ON ANISOTROPIC WAVE MOTIONS UNDER UNIFORM MAGNETIC AND CURRENT FIELDS

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An investigation is made into anisotropic and axisymmetric wave motions generated by an arbitrary harmonic pressure distribution acting on the free surface of an inviscid, incompressible and electrically conducting fluid in the presence of uniform magnetic and current fields. The problem is formulated as an initial value problem which is then solved by the joint Laplace and the generalized Hankel transforms treatment in conjunction with asymptotic methods. An integral as well as asymptotic solution of the problem related to certain pressure distributions of physical interest is presented. It is shown that the wave motions are anisotropic and dispersive and the anisotropy is entirely due to the inclusion of the magnetohydrodynamic body force. A detailed discussion of the axisymmetric wave motions including the effects of the imposed magnetic and current fields on the flow has been presented. In addition, it is shown that the present method of solution provides an interesting illustration of the applicability of generalized functions in hydromagnetic wave problems.

1. INTRODUCTION

The study of an electrically conducting fluid in the presence of a magnetic field was first initiated by Alfvén in 1942 and continued by many others available in the current literature on the subject. One of the striking features of a conducting flow under a magnetic field is that it is sometimes totally alike a non-conducting fluid. A careful observation reveals that on many occasions the magnetohydrodynamic effects on the classical problems of fluid dynamics are really interesting and have useful physical applications. From a mathematical and physical point of view, surface wave phenomena in an electrically conducting fluid in the presence of a magnetic field are very fascinating, and such wave problems in general deserve consideration on their own merit.

In recent years, Debnath (1968, 1969*a–c*) and Debnath and Rosenblat (1969) have made investigations into two-dimensional and axisymmetric wave motions produced by harmonically oscillating pressure distributions acting on the free surface of an inviscid, incompressible, non-rotating fluid (initially at rest or flows with uniform speed) of limited, unlimited and shallow depth. A rigorous mathematical and physical treatment of the problems in various situations is given with the aid of generalized function method together with asymptotic techniques. The characteristic features of these wave phenomena have been explored in detail.

The corresponding surface wave phenomena in two as well as in three dimensions in an electrically conducting fluid in the presence of a uniform magnetic and current field might be a little more complicated, but at the same time much more interesting from the mathematical and physical point of view. Furthermore, it is likely that they have important and useful applicability to magnetohydrodynamics.

In the case of an incompressible, electrically conducting fluid in the presence of uniform horizontal magnetic and current fields, anisotropic and dispersive wave motions can be generated by disturbances acting on the free surface of the fluid. This kind of situation is likely to occur due to the combined action of gravity and the vertical component of the imposed magnetohydrodynamic body force. The characteristic properties of the resulting wave motions are likely to be qualitatively and quantitatively modified presumably due to the imposed magnetic and current fields.

The present analysis is concerned with an extension of the axisymmetric wave problem to the magnetohydrodynamic situation in which the fluid is assumed to be electrically conducting with imposed magnetic and current fields. A linearized theory of transient development of anisotropic and axisymmetric surface waves generated by an oscillating pressure distribution acting on the free surface of a fluid of finite, infinite and shallow depth is developed in considerable detail. The problem is solved with the aid of generalized function technique in conjunction with asymptotic methods. An explicit and asymptotic solution of physical interest related to an arbitrary pressure distribution is presented so that the principal features of the flow can be explored. Careful attention has been given to examine the limiting behaviour of the asymptotic solution as time t tends to infinity. A detailed discussion of the axisymmetric wave motions including the effects of the imposed magnetic and current fields on the flow has been presented. It is shown that the wave motions are anisotropic as well as dispersive and the anisotropy is entirely due to the inclusion of the MHD body force. The solution of the corresponding wave problem in a non-conducting fluid has been recovered as a special case. In addition, it is shown that the present method of solution provides an interesting illustration of the applicability of the generalized function theory (Lighthill 1958, Jones 1966, Zemanian 1968) in wave problems of magnetohydrodynamics.

