ON THE VIBRATORY BEHAVIOUR OF FLUID IN A PARABOLIC CONTAINER

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The advent of liquid propellent launch vehicles has led to considerations of problems involving vibratory behaviour of fluids in rigid containers. Prediction of the dynamic behaviour of tanks, partially filled with liquid, is of paramount importance in the design of liquid-rocket-powered aerospace vehicles. Because of the development of kick stages and interplanetary vehicles, new and unusual container configurations need to be considered in the design of present-day missiles and aerospace vehicles. With this object in mind, the behaviour of liquid in a cylinder whose cross-section is bounded by four confocal parabolas is investigated in this paper. The main object is to determine the eigenvalues and eigenfunctions associated with the problem. An approximate range of eigenvalues is given by using the maximum-minimum principle. Also by solving a system of transcendental equations, some exact values of the eigenvalues are determined. Curves of lowest natural frequencies are plotted as a function of a non-dimensional tank parameter.

1. INTRODUCTION

This study is concerned with the vibratory behaviour of fluids in a rigid cylindrical vessel. We shall consider a parabolic cylinder Σ whose walls are bounded by four confocal orthogonal parabolas. The cross-section of such a cylindrical container is shown in Fig. 1. We shall consider a mass of liquid which is contained in the cylinder Σ. The cylinder is closed by a horizontal plane at a depth h below the undisturbed free surface (Fig. 2). The vessel is subjected to a prescribed vertical acceleration. We take Cartesian axes (x, y, z) moving with the vessel, such that the equation of the undisturbed free surface is $z = 0$, and the equation of the base of the vessel is $z = -h$. Curve C is the boundary of the free surface at the rigid wall and τ is the volume which the fluid occupies. We shall consider the fluid to be ideal, homogeneous and incompressible making a constant angle of contact of 90° with the walls of the vessel and assume that the amplitudes of the irrotational fluid motion are small. The behaviour of the resulting surface wave system is

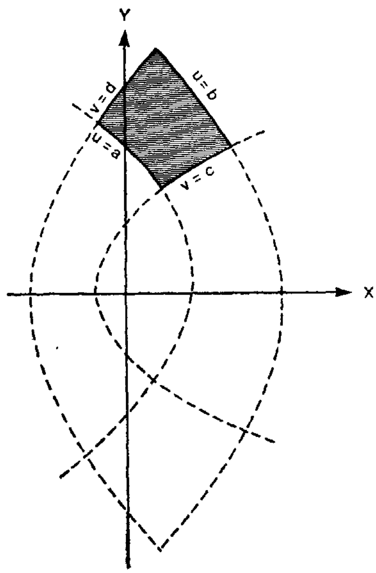

FIG. 1. Cross-section bounded by four confocal orthogonal parabolas.

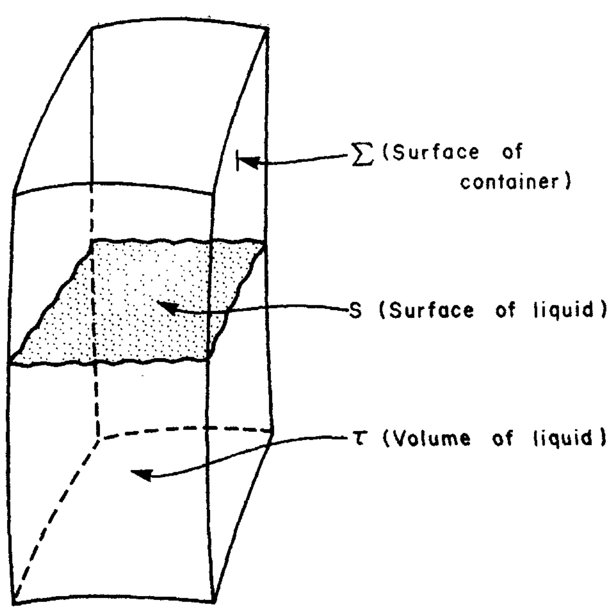

FIG. 2. Parabolic cylindrical container.

