SOME CONSEQUENCES OF MILLIONSHTCHIKOV'S HYPOTHESIS IN THE EARLY-PERIOD DECAY PROCESS OF TURBULENCE

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In the present work the dynamical equation for the early-period decay of most general form of turbulence in the transformed (energy) space has been obtained by using Millionshtchikov's hypothesis of quasi-normality in the fluctuating components of velocities at several points within the turbulent fluid. Two lemmas have been established to obtain the limiting forms of spectrum tensors and of Millionshtchikov's hypothesis when two or more points under reference coincide. These lemmas have been freely used to derive the final equation for early-period decay of general form of turbulence. The above equation has been firstly simplified for the case of homogeneous turbulence and subsequently for the case of isotropic turbulence obtaining thereby the well-known decay equation due to Reid and Proudman (1954).

1. Introduction

It is known that even in the simple case of isotropic turbulence, one cannot straightway solve Karman-Howarth equation without any additional assumption. This is due to the fact that one arrives at this single equation with two unknown functions at the expense of all of the fundamental equations, viz. equations of motion and equations of continuity. If one tries to form an equation with moments one order higher than those which appear in Karman-Howarth equation, one obtains an equation with third and fourth order velocity moments. It is to be noted that Karman-Howarth equation contains second and third order moments while this new equation will contain third and fourth order moments. Thus two equations are obtained with three unknowns and so problem of indeterminacy remains. Constructions of still higher order moments and derivation of equations relating them cannot remove this indeterminacy as, in each successive stage, the number of unknown functions will be greater than the number of equations by one.

To overcome the above indeterminacy, one has to introduce a certain hypothesis which, in its turn, is consistent with the physical picture of turbulence. Accordingly, we find that in the works of Obukhov (1941), Heisenberg (1948*b*), Karman (1948) the respective author has put forward forms for the transfer term in the equation of decay of turbulent energy in case of isotropic turbulence. Besides this, Millionshtchikov (1941*a*, *b*) advances his hypothesis of quasi-normality in velocity distributions of turbulent fluid. Millionshtchikov's hypothesis states that the fourth order velocity correlations are related to the second order velocity correlations as in a normal distribution. There are experimental evidences both in favour of and against this hypothesis. It is useful to take note of the experiments conducted by Uberoi (1953) which confirm the complete validity of this hypothesis in case of multiple-point correlations (except for the particular case of two-point correlations). The applications of Millionshtchikov's hypothesis in the present work is confined to multiple-point correlations and the validity of this hypothesis in the problem considered here may be acceptable.

2. Velocity Correlations and Their Transforms

Consider a turbulent fluid medium extending to infinity in all directions. Let $u_i(\vec{X}, t)$ and $u'_j(\vec{X}', t)$ be the fluctuating components of velocities at the points $P(\vec{X})$ and $P'(\vec{X}')$ respectively at time t in this medium, the positions of P and P' being specified by the position vectors $\vec{X}$ and $\vec{X}'$ respectively with reference to a fixed coordinate frame inside the medium. Resolve the velocity components to obtain respective space-Fourier expansions as

$$u_i(\vec{X}, t) = \int e^{-i\vec{K}\cdot\vec{X}}\, dz_i(\vec{K}, t) \quad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

$$u'_j(\vec{X}', t) = \int e^{-i\vec{K}'\cdot\vec{X}'}\, dz_j(\vec{K}', t) \quad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

where the integrations are taken over the whole of respective wavenumber spaces. Here $dz_i(\vec{K}, t)$ and $dz_j(\vec{K}', t)$ are not differentiable with respect to $\vec{K}$ and $\vec{K}'$ respectively, but are differentiable with respect to t. Similarly, considering a third point P'' with position vector $\vec{X}''$, one may write

$$u''_k(\vec{X}'', t) = \int e^{-i\vec{K}''\cdot\vec{X}''}\, dz_k(\vec{K}'', t) \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

where $u''_k(\vec{X}'', t)$ is the kth component of the fluctuating part of the velocity at P'' and $dz_k(\vec{K}'', t)$ is the same type of function as $dz_i(\vec{K}, t)$ or $dz_j(\vec{K}', t)$. Introduce the two-point second order velocity correlations as

$$\left.\begin{aligned} \overline{u_i u'_j} &= F_{i,j}(\vec{X}, \vec{X}', t) \\ \overline{u'_j u''_k} &= F_{j,k}(\vec{X}', \vec{X}'', t) \\ \overline{u_i u''_k} &= F_{i,k}(\vec{X}, \vec{X}'', t) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2.4)$$

where the bar over any expression means ensemble average or expectation of the quantities involved and comma in the suffix distinguishes components associated with different points. In the same manner, introduce the three-point third order velocity correlation as

$$\overline{u_i u'_j u''_k} = F_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right). \quad \ldots \quad (2.5)$$

Using relations (2.1) and (2.2) in the first one of the correlation functions defined in (2.4), one obtains

$$\begin{aligned}\overline{u_i u'_j} &= F_{i,j}\left(\vec{X}, \vec{X}', t\right)\\ &= \int\int e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\,\overline{dz_i\left(\vec{K}, t\right) dz_j\left(\vec{K}', t\right)}\\ &= \int\int e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\,d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right) \quad \ldots \quad (2.6)\end{aligned}$$

where

$$d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right) = \overline{dz_i\left(\vec{K}, t\right) dz_j\left(\vec{K}', t\right)}. \quad \ldots \quad (2.7)$$

Similarly, two such relations can be written for $F_{j,k}\left(\vec{X}', \vec{X}'', t\right)$ and $F_{i,k}\left(\vec{X}, \vec{X}'', t\right)$. Applying the relations (2.1), (2.2) and (2.3) in the correlation function defined by (2.5), one obtains

$$\begin{aligned}\overline{u_i u'_j u''_k} &= F_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right)\\ &= \int\int\int e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,\overline{dz_i\left(\vec{K}, t\right) dz_j\left(\vec{K}', t\right) dz_k\left(\vec{K}'', t\right)}\\ &= -i\int\int\int e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) \quad \ldots \quad (2.8)\end{aligned}$$

where

$$-i\,d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \overline{dz_i\left(\vec{K}, t\right) dz_j\left(\vec{K}', t\right) dz_k\left(\vec{K}'', t\right)}. \quad \ldots \quad (2.9)$$

Assume the Fourier inverses of $d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ and $d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)$ and of such other functions appearing in (2.6) and (2.8) respectively as

$$\frac{d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right)}{d\vec{K}\,d\vec{K}'} = \frac{1}{(2\pi)^6}\int\int e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\,dF_{i,j}\left(\vec{X}, \vec{X}', t\right) \quad \ldots \quad (2.10)$$

and

$$\frac{d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)}{d\vec{K}\,d\vec{K}'\,d\vec{K}''} = \frac{i}{(2\pi)^9}\int\int\int e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,dF_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) \quad \ldots \quad (2.11)$$

where the integrations are considered over the whole of real spaces $d\tau\left(\vec{X}\right) d\tau\left(\vec{X}'\right)$ (six-fold) in (2.10), and $d\tau\left(\vec{X}\right) d\tau\left(\vec{X}'\right) d\tau\left(\vec{X}''\right)$ (nine-fold) in (2.11) respectively.

It may be noted that there will be two other relations like (2.10) corresponding to $F_{j,k}\left(\vec{X}', \vec{X}'', t\right)$ and $F_{i,k}\left(\vec{X}, \vec{X}'', t\right)$ respectively.

3. Two Essential Lemmas

In what follows, two lemmas have been stated and proved. These lemmas will be necessary for subsequent calculations. These lemmas are in regard to

(i) the behaviour of the correlation tensors in the energy space when two or more points under consideration coincide; and

(ii) the representation of the hypothesis of Millionshtchikov in the energy space.

Lemma 1—If one puts

$$d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right) = \psi_{i,j}\left(\vec{K}, \vec{K}', t\right) d\vec{K}\, d\vec{K}' \quad .. \quad .. \quad (3.1)$$

and

$$d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) d\vec{K}\, d\vec{K}'\, d\vec{K}'' \quad .. \quad (3.2)$$

where $d\vec{K}$, $d\vec{K}'$, and $d\vec{K}''$ represent the elements of volume in the three respective wavenumber spaces $\vec{K}$, $\vec{K}'$ and $\vec{K}''$, and $\psi_{i,j}$; $\psi_{i,j,k}$ are suitable spectrum tensors, then

$$\int \psi_{i,j,k}\left(\overrightarrow{\lambda - K''}, \vec{K}', \vec{K}'', t\right) d\vec{K}'' = \psi_{ik,j}\left(\vec{\lambda}, \vec{K}', t\right). \quad .. \quad (3.3)$$

In fact, the relation (3.3) follows when the third point merges with the first point.