2. Mathematical Formulation

We consider a linearized problem of axisymmetric wave propagation in an inviscid, incompressible, homogeneous and electrically conducting fluid with a horizontal-free surface. We take the origin of the cylindrical polar coordinates (r, θ, z) with the z-axis vertically upwards and the undisturbed free surface at $z = 0$. The problem is investigated under the following assumptions:

(i) In the undisturbed state, the fluid is of infinite radial extent and is subjected to a uniform magnetic field $\mathbf{B} = (0, B_\theta, 0)$ and a uniform current field of density $\mathbf{J} = (J_r, 0, 0)$; and the gravity g. The vertical component of MHD force $\mathbf{J}\times\mathbf{B}$ along with the gravity constitutes the total body force acting on the system. If α is the angle between the co-planar vectors $\mathbf{J}$ and $\mathbf{B}$, and the latter makes an angle β with the wave propagation vector and the r-axis, it readily turns out that $J_r = J\cos(\alpha-\beta)$, $J_\theta = J\sin(\alpha-\beta)$, $B_r = B\cos\beta$ and $B_\theta = B\sin\beta$.

(ii) All perturbations of the magnetic field due to the imposed and the induced currents are negligible so that curl $\mathbf{E} = 0$, where $\mathbf{E}$ is the electric field. The wave damping due to Ohmic dissipation is also neglected, that is the term $\mathbf{u}\times\mathbf{B}$ involved in the generalized Ohm's law must vanish.

(iii) The assumption of small wave-amplitude is made so that this together with (ii) leads us to linearize the governing field equations and the boundary conditions.

(iv) Effects of the surface tension and viscosity of the fluid are neglected.

(v) The problem will be treated as an initial value problem. The wave motions are generated by a periodic pressure

$$\left.\begin{aligned} p(r,t) &= P\,p(r)\,e^{i\omega t}H(t), \quad 0<r<a \\ &= 0 \qquad\qquad\qquad\quad , \quad r>a \end{aligned}\right\} \qquad (2.1)$$

acting on the undisturbed free surface $z = 0$, where P is constant, $p(r)$ is an arbitrary function of r and $H(t)$ is the Heaviside step function of time t and ω is the fixed frequency of oscillations.

The unsteady flow equations for the conducting fluid (Ferraro and Plumpton 1966 and Cambel 1961) are

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\,.\,\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - g\mathbf{k}+\frac{1}{\rho}\mathbf{J}\times\mathbf{B} \qquad (2.2)$$

$$\text{div}\,\mathbf{u} = 0 \qquad (2.3)$$

$$\text{div}\,\mathbf{B} = 0 \qquad (2.4)$$

$$\text{Curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (2.5)$$

$$\text{Curl}\,\mathbf{B} = \mu\mathbf{J} \qquad (2.6)$$

$$\mathbf{J} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}) \qquad (2.7)$$

where $\mathbf{u} = (u, v, w)$ is the velocity field, ρ the density, μ the magnetic permeability, σ is the electrical conductivity of the fluid and ρ, μ, σ are all constants throughout the flow field.

In view of the assumption that the motion is irrotational, there exists a wave potential $\phi(r, z;\ t)$ which is governed by the Laplace equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0, \quad -h\leq z\leq 0,\ 0\leq r<\infty. \qquad (2.8)$$

It follows from the linearized equation of motion (2.2) with the above assumptions that the dynamic condition at the free surface takes the form

$$\frac{\partial\phi}{\partial t}+\left(g+\frac{B_r J_\theta}{\rho}\right)\eta=-\frac{P}{\rho}p(r)\,e^{i\omega t}, \quad z=0,\ t>0 \qquad (2.9)$$

where $\eta(r, t)$ is the vertical surface elevation function.