governed by the solution of the following linearized boundary value problem

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \qquad (1)$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on the walls} \qquad (2)$$

$$\frac{\partial\phi}{\partial z} = 0 \text{ on the base, } z = -h \qquad (3)$$

$$\frac{\partial\zeta}{\partial t} = -\frac{\partial\phi}{\partial z} \text{ at the free surface } (z = 0) \qquad (4)$$

$$\frac{\gamma}{\rho}\left[\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right] = -\frac{\partial\phi}{\partial t}+(g+\ddot{F})\zeta \text{ at the free surface } (z = 0). \qquad (5)$$

The derivation of (1) through (5) is given in detail in Lamb (1962) and by Benjamin and Ursell (1954). Here $\frac{\partial}{\partial n}$ denotes the normal derivative, g the gravitational acceleration, $\ddot{F}$ the arbitrary vertical acceleration, ρ the density, γ the surface tension and ϕ the velocity potential. The free surface of the fluid is described by $z = \zeta(x, y, t)$. The curvature of the free surface is approximated by $\zeta_{xx}+\zeta_{yy}$. The approximation used here for the curvature is familiar in membrane theory given by Rayleigh (1894). For cylinders of constant cross-section it has been shown by Benjamin and Ursell (1954) that ϕ, ζ and $\left[\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right]$ can be expanded in terms of the complete orthogonal set of eigenfunctions. Our object is then to obtain these eigenfunctions and the eigenvalues associated with this problem. The eigenvalues and eigenfunctions for a similar and much simpler problem have been given in a recent paper by Lomen and Fontenot (1967). In this paper we wish to discuss the problem for a more general cylinder. This problem is more general in nature and as expected is more complicated. Similar problems in electronmagnetic propagation through wave guides of orthogonal parabolic cross-section have been discussed by a number of authors (Spence and Wells 1942, Tung 1950, Tung and Higgins 1955).

2. The Parabolic Coordinates

Introduce new real variables u, v defined by the complex equation $x+iy = (1/2)(u+iv)^2$ so that the parabolic cylinder coordinates (u, v, z) are given by

$$x = (1/2)(u^2-v^2),\ y = uv,\ z = z. \qquad (6)$$

The curves on which u or v is constant are parabolas confocal with the boundary (Fig. 3). The lines $u = c$, where c is an arbitrary positive constant, comprise a family of parabolas symmetrical about the x-axis and concave to the left; the lines $v = c'$ comprise a family of parabolas, also symmetrical about the x-axis, but concave to the right. The two families are both confocal and orthogonal. Their common focus is the origin $x = y = 0$.

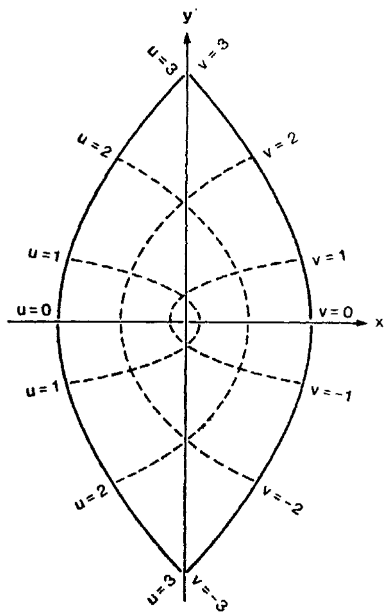

FIG. 3. Parabolic cylindrical coordinates.

While the function $\frac{1}{2}(u+iv)^2$ is analytic everywhere except at infinity, $u+iv = 0$ is nevertheless a critical point because the scale factor $|u+iv|$ goes to zero there. To eliminate double values in the inverse function, we require that $u > 0$ and $-\infty < v < \infty$. The detailed analysis is given by Morse and Feshback (1953) and Lebedev (1965). The line $\xi = 0$, which is the map of the negative half of the x-axis, thus is made the map of the branch cut, positive values if v being above this line and negative values below it. We shall study the most general case in which the cross-section of the cylinder is formed of four confocal orthogonal parabolas as in Fig. 1. Let the bounding parabolas be defined as $u = a, b$; $v = c, d$.