Proof: Rewrite the Fourier inverse of the relation (2.8) and use the relation (3.2) to obtain

$$\psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \frac{i}{(2\pi)^9} \int\int\int F_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) e^{i\left[\vec{K}.\vec{X} + \vec{K}'.\vec{X}' + \vec{K}''.\vec{X}''\right]} d\vec{X}\, d\vec{X}'\, d\vec{X}''$$

$$= \frac{i}{(2\pi)^9} \int\int\int F_{i,j,k}\left(\vec{X}, \vec{X}', \overrightarrow{X'' - X} + \vec{X}, t\right) e^{i\left[\left(\vec{K} + \vec{K}''\right).\vec{X} + \vec{K}'.\vec{X}' + \vec{K}''.\left(\vec{X}'' - \vec{X}\right)\right]} d\vec{X}\, d\vec{X}'\, d\vec{X}''$$

$$.. \quad (3.4)$$

where $d\vec{X}$, $d\vec{X}'$, $d\vec{X}''$ are respective volume elements of three real spaces $\tau\left(\vec{X}\right)$, $\tau\left(\vec{X}'\right)$, $\tau\left(\vec{X}''\right)$. Now, for the asymptotic case, when the point P'' coincides with P, $\overrightarrow{X'' - X}$ tends to zero. Replacing $\overrightarrow{X'' - X}$ by $\vec{Y}$, one may write the Fourier inverse of (3.4) as

$$F_{i,j,k}\left(\vec{X}, \vec{X}', \overrightarrow{Y + X}, t\right) = -i \int\int\int \psi_{i,j,k}\left(\overrightarrow{K + K''} - \vec{K}'', \vec{K}', \vec{K}'', t\right) e^{-i\left[\left(\overrightarrow{K + K''}\right).\vec{X} + \vec{K}'.\vec{X}' + \vec{K}''.\vec{Y}\right]} d\vec{\lambda}\, d\vec{K}'\, d\vec{K}''$$

$$.. \quad (3.5)$$

where

$$\vec{\lambda} = \vec{K} + \vec{K}''.$$

Thus, for the asymptotic case $\vec{Y} \to 0$, one obtains

$$F_{ik,j}\left(\vec{X}, \vec{X}', t\right) = -i \int\int\int \underline{\psi_{i,j,k}\left(\overrightarrow{\lambda - K''}, \vec{K}', \vec{K}'', t\right)} e^{-i\left[\vec{\lambda}.\vec{X} + \vec{K}'.\vec{X}'\right]} d\lambda \, d\vec{K}' \, \underline{d\vec{K}''}.$$

Now taking the two-fold Fourier inverse of the integrand involving $d\vec{K}''$ (underlined above), one derives

$$\int \psi_{i,j,k}\left(\overrightarrow{\lambda - K''}, \vec{K}', \vec{K}'', t\right) dK'' = \frac{i}{(2\pi)^6} \int\int F_{ik,j}\left(\vec{X}, \vec{X}', t\right) e^{i\left[\vec{\lambda}.\vec{X} + \vec{K}'.\vec{X}'\right]} d\vec{X} \, d\vec{X}'$$

$$= \psi_{ik,j}\left(\vec{\lambda}, \vec{K}', t\right), \text{ (from definition)}.$$

Lemma 2—Consider the consequences of the assumption that the fourth order velocity correlations are related to the second order velocity correlations as in a normal distribution. Taking the Fourier inverse of the respective velocity correlations, it is easy to rewrite this hypothesis in terms of quantities of the energy space and then apply lemma 1 to obtain the asymptotic behaviour of such relation when some of the points under consideration coincide.

In addition to components of velocities at three points P, P', P'', let us introduce the lth component of fluctuating velocity at a fourth point $P'''\left(\vec{X}'''\right)$ as $u_l\left(\vec{X}''', t\right)$. Then, as required by the present hypothesis,

$$\overline{u_i u'_j u''_k u'''_l} = \overline{u_i u'_j}\, \overline{u''_k u'''_l} + \overline{u_i u''_k}\, \overline{u'_j u'''_l} + \overline{u_i u'''_l}\, \overline{u'_j u''_k}. \quad .. \quad (3.6)$$

In (3.6), the terms like $\overline{u_i u'_j u''_k u'''_l}$ are not shown as they are all zero because of $\bar{u}_i = 0$, etc. Let us now introduce the integral forms of $\overline{u_i u'_j}$, $\overline{u''_k u'''_l}$, $\overline{u_i u''_k}$, . . . etc. from (2.6) and (3.1) in (3.6). Further, we require the integral form for $\overline{u_i u'_j u''_k u'''_l}$ which can be easily introduced in analogy to (2.6) and (2.8), as

$$\overline{u_i u'_j u''_k u'''_l} = \int\int\int\int \psi_{i,j,k,l}\left(\vec{K}, \vec{K}', \vec{K}'', \vec{K}''', t\right) e^{-i\left[\vec{K}.\vec{X} + \vec{K}'.\vec{X}' + \vec{K}''.\vec{X}'' + \vec{K}'''.\vec{X}'''\right]} d\vec{K} \, d\vec{K}' \, d\vec{K}'' \, d\vec{K}'''. \quad .. \quad (3.7)$$

Introducing the integral forms (2.6), (2.8), (3.7), etc., in (3.6), we obtain, for arbitrary volume elements $d\vec{K}\, d\vec{K}'\, d\vec{K}''\, d\vec{K}'''$,

$$\begin{aligned}\psi_{i,j,k,l}&\left(\vec{K}, \vec{K}', \vec{K}'', \vec{K}''', t\right)\\ &= \psi_{i,j}\left(\vec{K}, \vec{K}', t\right)\psi_{k,l}\left(\vec{K}'', \vec{K}''', t\right)\\ &\quad + \psi_{i,k}\left(\vec{K}, \vec{K}'', t\right)\psi_{j,l}\left(\vec{K}', \vec{K}''', t\right)\\ &\quad + \psi_{i,l}\left(\vec{K}, \vec{K}''', t\right)\psi_{j,k}\left(\vec{K}', \vec{K}'', t\right). \quad .. \quad .. \quad (3.8)\end{aligned}$$

The relation (3.8) is the basic relation of lemma 2 and this result can be rewritten when two or more points of reference coincide. In analogy to (3.3), we may rewrite eqn. (3.8) when the point P''' coincides with the point P as

$$\begin{aligned}\int \psi_{i,j,k,l}\left(\overrightarrow{K-K'''}, \vec{K}', \vec{K}'', \vec{K}''', t\right) d\vec{K}''' \\ = \int \psi_{i,j}\left(\overrightarrow{K-K'''}, \vec{K}', t\right)\psi_{k,l}\left(\vec{K}'', \vec{K}''', t\right) d\vec{K}''' \\ + \int \psi_{i,k}\left(\overrightarrow{K-K'''}, \vec{K}'', t\right)\psi_{j,l}\left(\vec{K}', \vec{K}''', t\right) d\vec{K}''' \\ + \int \psi_{i,l}\left(\overrightarrow{K-K'''}, \vec{K}''', t\right)\psi_{j,k}\left(\vec{K}', \vec{K}'', t\right) d\vec{K}'''. \quad \ldots \quad (3.9)\end{aligned}$$

Performing the integration over the whole of $\vec{K}'''$-space, we obtain

$$\begin{aligned}\psi_{il,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \int \psi_{i,j}\left(\overrightarrow{K-K'''}, \vec{K}', t\right)\psi_{k,l}\left(\vec{K}'', \vec{K}''', t\right) d\vec{K}''' \\ + \int \psi_{i,j}\left(\overrightarrow{K-K'''}, \vec{K}'', t\right)\psi_{j,l}\left(\vec{K}', \vec{K}''', t\right) d\vec{K}''' \\ + \psi_{il}\left(\vec{K}, t\right)\psi_{j,k}\left(\vec{K}', \vec{K}'', t\right) \quad \ldots \quad \ldots \quad \ldots \quad (3.10)\end{aligned}$$