The kinematical-free surface condition is

$$\frac{\partial\phi}{\partial z}=\frac{\partial\eta}{\partial t}, \quad z=0,\ t>0. \qquad (2.10)$$

The boundary condition at the rigid bottom surface is

$$\frac{\partial\phi}{\partial z}=0 \quad \text{on } z=-h. \qquad (2.11)$$

The initial conditions are

$$\phi=\eta=0 \quad \text{at } t=0 \qquad (2.12a)$$

$$\phi_t=-\frac{P}{\rho}p(r) \text{ at } z=0,\ t=0. \qquad (2.12b)$$

In addition, we assume the existence of the Hankel transform of $\phi(r, z;\ t)$, $\eta(r, t)$ and $p(r)$ with respect to r in the generalized sense of Lighthill (1958). Thus the initial value problem has to be solved for $\phi(r, z;\ t)$ as well as $\eta(r, t)$ subject to the boundary and initial conditions (2.9)–(2.12).

Remarks

(1) The problem could be solved as a steady state by omitting the initial conditions (2.12) and imposing the radiation condition at infinity or an equivalent condition (Lamb 1905, Lighthill 1960). However, the present investigation assumes no such radiation condition or an equivalent device to derive a unique solution of physical interest.

(2) It is interesting to observe that the anisotropic character of the wave motions is essentially related to the β-dependence of the vertical component of the imposed MHD force

$$B_r J_\theta = BJ\cos\beta\sin(\alpha-\beta). \qquad (2.13)$$

In engineering applications, two cases (i) $\alpha=\pi/2$ and (ii) $\alpha=0$ are of particular interest.

(3) Two possible methods, such as (i) power series method and (ii) asymptotic method, can be employed to obtain the solution of physical interest of the problem. With regard to the power series method used by Lamb (1932),

the following remarks may be worth noting. In connection with a steady state axisymmetric wave problem due to local disturbances, a series solution for the surface elevation $\eta(r, t)$ was obtained by Lamb by expanding the integrand of the integral solution in a power series together with term-by-term integration. One such solution for $\eta(r, t)$ is

$$\eta(r, t) = \frac{P}{2\pi r^2}\left\{\frac{1^2}{2!}\left(\frac{gt^2}{r}\right) - \frac{1^2 3^2}{6!}\left(\frac{gt^2}{r}\right)^3 + \frac{1^2 3^2 5^2}{10!}\left(\frac{gt^2}{r}\right)^5 + \ldots\right\}. \quad \ldots (2.14)$$

This series is convergent for all values of $\frac{gt^2}{r}$, but, unfortunately, its use is restricted to small values of the parameter $\frac{gt^2}{r}$, that is small values of t. The series converges very slowly for large values of the parameter $\frac{gt^2}{r}$ and hence it is not very useful for large t.

This leads us to give preference to the asymptotic methods for the investigation of the problem stated above. The asymptotic methods cannot only eliminate the deficiencies of the classical series solution but also enables us to derive a very useful solution valid for large t. Moreover, the asymptotic solution is powerful enough to describe all qualitative features of wave phenomena in addition to its usefulness from the computational point of view. It is sometimes also suitable for small values of the parameter involved. Finally, the power and success of the asymptotic treatment has recently been well established on its own merit.

3. Solution of the Problem

It is convenient to solve the problem by the joint Laplace and generalized Hankel transforms technique. We introduce the Laplace transforms $\tilde{\phi}$, $\tilde{\eta}$ of ϕ, η respectively with respect to t by an integral (Sen 1969)

$$\tilde{\phi} = \tilde{\phi}(r, z; s) = \int_0^\infty e^{-st}\,\phi(r\ \ z; t)\,dt. \quad \ldots \quad \ldots \quad \ldots (3.1)$$

We next introduce the generalized Hankel transformations $\bar{\tilde{\phi}}$, $\bar{\tilde{\eta}}$ of $\tilde{\phi}$, $\tilde{\eta}$ respectively with respect to r by an integral

$$\bar{\tilde{\phi}} = \bar{\tilde{\phi}}(k, z; s) = \int_0^\infty r\,J_0(kr)\,\tilde{\phi}(r, z; s)\,dr \quad \ldots \quad \ldots (3.2)$$

where $J_0(kr)$ is the Bessel function of the first kind and order zero.

