform method
3·0	−0·0029648	−0·003182
4·0	−0·0025459	−0·003016
5·0	−0·0016901	−0·001505
6·0	−0·0010730	−0·001146
7·0	−0·000700	−0·000762

Although the computations were made for a number of other elementary functions such as exp $(-X^2)$, sin (X), cos (X), $1/(1+X^2)$, only those for exp $(-X)$ are reported in this paper. These computations were made using the double-precision arithmetic (equivalent to about 16 decimals) throughout. Table II compares the values of $f(x)$ obtained by the methods of §§ 2 and 5 with $F(X) = \exp(-X)$.

Tables I and II indicate that the generalized trapezoidal rule yields sufficiently accurate solution for use in practical problems.

Acknowledgements

Sections 2 and 5 of this paper are taken from the Project Work submitted to the University of Edinburgh, and the author wishes to express his sincere appreciation to Mr. D. Kershaw for his useful advice and supervision throughout the work. He also thanks the referee for his most helpful comments and the Head of the Computer Centre, I.I.T., Kanpur, for facilities to use the IBM-7044 computer. Grateful acknowledgement is made of a CSIR research scheme for carrying out the computations at Kanpur.

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