By a similar procedure, writing $\overrightarrow{K'-K''}$ for $\vec{K}'$ throughout in (3.10), and integrating term by term over the whole of $\vec{K}''$-space, we obtain the asymptotic form of (3.10) when P'' coincides with P' as

$$\begin{aligned}\psi_{il,jk}\left(\vec{K}, \vec{K}', t\right) = \int\int \Big[\psi_{i,j}\left(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t\right)\psi_{k,l}\left(\vec{K}'', \vec{K}''', t\right) \\ + \psi_{i,k}\left(\overrightarrow{K-K'''}, \vec{K}'', t\right)\psi_{j,l}\left(\overrightarrow{K'-K''}, \vec{K}''', t\right)\Big] d\vec{K}''\, d\vec{K}''' \\ + \psi_{il}\left(\vec{K}, t\right)\psi_{jk}\left(\vec{K}', t\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.11)\end{aligned}$$

4. Dynamical Equations

If u_i be the fluctuating component of velocity and p be the fluctuating part of the pressure at the point $P\left(\vec{X}\right)$, then the equations of motion at P are given by

$$\frac{\partial u_i}{\partial t} + u_l \frac{\partial}{\partial x_l} u_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla_X^2 u_i \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

where t is measured from any fixed instant of time, ρ is the density of the fluid, ν the kinematic viscosity of the fluid,

$$\vec{X} \equiv \{x_1, x_2, x_3\} \text{ and } \nabla_X^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

If we introduce the incompressibility condition

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

in (4.1), we obtain

$$\frac{\partial u_i}{\partial t}+\frac{\partial}{\partial x_l}u_iu_l=-\frac{\partial}{\partial x_i}\left(\frac{p}{\rho}\right)+\nu\nabla_X^2u_i. \qquad (4.3)$$

Similarly, the equations of motion for the points $P'(\vec{X}')$ and $P''(\vec{X}'')$ can be obtained in the forms

$$\frac{\partial u'_j}{\partial t}+\frac{\partial}{\partial x'_m}u'_ju'_m=-\frac{\partial}{\partial x'_j}\left(\frac{p'}{\rho}\right)+\nu\nabla_{X'}^2u'_j \qquad (4.4)$$

and

$$\frac{\partial u''_k}{\partial t}+\frac{\partial}{\partial x''_n}u''_ku''_n=-\frac{\partial}{\partial x''_k}\left(\frac{p''}{\rho}\right)+\nu\nabla_{X''}^2u''_k \qquad (4.5)$$

where u'_j and u''_k are the respective components of fluctuating parts of velocities at $P'(x'_i)$ and $P''(x''_i)$ respectively, while p', p'' are the fluctuating parts of the pressures at those points, and

$$\nabla_{X'}^2\equiv\frac{\partial^2}{\partial x'^2_1}+\frac{\partial^2}{\partial x'^2_2}+\frac{\partial^2}{\partial x'^2_3}=\frac{\partial^2}{\partial x'_i\partial x'_i}$$

and

$$\nabla_{X''}^2\equiv\frac{\partial^2}{\partial x''^2_1}+\frac{\partial^2}{\partial x''^2_2}+\frac{\partial^2}{\partial x''^2_3}=\frac{\partial^2}{\partial x''_i\partial x''_i}.$$

Multiplying (4.3) by u'_j, (4.4) by u_i, and adding we obtain on averaging

$$\frac{\partial}{\partial t}(\overline{u_iu'_j})+\frac{\partial}{\partial x_l}(\overline{u_iu_lu'_j})+\frac{\partial}{\partial x'_m}(\overline{u'_ju'_mu_i})$$

$$=-\frac{\partial}{\partial x_i}\left(\frac{\overline{pu'_j}}{\rho}\right)-\frac{\partial}{\partial x'_j}\left(\frac{\overline{p'u_i}}{\rho}\right)+\nu(\nabla_X^2+\nabla_{X'}^2)(\overline{u_iu'_j}). \qquad (4.6)$$

Equation (4.6) can be written in terms of correlation tensors as

$$\frac{\partial}{\partial t}F_{i,j}+\frac{\partial}{\partial x_l}F_{il,j}+\frac{\partial}{\partial x'_m}F_{i,jm}=-\frac{\partial}{\partial x_i}P_{o,j}-\frac{\partial}{\partial x'_j}P_{i,o}+\nu(\nabla_X^2+\nabla_{X'}^2)F_{i,j} \qquad (4.7)$$

where

$$\overline{u_iu'_j}=F_{i,j}(\vec{X},\vec{X}',t)\,;\quad \overline{u_iu_lu'_j}=F_{il,j}(\vec{X},\vec{X}',t)\,;$$

$$\overline{u'_ju'_mu_i}=F_{i,jm}(\vec{X},\vec{X}',t)\,;\quad \frac{\overline{pu'_j}}{\rho}=P_{o,j}(\vec{X},\vec{X}',t)\,;\quad \frac{\overline{p'u_i}}{\rho}=P_{i,o}(\vec{X},\vec{X}',t).$$

Again, multiplying (4.3) by $u'_ju''_k$, (4.4) by u''_ku_i, (4.5) by $u_iu'_j$ and adding we obtain on averaging

$$\frac{\partial}{\partial t}\overline{u_iu'_ju''_k}+\frac{\partial}{\partial x_l}\overline{u_iu_lu'_ju''_k}+\frac{\partial}{\partial x'_m}\overline{u_iu'_ju'_mu''_k}+\frac{\partial}{\partial x''_n}\overline{u_iu'_ju''_ku''_n}$$

$$=-\frac{\partial}{\partial x_i}\left(\frac{\overline{pu'_ju''_k}}{\rho}\right)-\frac{\partial}{\partial x'_j}\left(\frac{\overline{p'u_iu''_k}}{\rho}\right)-\frac{\partial}{\partial x''_k}\left(\frac{\overline{p''u_iu'_j}}{\rho}\right)$$

$$+\nu(\nabla_X^2+\nabla_{X'}^2+\nabla_{X''}^2)\overline{u_iu'_ju''_k}. \qquad (4.8)$$

Equation (4.8) can be written in terms of correlation tensors as

$$\frac{\partial}{\partial t}F_{i,j,k}+\frac{\partial}{\partial x_l}F_{il,j,k}+\frac{\partial}{\partial x'_m}F_{i,jm,k}+\frac{\partial}{\partial x''_n}F_{i,j,kn}$$
$$=-\frac{\partial}{\partial x_i}P_{o,j,k}-\frac{\partial}{\partial x'_j}P_{i,o,k}-\frac{\partial}{\partial x''_k}P_{i,j,o}$$
$$+\nu(\nabla^2_X+\nabla^2_{X'}+\nabla^2_{X''})F_{i,j,k} \quad \ldots \quad \ldots \quad \ldots \quad (4.9)$$

where

$$\overline{u_i u'_j u''_k}=F_{i,j,k}(\vec{X},\vec{X}',\vec{X}'',t)\,;\quad \overline{u_i u_l u'_j u''_k}=F_{il,j,k}(\vec{X},\vec{X}',\vec{X}'',t)\,;$$
$$\overline{u_i u'_j u'_m u''_k}=F_{i,jm,k}(\vec{X},\vec{X}',\vec{X}'',t)\,;\quad \overline{u_i u'_j u''_k u''_n}=F_{i,j,kn}(\vec{X},\vec{X}',\vec{X}'',t)\,;$$
$$\frac{\overline{pu'_j u''_k}}{\rho}=P_{o,j,k}(\vec{X},\vec{X}',\vec{X}'',t)\,;\quad \frac{\overline{p'u_i u''_k}}{\rho}=P_{i,o,k}(\vec{X},\vec{X}',\vec{X}'',t)\,;\quad \frac{\overline{p''u_i u'_j}}{\rho}=P_{i,j,o}(\vec{X},\vec{X}',\vec{X}'',t).$$

It is to be noted that the correlations that appear in (4.7) are all two-point correlations whereas those appearing in (4.9) are all three-point correlations. Hence correlation tensors in (4.7) are, in general, functions of

$$\vec{X}\equiv\{x_1,x_2,x_3\},\ \vec{X}'\equiv\{x'_1,x'_2,x'_3\}\text{ and of time }t,$$

and correlation tensors in (4.9) are functions of

$$\vec{X}\equiv\{x_1,x_2,x_3\},\ \vec{X}'\equiv\{x'_1,x'_2,x'_3\},\ \vec{X}''\equiv\{x''_1,x''_2,x''_3\}\text{ and of }t.$$

We note that the suffix at the feet of the tensors have been separated by commas to distinguish components that relate to different points under consideration.