Making use of the integral transform methods, the solution of (2.8) subject to the boundary and initial conditions (2.9)–(2.12) can be obtained as

$$\bar{\tilde{\phi}}(k, z;\ s) = -\frac{P}{\rho}\,\frac{s\,\bar{p}(k)}{(s-i\omega)}\,\frac{\cosh k(z+h)}{(s^2+\alpha^2)\cosh kh} \quad \ldots \quad \ldots (3.3)$$

where

$$\alpha^2 = g'k\tanh kh,\ g' = g+b,\ b = \frac{B_r J_\theta}{\rho}. \quad \ldots \quad \ldots (3.4)$$

The expression for $\bar{\bar{\eta}}(k, s)$ can be written as

$$\bar{\bar{\eta}}(k, s) = -\frac{p}{\rho g'} \frac{\alpha^2 \bar{p}(k)}{(s-i\omega)(s^2+\alpha^2)}. \qquad (3.5)$$

The inversion theorem of the Laplace and the Hankel transformations together with the Faltung theorem for the former yields

$$\phi(r, z;\, t) = \frac{P}{\rho} \int_0^\infty \frac{k\,\bar{p}(k)}{(\alpha^2-\omega^2)} \frac{\cosh k(z+h)}{\cosh kh} (i\omega \cos \alpha t - \alpha \sin \alpha t - i\omega e^{i\omega t})\, J_0(kr)\, dk \qquad (3.6)$$

$$\eta(r, t) = \frac{P}{\rho g'} \int_0^\infty k\, \alpha(k)\, \bar{p}(k) J_0(kr)(\alpha^2-\omega^2)^{-1}(i\omega \sin \alpha t + \alpha \cos \alpha t - \alpha e^{i\omega t})\, dk. \qquad (3.7)$$

These integrals represent the integral solution for the wave potential $\phi(r, z; t)$ and the free surface elevation $\eta(r, t)$. In view of these complicated expressions, they cannot in general be evaluated exactly. Hence asymptotic methods are needed for their evaluation in cases of physically realistic pressure distributions.

In the case of an infinitely deep conducting fluid, that is when $h \to \infty$, the integral representation for $\phi(r, z; t)$ and $\eta(r, t)$ assumes the form

$$\phi(r, z;\, t) = \frac{P}{\rho} \int_0^\infty \frac{k\, e^{kz} J_0(kr)\, \bar{p}(k)}{(\alpha^2-\omega^2)} (i\omega \cos \alpha t - \alpha \sin \alpha t - i\omega e^{i\omega t})\, dk \qquad (3.8)$$

$$\eta(r, t) = \frac{P}{\rho g'} \int_0^\infty \frac{k\, \bar{p}(k)\, \alpha(k)\, J_0(kr)}{(\alpha^2-\omega^2)} (i\omega \sin \alpha t + \alpha \cos \alpha t - \alpha e^{i\omega t})\, dk \qquad (3.9)$$

where $\alpha^2(k)$ is to be replaced by its limiting value $g'k$ as $h \to \infty$.

4. Particular Cases of Interest

It has already been mentioned that the asymptotic analysis of the solution can be carried out with an arbitrary form of $p(r)$ involved in (2.1). However, it would be sufficient for the investigation of the characteristic features of the wave motions to take the simple form of $p(r)$, such as

$$\left.\begin{array}{l} \text{(a) } p(r) = \dfrac{\delta(r)}{r}, \quad \text{(b) } p(r) = 1, \quad \text{(c) } p(r) = e^{-r^2/r_0^2}, \quad \text{(d) } p(r) = (a^2-r^2)^n, \\ \text{(e) } p(r) = e^{-r^2} L_n(r^2), \quad \text{(f) } p(r) = J_0(\lambda r) \quad \text{and} \quad \text{(g) } p(r) = r^{n-1} e^{-mr^2} \end{array}\right\} \qquad (4.1)$$

where $n > -1$, $m > 0$, $\delta(r)$ is the Dirac function of distribution and $L_n(z)$ is the Laguerre polynomial of degree n.