Take $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x'_j}$ and $\frac{\partial}{\partial x''_k}$ of (4.3), (4.4) and (4.5) respectively and use the incompressibility conditions

$$\frac{\partial u_i}{\partial x_i}=\frac{\partial u'_j}{\partial x'_j}=\frac{\partial u''_k}{\partial x''_k}=0$$

to obtain

$$\frac{\partial^2}{\partial x_i\partial x_l}u_i u_l=-\frac{\partial^2}{\partial x_p\partial x_p}\left(\frac{p}{\rho}\right) \quad \ldots \quad \ldots \quad \ldots \quad (4.10)$$

$$\frac{\partial^2}{\partial x'_m\partial x'_j}u'_j u'_m=-\frac{\partial^2}{\partial x'_q\partial x'_q}\left(\frac{p'}{\rho}\right) \quad \ldots \quad \ldots \quad \ldots \quad (4.11)$$

and

$$\frac{\partial^2}{\partial x''_n\partial x''_k}u''_n u''_k=-\frac{\partial^2}{\partial x''_r\partial x''_r}\left(\frac{p''}{\rho}\right) \quad \ldots \quad \ldots \quad \ldots \quad (4.12)$$

(cf. equation (8.3.1), p. 178, of Batchelor 1953).

Multiplying (4.10) and (4.11) by u'_j and u_i respectively, we obtain on averaging

$$\frac{\partial^2}{\partial x_i\partial x_l}(\overline{u_i u_l u'_j})=-\frac{\partial^2}{\partial x_p\partial x_p}\left(\frac{\overline{pu'_j}}{\rho}\right)$$

and

$$\frac{\partial^2}{\partial x'_m \partial x'_j}(\overline{u'_j u'_m u_i}) = -\frac{\partial^2}{\partial x'_q \partial x'_q}\left(\frac{\overline{p' u_i}}{\rho}\right).$$

The same relations, in the notation of correlation tensors, are

$$\frac{\partial^2}{\partial x_i \partial x_l} F_{il,j}\left(\vec{X}, \vec{X}', t\right) = -\frac{\partial^2}{\partial x_p^2} P_{o,j}\left(\vec{X}, \vec{X}', t\right) \quad \ldots \quad \ldots (4.13)$$

and

$$\frac{\partial^2}{\partial x'_m \partial x'_j} F_{i,jm}\left(\vec{X}, \vec{X}', t\right) = -\frac{\partial^2}{\partial x'^2_q} P_{i,o}\left(\vec{X}, \vec{X}', t\right). \quad \ldots (4.14)$$

Similarly, multiplying (4.10), (4.11) and (4.12) by $u'_j u''_k$, $u_i u''_k$ and $u_i u'_j$ respectively, we get on averaging

$$\frac{\partial^2}{\partial x_i \partial x_l} F_{il,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = -\frac{\partial^2}{\partial x_p^2} P_{o,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) \quad \ldots (4.15)$$

$$\frac{\partial^2}{\partial x'_m \partial x'_j} F_{i,jm,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = -\frac{\partial^2}{\partial x'^2_q} P_{i,o,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) \quad \ldots (4.16)$$

and

$$\frac{\partial^2}{\partial x''_k \partial x''_n} F_{i,j,kn}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = -\frac{\partial^2}{\partial x''^2_r} P_{i,j,o}\left(\vec{X}, \vec{X}', \vec{X}'', t\right). \quad \ldots (4.17)$$

Let us now introduce the following integrals to obtain the Fourier inverses of the correlation functions that appear in (4.7) as

$$\left.\begin{aligned}
\psi_{i,j}\left(\vec{K}, \vec{K}', t\right) &= \frac{1}{(2\pi)^6}\int\int F_{i,j}\left(\vec{X}, \vec{X}', t\right) e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{X}\, d\vec{X}' \\
\psi_{ij,k}\left(\vec{K}, \vec{K}', t\right) &= \frac{i}{(2\pi)^6}\int\int F_{ij,k}\left(\vec{X}, \vec{X}', t\right) e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{X}\, d\vec{X}' \\
\pi_{o,j}\left(\vec{K}, \vec{K}', t\right) &= \frac{i}{(2\pi)^6}\int\int P_{o,j}\left(\vec{X}, \vec{X}', t\right) e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{X}\, d\vec{X}' \\
\pi_{i,o}\left(\vec{K}, \vec{K}', t\right) &= \frac{i}{(2\pi)^6}\int\int P_{i,o}\left(\vec{X}, \vec{X}', t\right) e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{X}\, d\vec{X}'
\end{aligned}\right\} \quad (4.18)$$

where the integrations are performed over the whole of $\vec{X}$, $\vec{X}'$-spaces and their respective volume elements are

$$d\vec{X} \equiv dx_1 dx_2 dx_3 \quad \text{and} \quad d\vec{X}' \equiv dx'_1 dx'_2 dx'_3.$$

The direct Fourier transform relations of the above are

$$\left.\begin{aligned}
F_{i,j}\left(\vec{X}, \vec{X}', t\right) &= \int\int \psi_{i,j}\left(\vec{K}, \vec{K}', t\right) e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{K}\, d\vec{K}' \\
F_{ij,k}\left(\vec{X}, \vec{X}', t\right) &= -i\int\int \psi_{ij,k}\left(\vec{K}, \vec{K}', t\right) e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{K}\, d\vec{K}' \\
P_{o,j}\left(\vec{X}, \vec{X}', t\right) &= -i\int\int \pi_{o,j}\left(\vec{K}, \vec{K}', t\right) e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{K}\, d\vec{K}' \\
P_{i,o}\left(\vec{X}, \vec{X}', t\right) &= -i\int\int \pi_{i,o}\left(\vec{K}, \vec{K}', t\right) e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]}\, d\vec{K}\, d\vec{K}'
\end{aligned}\right\} \quad (4.19)$$

where the integrations are performed over the whole of $\vec{K}$ and $\vec{K}'$-spaces, and their respective volume elements are $dK_1dK_2dK_3$ and $dK'_1dK'_2dK'_3$. Similarly, we may introduce the Fourier transforms of three-point correlations that appear in (4.9) as

$$\psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \frac{i}{(2\pi)^9}\int\int\int F_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right)e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{X}\,d\vec{X}'\,d\vec{X}'' \quad .. \ (4.20)$$

$$\psi_{il,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \frac{1}{(2\pi)^9}\int\int\int F_{il,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right)e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{X}\,d\vec{X}'\,d\vec{X}'' \quad .. \ (4.21)$$

and two other similar relations for $\psi_{i,jm,k}$ and $\psi_{i,j,kn}$. Proceeding in the same manner, we can have

$$\pi_{o,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \frac{1}{(2\pi)^9}\int\int\int P_{o,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right)e^{i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{X}\,d\vec{X}'\,d\vec{X}'' \quad .. \ (4.22)$$

and two other similar expressions for $\pi_{i,o,k}$ and $\pi_{i,j,o}$. The integrations appearing in (4.20), (4.21), (4.22) are to be performed over the whole of the real spaces $\vec{X}$, $\vec{X}'$, $\vec{X}''$, where their respective volume elements are $dx_1dx_2dx_3$, $dx'_1dx'_2dx'_3$, $dx''_1dx''_2dx''_3$. The Fourier inverses of the above relations are

$$F_{i,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = -i\int\int\int \psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{K}\,d\vec{K}'\,d\vec{K}'' \quad .. \ (4.23)$$

$$F_{il,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = \int\int\int \psi_{il,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{K}\,d\vec{K}'\,d\vec{K}'' \quad .. \ (4.24)$$

and two other similar expressions for $F_{i,jm,k}$ and $F_{i,j,kn}$, together with

$$P_{o,j,k}\left(\vec{X}, \vec{X}', \vec{X}'', t\right) = \int\int\int \pi_{o,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'+\vec{K}''.\vec{X}''\right]}\,d\vec{K}\,d\vec{K}'\,d\vec{K}'' \quad .. \ (4.25)$$

and two similar expressions for $P_{i,o,k}$ and $P_{i,j,o}$. The integrations appearing in (4.23), (4.24), (4.25) are taken over the whole of $\vec{K}$, $\vec{K}'$, $\vec{K}''$-spaces.