Making reference to Erdélyi (1954) and Sneddon (1951), the Hankel transforms $\bar{p}(k)$ related to (a)–(g), respectively, are

$$1,\ \frac{a}{k}J_1(ak),\ 2m^2\,e-k^2m^2\left(m=\frac{r_0^2}{4}\right),\ 2^n\,\Gamma(n+1)\,a^{2(n+1)}(ak)^{-(n+1)}\,J_{n+1}(ak),$$

$$(n\,!)^{-1}\,2^{-(2n+1)}\,k^{2n}\exp\left(-\frac{k^2}{4}\right),\ \int_0^a x\,J_0(\lambda x)\,J_0(kx)\,dx,\quad\text{and}$$

$$m^{-n/2}\,k^{-1}\,\Gamma\left(\frac{n+1}{2}\right)\exp\left(-\frac{k^2}{8m}\right)M_{n/2,\,0}\left(\frac{k^2}{4m}\right)$$

where $J_n(z)$ is the Bessel function of the first kind and order n, $M_{a,\,b}(z)$ is the Whittaker function (Whittaker and Watson 1965) and the above integral is a standard one (Watson 1922).

5. Asymptotic Analysis

To investigate the principal features of the axisymmetric wave motions in the conducting fluid, it is necessary to make an asymptotic approximation to (3.7) which is very much similar to that of the corresponding result in a non-conducting flow obtained by Debnath (1969*a*–*c*). This indicates that a similar asymptotic analysis with a slight modification can be made without any difficulty.

To avoid duplication, it may probably be fair to refer to the earlier work of the author where the details of the asymptotic analysis are available. So it would be of some help to the readers to have some familiarity with the papers mentioned above. However, in order to make the present discussion self-contained to some extent, we shall quote the necessary results from those papers without further proof and outline the asymptotic analysis with notations used earlier.

Thus it would be convenient to recall (3.7) and mention some of its properties. Integral solution (3.7) for $\eta(r,\,t)$ is convergent for all values of k in the range of integration $(0,\,\infty)$ provided the function $\bar{p}(k)$ is well behaved in $(0,\,\infty)$; which is really the case according to the choice of $p(r)$ stated in § 4.

Thus integral (3.7) can be written in a convenient form

$$\eta(r,\,t)=I_{tr}-I_{st} \qquad (5.1)$$

where the transient and steady state wave integrals are given by

$$I_{st}=\frac{P\,e^{i\omega t}}{\rho g'}\int_0^\infty\frac{k\,\alpha^2(k)\,\bar{p}(k)\,J_0(kr)\,dk}{\alpha^2-\omega^2} \qquad (5.2)$$

$$I_{tr}=\frac{P}{\rho g'}\int_0^\infty\frac{k\,\alpha(k)\,\bar{p}(k)\,J_0(kr)(i\omega\sin\alpha t+\alpha\cos\alpha t)\,dk}{(\alpha^2-\omega^2)}. \qquad (5.3)$$

It is important to indicate that the nature both of the ultimate wave system and of the asymptotic approach to it is determined by the singularities of the integrands in (5.2) and (5.3). The remainder of this section, therefore, is concerned with locating and characterizing these singularities.

It can be easily verified by the graphical method as used by Debnath and Rosenblat (1969) that the singularities of (5.2) and (5.3) are given by the roots of the transcendental equation

$$\alpha^2-\omega^2 = 0. \qquad (5.4)$$

A careful consideration suggests that eqn. (5.4) has only one positive real root at $k = k_0$ which is obtained as the point of intersection of the curves

$$\zeta = \sqrt{k \tanh kh} \text{ and } \zeta = \frac{\omega}{\sqrt{g'}}. \qquad (5.5)$$

The dominant contribution to I_{st} as $r \to \infty$ comes from the simple pole of the integrand at $k = k_0$. To obtain the contribution, it is convenient to replace $J_0(kr)$ by the pair of Hankel functions $H_0^{(1)}(kr)$ and $H_0^{(2)}(kr)$. Then the method of evaluation as used by Debnath and Rosenblat (1969) or by Debnath (1969*a–c*) enables us to write down

$$I_{st} \sim -\frac{Pi}{\rho g'}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}}\left\{\frac{\bar{p}(k_0)\, k_0\, \alpha^2(k_0)}{D'(k_0)}\right\} e^{i\left(\omega t - rk_0 + \frac{\pi}{4}\right)} + O\left(\frac{1}{r}\right) \qquad (5.6)$$

where the function $D(k)$ is given by

$$D(k) \equiv \alpha^2-\omega^2. \qquad (5.7)$$

The integral I_{tr} can be evaluated asymptotically for large values of t with the aid of the same technique as used by Debnath and Rosenblat (1969) or Debnath (1969*a–c*). It may be observed that the stationary and/or saddle points of (5.3) can be obtained by solving the equations

$$\frac{d\alpha}{dk} = \pm\frac{r}{t}. \qquad (5.8ab)$$

A consideration similar to that made earlier readily turns out that the eqns. (5.8*ab*) have a real root which represents the stationary point and may be denoted by $k = k_1$.

With the aid of the asymptotic technique as used by Debnath (1969*a–c*) or by Debnath and Rosenblat (1969), it readily turns out that the asymptotic representation of I_{tr} for large t has the form

$$I_{tr} \sim \frac{P}{\rho g'}\frac{k_1\,\bar{p}(k_1)}{\{4rt|f''_-(k_1)|\}^{\frac{1}{2}}}\left[\frac{\exp i\{rk_1-t\alpha(k_1)\}}{\alpha(k_1)+\omega}+\frac{\exp(-i)\{rk_1-t\alpha(k_1)\}}{\alpha(k_1)-\omega}\right] \qquad (5.9)$$

where

$$f''_-(k_1) \equiv \left(\alpha-\frac{r}{t}k\right). \qquad (5.10)$$

It remains to calculate the contribution to I_{tr} from its polar singularity which is the same as the pole of I_{st}. By a precisely similar argument made for the evaluation of I_{st}, it can be shown that the contribution to I_{tr} from the simple pole at $k = k_0$ is

$$I_{tr(\text{polar})} \sim \frac{Pi}{\rho g'}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \frac{\bar{p}(k_0)k_0\alpha^2(k_0)}{D'(k_0)} e^{i\left(\omega t - rk_0 + \frac{\pi}{4}\right)}. \quad \ldots \quad \ldots \quad (5.11)$$

The asymptotic form of the surface elevation $\eta(r, t)$ is therefore obtained as

$$\eta(r, t) = \eta_{st}(r, t) + \eta_{tr}(r, t) \quad \ldots \quad \ldots \quad \ldots \quad (5.12)$$

where η_{st}, η_{tr} represent the steady state and the transient components of $\eta(r, t)$ respectively.

It may be of interest to point out that the steady state solution $\eta_{st}(r, t)$ is actually made up of the contributions to I_{st} and I_{tr} from their simple pole at $k = k_0$; whereas the transient component of $\eta(r, t)$ comes from the stationary point at $k = k_1$. Therefore, the final form of $\eta_{st}(r, t)$ is given by

$$\eta_{st}(r, t) = \frac{iP}{g'\rho}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \frac{\bar{p}(k_0)\, k_0\, \alpha^2(k_0)}{D'(k_0)} e^{i\left(\omega t - rk_0 + \frac{\pi}{4}\right)}. \quad \ldots \quad \ldots \quad (5.13)$$

The transient solution $\eta_{tr}(r, t)$ is exactly identical with I_{tr} and hence it is represented by the asymptotic result obtained in (5.9) with (5.10).

Remark: It should be pointed out that solution (5.12) breaks down at $k_0 = k_1$. However, a special device advanced by Debnath can be adopted to obtain a valid solution near $k_0 = k_1$. Since our principal interest is in large values of t, so it seems reasonable to omit the derivation of the solution at the critical point in the present analysis.