Let us now rewrite (4.7), (4.13), (4.14) using the relations given in (4.19) as

$$\frac{\partial}{\partial t}\psi_{i,j}\left(\vec{K}, \vec{K}', t\right) = K_l\psi_{il,j}\left(\vec{K}, \vec{K}', t\right)+K'_m\psi_{i,jm}\left(\vec{K}, \vec{K}', t\right)+K_i\pi_{o,j}\left(\vec{K}, \vec{K}', t\right)$$
$$+K'_j\pi_{i,o}\left(\vec{K}, \vec{K}', t\right)-\nu(K^2+K'^2)\psi_{i,j}\left(\vec{K}, \vec{K}', t\right) \quad .. \ (4.26)$$

$$\frac{K_iK_sK_l\psi_{sl,j}\left(\vec{K}, \vec{K}', t\right)}{K^2} = -K_i\pi_{o,j}\left(\vec{K}, \vec{K}', t\right) \quad .. \quad .. \ (4.27)$$

and

$$\frac{K'_j K'_m K'_q \psi_{i,\,qm}\left(\vec{K},\,\vec{K}',\,t\right)}{K^2} = -K'_j \pi_{i,\,o}\left(\vec{K},\,\vec{K}',\,t\right). \qquad (4.28)$$

Similarly, applying the forms of three-point correlations from (4.23) to (4.25) in eqns. (4.9), (4.15), (4.16) and (4.17), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{i,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) = &-K_l\psi_{il,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - K'_m\psi_{i,\,jm,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ &-K''_n\psi_{i,\,j,\,kn}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - K_i\pi_{o,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ &-K'_j\psi_{i,\,o,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - K''_k\pi_{i,\,j,\,o}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ &-\nu(K^2+K'^2+K''^2)\psi_{i,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) \qquad (4.29)\end{aligned}$$

$$\frac{K_i K_s K_l \psi_{sl,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)}{K^2} = -K_i\pi_{o,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) \qquad (4.30)$$

$$\frac{K'_j K'_m K'_q \psi_{i,\,qm,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)}{K'^2} = -K'_j\pi_{i,\,o,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) \qquad (4.31)$$

and

$$\frac{K''_k K''_r K''_n \psi_{i,\,j,\,rn}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)}{K''^2} = -K''_k\pi_{i,\,j,\,o}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right). \qquad (4.32)$$

Substituting for $\pi_{o,\,j}\left(\vec{K},\,\vec{K}',\,t\right)$ and $\pi_{i,\,o}\left(\vec{K},\,\vec{K}',\,t\right)$ from (4.27) and (4.28) in (4.26), we finally obtain the equation

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{i,\,j}\left(\vec{K},\,\vec{K}',\,t\right) = &\,K_l\psi_{il,\,j}\left(\vec{K},\,\vec{K}',\,t\right) - \frac{K_iK_sK_l}{K^2}\psi_{sl,\,j}\left(\vec{K},\,\vec{K}',\,t\right)\\ &+K'_m\psi_{i,\,jm}\left(\vec{K},\,\vec{K}',\,t\right) - \frac{K'_jK'_mK'_q}{K'^2}\psi_{i,\,qm}\left(\vec{K},\,\vec{K}',\,t\right)\\ &-\nu(K^2+K'^2)\psi_{i,\,j}\left(\vec{K},\,\vec{K}',\,t\right). \qquad (4.33)\end{aligned}$$

Similarly substituting for $\pi_{o,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)$, $\pi_{i,\,o,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)$ and $\pi_{i,\,j,\,o}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)$ from (4.30) to (4.32) in (4.29), we get

$$\begin{aligned}\frac{\partial}{\partial t}\psi_{i,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) + K_l\psi_{il,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - \frac{K_iK_sK_l}{K^2}\psi_{sl,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ +K'_m\psi_{i,\,jm,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - \frac{K'_qK'_jK'_m}{K'^2}\psi_{i,\,qm,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ +K''_n\psi_{i,\,j,\,kn}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right) - \frac{K''_rK''_nK''_k}{K''^2}\psi_{i,\,j,\,rn}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right)\\ = -\nu(K^2+K'^2+K''^2)\psi_{i,\,j,\,k}\left(\vec{K},\,\vec{K}',\,\vec{K}'',\,t\right). \qquad (4.34)\end{aligned}$$

Here (4.33) is an equation connecting two-point second order correlations with two-point third order correlations in the wavenumber space and (4.34) is an equation which connects three-point third order correlations with three-point fourth order correlations in the wavenumber space. Further, (3.8) is a relation which connects a four-point fourth order correlation with products of two-point second order correlations, the final form of the said equation being written in the wavenumber space.

Now using Lemma 1 for merger of suitable sets of points in real space, we can obtain the reduced versions of eqns. (4.34) and (3.8) containing only two-point correlations of different orders. Thus (4.33) relates two-point second order velocity correlations to two-point third order velocity correlations in wavenumber space and reduced form of (4.34) relates two-point third order velocity correlations to two-point fourth order velocity correlations in wave-number space. Further, reduced form of eqn. (3.8) connects two-point fourth order velocity correlations to products of two-point second order velocity correlations in wavenumber space. So by the process of elimination, out of these three equations, we obtain an equation containing only transforms of two-point second order velocity correlations in the wavenumber space. Since we are here considering the early-period decay process of turbulence, so we can readily neglect the terms containing viscosity effects. Thus, we finally obtain the guiding equation for early-period decay process of turbulence as

$$\frac{\partial^2}{\partial t^2}\psi_{i,j}\left(\vec{K}, \vec{K}', t\right) = I_1\left(\vec{K}, \vec{K}', t\right)+I_2\left(\vec{K}, \vec{K}', t\right) \quad .. \quad .. \quad (4.35)$$

where

$$\begin{aligned}
I_1\left(\vec{K}, \vec{K}', t\right) = {} & 2K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]K'_m\left[\frac{K'_jK'_q}{K'^2}-\delta_{jq}\right] \\
& \cdot\int\int\left[\psi_{s,q}\left(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t\right)\psi_{m,l}\left(\vec{K}'', \vec{K}''', t\right)\right. \\
& \left.+\psi_{s,m}\left(\overrightarrow{K-K'''}, \vec{K}'', t\right)\psi_{q,l}\left(\overrightarrow{K'-K''}, \vec{K}''', t\right)\right] d\vec{K}''d\vec{K}''' \\
& +\int\left((K_t-K''_t)\left[\frac{(K_s-K''_s)(K_q-K''_q)}{(K-K'')^2}-\delta_{sq}\right]K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]\right. \\
& \cdot\int\left\{\psi_{q,j}\left(\overrightarrow{K-K''-K'''}, \vec{K}', t\right)\psi_{l,t}\left(\vec{K}'', \vec{K}''', t\right)\right. \\
& \left.\left.+\psi_{q,l}\left(\overrightarrow{K-K''-K'''}, \vec{K}'', t\right)\psi_{j,t}\left(\vec{K}', \vec{K}''', t\right)\right\} d\vec{K}'''\right) d\vec{K}'' \\
& +\int\left((K'_s-K''_s)\left[\frac{(K'_q-K''_q)(K'_t-K''_t)}{(K'-K'')^2}-\delta_{qt}\right]K'_m\left[\delta_{jq}-\frac{K'_jK'_q}{K'^2}\right]\right. \\
& \cdot\int\left\{\psi_{i,t}\left(\vec{K}, \overrightarrow{K'-K''-K'''}, t\right)\psi_{m,s}\left(\vec{K}'', \vec{K}''', t\right)\right.
\end{aligned}$$

$$+\psi_{i,s}\left(\vec{K}, \vec{K}''', t\right)\psi_{t,m}\left(\overrightarrow{K'-K''-K'''}, \vec{K}'', t\right)\Big\} d\vec{K}'''\Big) d\vec{K}''$$