6. Asymptotic Solution in Deep Conducting Fluid

In the limit $h \to \infty$ ($kh \gg 1$), it can be easily verified that the simple pole at $k = k_0$ and the stationary point at $k = k_1$ have the explicit form

$$k_0 = \frac{\omega^2}{g'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.1)$$

$$k_1 = \frac{g't^2}{4r^2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6.2)$$

Therefore the asymptotic solution for the surface elevation $\eta(r, t)$ in the present case has the following form:

$$\eta(r, t) \sim \begin{bmatrix} \dfrac{iP\omega^3}{\rho g'^2}\sqrt{\dfrac{2\pi}{rg'}}\,\bar{p}\left(\dfrac{\omega^2}{g'}\right) e^{i\left(\omega t - \frac{r\omega^2}{g'} + \frac{\pi}{4}\right)}, & g't \gg 2\omega r \\ 0 & , \; g't \ll 2\omega r \end{bmatrix}$$

$$+ \frac{P\, g't^3\, \bar{p}\left(\dfrac{g't^2}{4r^2}\right)}{\omega\rho\, 2^{\frac{1}{2}}\, r^4\left(\dfrac{g'^2t^2}{4r^2\omega^2} - 1\right)} \left\{ \frac{g't}{2r\omega} \cos\left(\frac{g't^2}{4r}\right) + i \sin\left(\frac{g't^2}{4r}\right) \right\} + O\left(\frac{1}{t^2}\right). \qquad (6.3)$$

Once again, it may be remarked that the point $k_0 = k_1$, at which solution (5.12) was invalid, becomes in this case the point $g't = 2r\omega$, where the solution stated above also breaks down. However, a special method devised by Debnath can be employed to compute a solution for $\eta(r, t)$ valid at the critical point. Following the procedure discussed in detail by the author as stated above, a solution valid at $g't = 2r\omega$ can be obtained in the form

$$\eta(r, t) \sim -\frac{3}{2}\frac{P\omega^3}{\rho g'^2}\left(\frac{\pi}{r}\right)^{\frac{1}{2}}\bar{p}\left(\frac{\omega^2}{g'}\right)\left(\cos\frac{\omega^2 r}{g'}+\sin\frac{\omega^2 r}{g'}\right). \quad .. \quad (6.4)$$

Thus the problem is completely solved for an arbitrary pressure distribution $p(r)$. The solution related to the particular form of $p(r)$ mentioned in (4.1a–g) can readily be written down.

7. Solution in Shallow Conducting Fluid

In the case of a very shallow conducting fluid, long wave approximation $kh \ll 1$ can be made. Consequently, it follows readily that $\alpha^2(k) \approx g'\, k^2 h$ to the first approximation; and the simple pole at $k = k_0$ takes the form $k_0 = \dfrac{\omega}{\sqrt{g'h}}$. Further, it may be noted that there is no stationary point of the transient wave integrals in a shallow fluid. This enables us to obtain an explicit solution for $\eta(r, t)$ as

$$\eta(r, t) \sim \frac{iP}{\rho}\sqrt{\frac{2\pi}{r}}\, h\, \bar{p}\left(\frac{\omega}{\sqrt{g'h}}\right)\left(\frac{\omega}{\sqrt{g'h}}\right)^{\frac{1}{2}} e^{i\left(\omega t - \frac{\omega r}{\sqrt{g'h}}+\frac{\pi}{4}\right)}. \quad .. \quad (7.1)$$

This solution is in perfect agreement with that of a non-conducting shallow fluid obtained by Debnath in the limit $b \to 0$. Thus the corresponding solution in a shallow fluid related to a particular pressure distribution stated in (4.1) can easily be obtained from (7.1).

8. Discussions and Conclusions

From the above axisymmetric analysis, it is clear that the problem has been explicitly and completely solved in a conducting fluid of an arbitrary depth with an arbitrary oscillating pressure distribution. Further an asymptotic solution for the free surface elevation $\eta(r, t)$ related to some physically realistic pressure distributions given in (4.1) can easily be obtained.

It is interesting to notice that the asymptotic solution (5.12) for the free surface elevation $n(r, t)$ contains a transient term $\eta_{tr}(r, t)$ in addition to the steady state component $\eta_{st}(r, t)$ which is the solution of physical interest of the corresponding steady state wave problem. As it has been indicated before, the steady state problem cannot, in general, be solved without the use of the radiation condition or its equivalent. Thus one of the significant conclusions of the present initial value investigation is that a unique solution of physical interest is achieved without having to resort to the use of the radiation condition or an equivalent device.

A careful examination of solution (5.12) as well as (6.3) for $\eta(r, t)$ reveals that the transient component does tend to zero in the limit $t \to \infty$ for fixed r, in all the cases (4.1b–g). This indicates clearly that the ultimate steady state is established in the limit. And the steady state solution $\eta(r, t)$ in the case of finite depth is given by

$$\eta(r, t) \sim -\frac{Pi}{\rho g'}\left(\frac{2\pi}{rk_0}\right)^{\frac{1}{2}} \frac{\bar{p}(k_0)\, k_0\, \alpha^2(k_0)}{D'(k_0)} e^{i\left(\omega t - rk_0 + \frac{\pi}{4}\right)}. \qquad (8.1)$$

In the case of a very deep conducting fluid, the steady state solution has the form

$$\eta(r, t) \sim \frac{iP\omega^3}{\rho g'^2}\sqrt{\frac{2\pi}{rg'}}\,\bar{p}\left(\frac{\omega^2}{g'}\right) e^{i\left(\omega t - \frac{r\omega^2}{g'} + \frac{\pi}{4}\right)}. \qquad (8.2)$$

Both the solutions (8.1) and (8.2) represent the progressive circular waves propagating in the conducting medium with the group velocity, $\frac{\omega}{2k_0}$, $\frac{g'}{2\omega}$ according as the fluid is of finite or infinite depth. The amplitude of the wave motions is proportional to $r^{-\frac{1}{2}}$ which is irrespective of the depth of the fluid.

It may be recognized that in the presence of the magnetic and current fields, the phase as well as the group velocity of the progressive waves is much greater than their corresponding values in a conducting fluid flow. Indeed this significant modification is solely due to the external fields.

In the case of a very shallow fluid, the solution for $\eta(r, t)$ is given by (7.1). This solution corresponds to the axisymmetric waves travelling with the group velocity $\frac{1}{2}\sqrt{g'h}$ which is independent of the wavelength, but is affected by the external magnetic and current fields. This implies that the waves are evidently non-dispersive in character. The amplitude of the wave-trains decays like $r^{-\frac{1}{2}}$.

Another interesting point is that the solution for $\eta(r, t)$ related to case (4.1a) does not tend to the corresponding steady state in the limit $t \to \infty$ for fixed r. The above analysis reveals a significant difference between the wave motions produced by oscillating pressure distributions confined over a finite region, however small, and at a single point on the free surface of the fluid. Several convincing mathematical and physical reasons similar to those suggested by Debnath (1968, 1969*a–c*) can be advanced in favour of this analysis.

It is also pertinent to point out that the solution for $\eta(r, t)$ associated with (4.1a) readily follows from that of (4.1b) as a limit $P \to \infty$ $a \to 0$ provided Pa^2 is a finite constant.

The most striking feature of the axisymmetric wave-systems in a conducting fluid of finite and infinite depth is that they are anisotropic as well as dispersive in character. The anisotropic character is essentially introduced through the β-dependence of the imposed MHD force as indicated in eqn. (2.13). Similar conclusions can also be drawn about the waves in a very shallow conducting fluid except of the fact that these waves are non-dispersive. This can easily be verified from the phase velocity of the waves.

In the absence of the imposed magnetic and current fields, that is in the non-conducting limit $b \to 0$, the present analysis shows an excellent agreement with that of non-conducting fluid considered earlier by Debnath (1969*a*–*c*) and Debnath and Rosenblat (1969).

Finally, it may be natural to ask some questions about the stability of the flow when the MHD force and gravity are equal or unequal in both magnitude and direction. An axisymmetric stability problem similar to Murty's (1961) stability analysis may be investigated to answer these questions. This will be considered in a subsequent paper.

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