$$+\int \Big(K''_s\left[\frac{K''_rK''_m}{K''^2} - \delta_{rm}\right]K'_m\left[\delta_{jq} - \frac{K'_jK'_q}{K'^2}\right]$$

$$\cdot \int \Big\{\psi_{i,r}\left(\vec{K}, \overrightarrow{K''-K'''}, t\right)\psi_{q,s}\left(\overrightarrow{K'-K''}, \vec{K}''', t\right)$$

$$+\psi_{i,s}\left(\vec{K}, \vec{K}''', t\right)\psi_{q,r}\left(\overrightarrow{K'-K''}, \overrightarrow{K''-K'''}, t\right)\Big\} d\vec{K}'''\Big) d\vec{K}'' \qquad .. \; (4.36)$$

and

$$I_2\left(\vec{K}, \vec{K}', t\right) = 2K_l\left[\delta_{is} - \frac{K_iK_s}{K^2}\right]K'_m\left[\frac{K'_jK'_q}{K'^2} - \delta_{jq}\right]\psi_{sl}\left(\vec{K}, t\right)\psi_{qm}\left(\vec{K}', t\right)$$

$$+\int (K_t - K''_t)\left[\frac{(K_s-K''_s)(K_q-K''_q)}{(K-K'')^2} - \delta_{sq}\right]K_l\left[\delta_{is} - \frac{K_iK_s}{K^2}\right]$$

$$\cdot\, \psi_{qt}\left(\overrightarrow{K-K''}, t\right)\psi_{j,l}\left(\vec{K}', \vec{K}'', t\right) d\vec{K}''$$

$$+\int K''_n\left[\frac{K''_rK''_l}{K''^2} - \delta_{rl}\right]K_l\left[\delta_{is} - \frac{K_iK_s}{K^2}\right]\psi_{s,j}\left(\overrightarrow{K-K''}, \vec{K}', t\right)\psi_{rn}\left(\vec{K}'', t\right) d\vec{K}''$$

$$+\int (K'_s - K''_s)\left[\frac{(K'_q-K''_q)(K'_t-K''_t)}{(K'-K'')^2} - \delta_{qt}\right]K'_m\left[\delta_{jq} - \frac{K'_jK'_q}{K'^2}\right]$$

$$\cdot\, \psi_{i,m}\left(\vec{K}, \vec{K}'', t\right)\psi_{ts}\left(\overrightarrow{K'-K''}, t\right) d\vec{K}''$$

$$+\int K''_s\left[\frac{K''_rK''_m}{K''^2} - \delta_{rm}\right]K'_m\left[\delta_{jq} - \frac{K'_jK'_q}{K'^2}\right]\psi_{i,q}\left(\vec{K}, \overrightarrow{K'-K''}, t\right)\psi_{rs}\left(\vec{K}'', t\right) d\vec{K}''$$

$$.. \; (4.37)$$

The relation (4.35) is the final form of the resultant tensor equation connecting the second order velocity correlations in the transformed space. This equation represents the process of transfer of energy represented by the non-linear terms of Navier-Stokes equation under the assumption of quasi-normality hypothesis of Millionshtchikov. It is to be noted that eqn. (4.35) is valid for all types of turbulence and represents the early-period decay phenomena under the assumption of Millionshtchikov's hypothesis. It will be shown in the next article that in case of homogeneous turbulence eqn. (4.35) reduces to the form

$$\frac{\partial^2}{\partial t^2}\psi_{i,j}\left(\vec{K}, \vec{K}', t\right) = I_1\left(\vec{K}, \vec{K}', t\right) \qquad .. \quad .. \quad .. \; (4.38)$$

where $I_1\left(\vec{K}, \vec{K}', t\right)$ is given by (4.36). It is proposed that in section 5, (4.38) will be further simplified for the case of homogeneous and isotropic turbulence to give the simple equation that has been obtained by Reid and Proudman (1954) and also by Tatsumi (1957) in a different but equivalent form.

5. Reduction of Equation (4.35) for the Case of Homogeneous Isotropic Turbulence

(i) *Homogeneity Condition:*

Let us define the velocity correlation of second order by

$$F_{i,j} = \overline{u_i\left(\vec{X}, t\right)u'_j\left(\vec{X}', t\right)} \qquad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

where u_i and u'_j are the fluctuating parts of the velocity components at $P\left(\vec{X}\right)$ and $P'\left(\vec{X}'\right)$ respectively. Then the space-Fourier inverse relation for $F_{i,j}\left(\vec{X}, \vec{X}', t\right)$ is given by

$$F_{i,j}\left(\vec{X}, \vec{X}', t\right) = \frac{1}{(2\pi)^6}\int_{\tau(K)}\int_{\tau(K')} e^{-i\left[\vec{K}.\vec{X}+\vec{K}'.\vec{X}'\right]} d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right) \quad \ldots \quad (5.2)$$

where $\vec{K}$, $\vec{K}'$ are the wave-vectors associated with the real space-vectors $\vec{X}$ and $\vec{X}'$ respectively and the integrations are taken over the whole of $\vec{K}$ and $\vec{K}'$-spaces. In case of homogeneous turbulence $F_{i,j}$ should necessarily be function of the difference vector $\vec{X}'-\vec{X}\left(\equiv \vec{\xi}\right)$ only and of t. So the form of $d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ must be represented by

$$d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right) = \psi_{i,j}\left(\vec{K}, \vec{K}', t\right)\delta\left(\vec{K}+\vec{K}'\right) d\vec{K}\, d\vec{K}' \qquad \ldots \quad \ldots \quad (5.3)$$

which is quite clear from the form of the integral given in (5.2). In (5.3) $\delta\left(\vec{K}+\vec{K}'\right)$ is Dirac's three-dimensional delta function. We may compare the general form of $d\phi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ as given in (3.1) with that of $d\phi_{i,j}$ as written in (5.3) and see that in subsequent calculation whenever $\psi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ appears in the general type of turbulence, we need substitute $\psi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ $\delta\left(\vec{K}+\vec{K}'\right)$ in place of $\psi_{i,j}\left(\vec{K}, \vec{K}', t\right)$ in consideration of homogeneous turbulence. Further in place of relation (3.2), viz.

$$d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) d\vec{K}\, d\vec{K}'\, d\vec{K}''$$

we need consider

$$d\phi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right) = \psi_{i,j,k}\left(\vec{K}, \vec{K}', \vec{K}'', t\right)\delta\left(\overrightarrow{K+K'+K''}\right) d\vec{K}\, d\vec{K}'\, d\vec{K}''$$

when the turbulence is considered homogeneous. Thus the modified form of eqn. (4.35) in case of homogeneous turbulence is

$$\frac{\partial^2}{\partial t^2}\int\int \psi_{i,j}\left(\vec{K}, \vec{K}', t\right)\delta\left(\vec{K}+\vec{K}'\right)\,d\vec{K}\,d\vec{K}'$$

$$= \int\int \Bigg(2K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]K'_m\left[\frac{K'_jK'_q}{K'^2}-\delta_{jq}\right]$$

$$.\left\{\int\int \psi_{s,q}\left(\overrightarrow{K-K'''}, \overrightarrow{K'-K''}, t\right)\psi_{m,l}\left(\vec{K}'', \vec{K}''', t\right)\delta\left(\overrightarrow{K-K'''}+\overrightarrow{K'-K''}\right)\delta\left(\vec{K}''+\vec{K}'''\right)\,d\vec{K}''\,d\vec{K}'''\right.$$

$$\left.+\int\int \psi_{s,m}\left(\overrightarrow{K-K'''}, \vec{K}'', t\right)\psi_{q,l}\left(\overrightarrow{K'-K''}, \vec{K}''', t\right)\delta\left(\overrightarrow{K-K'''}+\vec{K}''\right)\delta\left(\overrightarrow{K'-K''}+\vec{K}'''\right)\,d\vec{K}''d\vec{K}'''\right\}\Bigg)\,d\vec{K}\,d\vec{K}'$$

$$+\int\int\int \Bigg((K_t-K''_t)\left[\frac{(K_s-K''_s)(K_q-K''_q)}{(K-K'')^2}-\delta_{sq}\right]K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]$$

$$.\left\{\int \psi_{q,j}\left(\overrightarrow{K-K''-K'''}, \vec{K}', t\right)\psi_{l,t}\left(\vec{K}'', \vec{K}''', t\right)\delta\left(\overrightarrow{K-K''-K'''}+\vec{K}'\right)\delta\left(\vec{K}''+\vec{K}'''\right)\,d\vec{K}'''\right.$$

$$\left.+\int \psi_{q,l}\left(\overrightarrow{K-K''-K'''}, \vec{K}'', t\right)\psi_{j,t}\left(\vec{K}', \vec{K}''', t\right)\delta\left(\overrightarrow{K-K''-K'''}+\vec{K}''\right)\delta\left(\vec{K}'+\vec{K}'''\right)\,d\vec{K}'''\right\}\Bigg)\,d\vec{K}\,d\vec{K}'\,d\vec{K}''$$

$$+\int\int\int \Bigg(K''_n\left[\frac{K''_rK''_l}{K''^2}-\delta_{rl}\right]K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]$$

$$.\left\{\int \psi_{s,r}\left(\overrightarrow{K-K''}, \overrightarrow{K''-K'''}, t\right)\psi_{j,n}\left(\vec{K}', \vec{K}''', t\right)\delta\left(\overrightarrow{K-K''}+\overrightarrow{K''-K'''}\right)\delta\left(\vec{K}'+\vec{K}'''\right)\,d\vec{K}'''\right.$$

$$\left.+\int \psi_{s,n}\left(\overrightarrow{K-K''}, \vec{K}''', t\right)\psi_{j,r}\left(\vec{K}', \overrightarrow{K''-K'''}, t\right)\delta\left(\overrightarrow{K-K''}+\vec{K}'''\right)\delta\left(\vec{K}'+\overrightarrow{K''-K'''}\right)\,d\vec{K}'''\right\}\Bigg)\;.\,d\vec{K}\,d\vec{K}'\,d\vec{K}''$$

$$+\int\int\int \Bigg((K'_s-K''_s)\left[\frac{(K'_q-K''_q)(K'_t-K''_t)}{(K'-K'')^2}-\delta_{qt}\right]K'_m\left[\delta_{jq}-\frac{K'_jK'_q}{K'^2}\right]$$

$$.\left\{\int \psi_{i,t}\left(\vec{K}, \overrightarrow{K'-K''-K'''}, t\right)\psi_{m,s}\left(\vec{K}'', \vec{K}''', t\right)\delta\left(\vec{K}+\overrightarrow{K'-K''-K'''}\right)\delta\left(\vec{K}''+\vec{K}'''\right)\,d\vec{K}'''\right.$$

$$\left.+\int \psi_{i,s}\left(\vec{K}, \vec{K}''', t\right)\psi_{t,m}\left(\overrightarrow{K'-K''-K'''}, \vec{K}'', t\right)\delta\left(\vec{K}+\vec{K}'''\right)\delta\left(\overrightarrow{K'-K''-K'''}+\vec{K}''\right)\,d\vec{K}'''\right\}\Bigg)\,d\vec{K}\,d\vec{K}'\,d\vec{K}''$$

$$+\int\int\int \Bigg(K''_s\left[\frac{K''_rK''_m}{K''^2}-\delta_{rm}\right]K'_m\left[\delta_{jq}-\frac{K'_jK'_q}{K'^2}\right]$$

$$.\left\{\int \psi_{i,r}\left(\vec{K}, \overrightarrow{K''-K'''}, t\right)\psi_{q,s}\left(\overrightarrow{K'-K''}, \vec{K}''', t\right)\delta\left(\vec{K}+\overrightarrow{K''-K'''}\right)\delta\left(\overrightarrow{K'-K''}+\vec{K}'''\right)\,d\vec{K}'''\right.$$

$$\left.+\int \psi_{i,s}\left(\vec{K}, \vec{K}''', t\right)\psi_{q,r}\left(\overrightarrow{K'-K''}, \overrightarrow{K''-K'''}, t\right)\delta\left(\vec{K}+\vec{K}'''\right)\delta\left(\overrightarrow{K'-K''}+\overrightarrow{K''-K'''}\right)\,d\vec{K}'''\right\}\Bigg)\,d\vec{K}\,d\vec{K}'\,d\vec{K}''$$

$$+\int\int 2K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]K'_m\left[\frac{K'_jK'_q}{K'^2}-\delta_{jq}\right]\psi_{sl}\left(\vec{K}, t\right)\psi_{qm}\left(\vec{K}', t\right)\delta\left(\vec{K}\right)\delta\left(\vec{K}'\right)\,d\vec{K}\,d\vec{K}'$$

$$+\int\int\int (K_t-K''_t)\left[\frac{(K_s-K''_s)(K_q-K''_q)}{(K-K'')^2}-\delta_{sq}\right]K_l\left[\delta_{is}-\frac{K_iK_s}{K^2}\right]$$

$$\cdot\ \psi_{qt}\left(\overrightarrow{K-K''}, t\right)\psi_{j,l}\left(\vec{K'}, \vec{K''}, t\right)\delta\left(\overrightarrow{K-K''}\right)\delta\left(\vec{K'}+\vec{K''}\right) d\vec{K}\, d\vec{K'}\, d\vec{K''}$$

$$+\int\int\int K''_n\left[\frac{K''_r K''_l}{K''^2}-\delta_{rl}\right]K_l\left[\delta_{is}-\frac{K_i K_s}{K^2}\right]$$

$$\cdot\ \psi_{s,j}\left(\overrightarrow{K-K''}, \vec{K'}, t\right)\psi_{rn}\left(\vec{K''}, t\right)\delta\left(\overrightarrow{K-K''}+\vec{K'}\right)\delta\left(\vec{K''}\right) d\vec{K}\, d\vec{K'}\, d\vec{K''}$$

$$+\int\int\int (K'_s-K''_s)\left[\frac{(K'_q-K''_q)(K'_t-K''_t)}{(K'-K'')^2}-\delta_{qt}\right]K'_m\left[\delta_{jq}-\frac{K'_j K'_q}{K'^2}\right]$$

$$\cdot\ \psi_{i,m}\left(\vec{K}, \vec{K''}, t\right)\psi_{ts}\left(\overrightarrow{K'-K''}, t\right)\delta\left(\vec{K}+\vec{K''}\right)\delta\left(\overrightarrow{K'-K''}\right) d\vec{K}\, d\vec{K'}\, d\vec{K''}$$

$$+\int\int\int K''_s\left[\frac{K''_r K''_m}{K''^2}-\delta_{rm}\right]K'_m\left[\delta_{jq}-\frac{K'_j K'_q}{K'^2}\right]$$

$$\cdot\ \psi_{i,q}\left(\vec{K}, \overrightarrow{K'-K''}, t\right)\psi_{rs}\left(\vec{K''}, t\right)\delta\left(\vec{K}+\overrightarrow{K'-K''}\right)\delta\left(\vec{K''}\right) d\vec{K}\, d\vec{K'}\, d\vec{K''}. \quad \ldots \quad (5.4)$$

After two sets of integrations, eqn. (5.4) simplifies to

$$\frac{\partial^2}{\partial t^2}\int \psi_{i,j}\left(\vec{K}, -\vec{K}, t\right) d\vec{K}$$

$$= 2\int\int K_l^{\mathrm{IV}}\left[\delta_{is}-\frac{K_i^{\mathrm{IV}}K_s^{\mathrm{IV}}}{K^{\mathrm{IV}2}}\right]K_m^{\mathrm{IV}}\left[\delta_{jq}-\frac{K_j^{\mathrm{IV}}K_q^{\mathrm{IV}}}{K^{\mathrm{IV}}}\right]\cdot\left\{\psi_{s,q}\left(\vec{K}, -\vec{K}, t\right)\psi_{m,l}\left(-\vec{K'}, \vec{K'}, t\right)\right.$$

$$\left.+\psi_{s,m}\left(\vec{K}, -\vec{K}, t\right)\psi_{q,l}\left(-\vec{K'}, \vec{K'}, t\right)\right\} d\vec{K}\, d\vec{K'}$$

$$+\int\int K_t^{\mathrm{IV}}\left[\frac{K_s^{\mathrm{IV}}K_q^{\mathrm{IV}}}{K^{\mathrm{IV}2}}-\delta_{sq}\right]K_l\left[\delta_{is}-\frac{K_i K_s}{K^2}\right]\cdot\left\{\psi_{q,j}\left(\vec{K}, -\vec{K}, t\right)\psi_{l,t}\left(-\vec{K'}, \vec{K'}, t\right)\right.$$

$$\left.+\psi_{q,l}\left(\vec{K'}, -\vec{K'}, t\right)\psi_{j,t}\left(-\vec{K}, \vec{K}, t\right)\right\} d\vec{K}\, d\vec{K'}$$

$$+\int\int K_n^{\mathrm{IV}}\left[\frac{K_r^{\mathrm{IV}}K_l^{\mathrm{IV}}}{K^{\mathrm{IV}2}}-\delta_{rl}\right]K_l\left[\delta_{is}-\frac{K_i K_s}{K^2}\right]\cdot\left\{\psi_{s,r}\left(-\vec{K'}, \vec{K'}, t\right)\psi_{j,n}\left(-\vec{K}, \vec{K}, t\right)\right.$$

$$\left.+\psi_{s,n}\left(-\vec{K'}, \vec{K'}, t\right)\psi_{j,r}\left(-\vec{K}, \vec{K}, t\right)\right\} d\vec{K}\, d\vec{K'}$$

$$+\int\int K_s^{\mathrm{IV}}\left[\frac{K_q^{\mathrm{IV}}K_t^{\mathrm{IV}}}{K^{\mathrm{IV}2}}-\delta_{qt}\right]K_m\left[\delta_{jq}-\frac{K_j K_q}{K^2}\right]\cdot\left\{\psi_{i,t}\left(\vec{K}, -\vec{K}, t\right)\psi_{m,s}\left(K', -K', t\right)\right.$$

$$\left.+\psi_{i,s}\left(\vec{K}, -\vec{K}, t\right)\psi_{t,m}\left(-\vec{K'}, \vec{K'}, t\right)\right\} d\vec{K}\, d\vec{K'}$$

$$+\int\int K_s^{\mathrm{IV}}\left[\frac{K_r^{\mathrm{IV}}K_m^{\mathrm{IV}}}{K^{\mathrm{IV}2}}-\delta_{rm}\right]K_m\left[\delta_{jq}-\frac{K_j K_q}{K^2}\right]\cdot\left\{\psi_{i,r}\left(\vec{K}, -\vec{K}, t\right)\psi_{q,s}\left(K', -K', t\right)\right.$$

$$\left.+\psi_{i,s}\left(\vec{K}, -\vec{K}, t\right)\psi_{q,r}\left(\vec{K'}, -\vec{K'}, t\right)\right\} d\vec{K}\, d\vec{K'} \quad \ldots\ \ldots\ \ldots\ \ldots \quad (5.5)$$

where the integrations over $\vec{K}''$ and $\vec{K}'''$-spaces have been performed and the symbol $\vec{K}^{IV}$ has been introduced by the relation

$$\vec{K}^{IV} = \vec{K}+\vec{K}'. \qquad \qquad (5.6)$$

(ii) *Homogeneous and Isotropic Turbulence:*

Let us now simplify equation (5.5) for the case of homogeneous and isotropic turbulence by introducing the well-known form for the second rank spectrum tensor $\psi_{s,q}\left(\vec{K}, -\vec{K}, t\right)$ as

$$\psi_{s,q}\left(\vec{K}, -\vec{K}, t\right) = \frac{F(K, t)}{4\pi K^2}\left\{\delta_{sq}-\frac{K_sK_q}{K^2}\right\} \qquad \qquad (5.7)$$

(cf. eqn. (48), Reid and Proudman 1954)
where $F(K, t)$ is the energy spectrum function of homogeneous and isotropic turbulence. Similarly, introducing for $\psi_{m,l}\left(-\vec{K}', \vec{K}', t\right)$ the isotropic form

$$\psi_{m,l}\left(-\vec{K}', \vec{K}', t\right) = \frac{F(K', t)}{4\pi K'^2}\left\{\delta_{ml}-\frac{K'_mK'_l}{K'^2}\right\} \qquad \qquad (5.8)$$

and similar other expressions for all other second rank tensors appearing in (5.5), we obtain after putting the suffix $i = j$ and summing over all the values of the indices,

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\int_0^\infty 2K^2F(K, t)\,dK \\
&= \int_0^\infty\int_0^\infty\int_{-1}^1 \Bigg\{-\frac{K^{IV2}}{2K'^2}\cdot Q\cdot\left[1+\frac{(K^2+K^{IV2}-K'^2)^2}{4K^2K^{IV2}}\right]+\frac{Q}{8K^2K'^2}\cdot(K'^2+K^{IV2}-K^2)(K^2+K^{IV2}-K'^2) \\
&+\frac{K^2Q}{4K'^2}\left[1+\frac{(K^2+K^{IV2}-K'^2)^2}{4K^2K^{IV2}}\right]+\frac{Q}{16K^{IV2}K'^2}\cdot(K^2+K^{IV2}-K'^2)(K^2-K'^2-K^{IV2}) \\
&+\frac{Q}{16K'^2K^{IV2}}[-Q-2K'^2(K^2+K^{IV2}-K'^2)]+\frac{Q}{16K'^2K^{IV2}}\cdot(K^2+K^{IV2}-K'^2)(K^{IV2}-K^2-K'^2) \\
&+\frac{K^2Q}{4K'^2}\left[1+\frac{(K^2+K^{IV2}-K'^2)^2}{4K^2K^{IV2}}\right]-\frac{Q}{16K'^2K^{IV2}}\cdot(K^2+K^{IV2}-K'^2)(K^{IV2}+K'^2-K^2) \\
&+\frac{Q}{8K'^2K^{IV2}}\cdot[(K^2+K^{IV2}-K'^2)(K^{IV2}-K'^2-K^2)-K^{IV2}(K^{IV2}-K^2-K'^2)]\Bigg\} \\
&\cdot\frac{F(K, t)F(K', t)}{2}\cdot d\mu\, dK\, dK' \qquad \qquad (5.9)
\end{aligned}$$

where the wavenumber space integral $d\vec{K}$ has been replaced by the line-integral $4\pi K^2\,dK$ and the multiple integral $d\vec{K}\,d\vec{K}'$ has been replaced by

$4\pi K^2 dK 2\pi K'^2 \, dK' d\mu$, μ being the cosine of the angle between the wave-vectors $\vec{K}$, $\vec{K}'$ represented by $\frac{\left(\vec{K}\,.\,\vec{K}'\right)}{|K|\,|K'|}$ and Q is a symmetric function of $\vec{K}$, $\vec{K}'$, given by

$$Q = K^4 + K'^4 + K^{\mathrm{IV}4} - 2K^2K'^2 - 2K'^2K^{\mathrm{IV}2} - 2K^{\mathrm{IV}2}K^2. \quad .. \; (5.10)$$

After integrations, (5.9) reduces to the simple form

$$\frac{\partial^2}{\partial t^2}\int_0^\infty 2K^2F(K,t)\,dK = \tfrac{4}{3}\left[\int_0^\infty K^2F(K,t)\,dK\right]^2. \quad .. \; (5.11)$$

If $\bar{\omega}^2$ denotes the mean square variation in one component of vorticity in case of homogeneous and isotropic turbulence, then we can write the usual expression

$$\tfrac{3}{2}\bar{\omega}^2 = \int_0^\infty K^2F(K,t)\,dK \quad .. \quad .. \quad .. \quad .. \; (5.12)$$

[cf. Batchelor 1953, eqn. (3.2.4), p. 39].
Introducing the relation (5.12) in the equation (5.11), we obtain

$$\frac{d^2}{dt^2}\bar{\omega}^2 = (\bar{\omega}^2)^2 \quad .. \quad .. \quad .. \quad .. \; (5.13)$$

which is the same as the decay equation given by Reid and Proudman (1954). It may be noted that Tatsumi (1957) also obtained the same relation as (5.13) but in a different although equivalent form.

The equation (5.5) has got potentialities. If any special type of symmetry in homogeneous turbulence be conceived and the corresponding $\psi_{ij}\left(\vec{K}, -\vec{K}, t\right)$ for this special type of turbulence can be written down, then it will be possible to effect the summations involved in the integrals in (5.5). This equation may thus be much simplified and give us the decay equation of the specific type of turbulence in simple form.

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