3. Basic Equations in Parabolic Coordinates

Using the coordinate transformation as given in eqn. (6), Laplace's eqn. (1) may be written as

$$\frac{1}{(u^2+v^2)}\left[\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right]+\frac{\partial^2\phi}{\partial z^2} = 0. \qquad (7)$$

Let us put $\phi(u, v, z, t) = \psi(u, v, t)\, f(z)$ in (7). We obtain

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}+k^2(u^2+v^2)\psi = 0 \qquad (8)$$

and

$$f(z) = \cosh k(z+h). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The function $f(z)$ has been chosen so that the boundary condition (3) is satisfied. The separation constant k^2 which occurs in (8) has been shown to be real and positive (Courant and Hilbert 1953). The substitution of $\psi = U(u).V(v).T(t)$ in (8) and subsequent separation yields

$$\frac{d^2U}{du^2}+(k^2u^2+m)U = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and

$$\frac{d^2V}{dv^2}+(k^2v^2-m)V = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where m is the separation constant, arbitrary at this point.

The boundary conditions associated with these differential equations may be obtained by introducing (6) into (2). This yields

$$\frac{dU}{du} = 0 \quad (\text{at } u = a,\, b) \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$\frac{dV}{dv} = 0 \quad (\text{at } v = c,\, d). \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Since the boundary value problem, defined by (10) or (11) subject to the Neumann conditions (12) or (13), comprises a Sturm-Liouville system, it can be shown that m is also real (Sommerfeld 1949). The differential equations for both u and v factors are identical except for the sign of m. Thus the solutions for $U(u)$ and $V(v)$ may be expressed in terms of the functions $H(\alpha,\, uk^{1/2})$ and $H(-\alpha,\, vk^{1/2})$, where $H(\alpha,\, \lambda)$ is a solution of the differential equation

$$\frac{d^2H}{d\lambda^2}+(\alpha+\lambda^2)H = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

with $\lambda^2 = ku^2$ or kv^2 and $\alpha = m/k$.

4. On the Solution of Equation (14)

Solutions of (14) were obtained originally in definite integral form by Weber (1869). The even and odd solutions of (14) were discussed by Spence and Wells (1942) in connection with electromagnetic wave propagation in parabolic pipe where the separation constant is restricted to positive integral value. In our problem α is any real number. The behaviour of the odd and even functions and the choice of the form of the linearly independent solutions is discussed by Darwin (1949), Tung (1950) and Miller (1952). The asymptotic series for the even and odd solutions of (14) for λ or α becoming very large are given by Tung (1950). Let us designate the even and odd solutions of (14) as $H_e(\alpha,\, \lambda)$ and $H_0(\alpha,\, \lambda)$ respectively. These we consider to be the fundamental set of solutions.

Representative plots of the functions and their derivatives are given by Wells and Spence (1945), Tung (1950), Lomen and Fontenot (1967) for various values of α. For $\alpha = \pm 1, \pm 2, \pm 3$, and $0 \leqslant \lambda \leqslant 3$, values of the functions are given by Wells and Spence (1945). For $-10 \leqslant \alpha \leqslant 10$ and $0 \leqslant \lambda \leqslant 5/\surd 2$, values of the functions may be obtained from Abramowitz and Stegun (1964) provided we make use of the relations

$$H_e(\alpha, \lambda) = (\tfrac{1}{2})W(-\alpha/2, 0)[W(-\alpha/2, 2^{1/2}\lambda)+W(-\alpha/2, -2^{1/2}\lambda)], \quad .. \ (15a)$$

$$H_0(\alpha, \lambda) = 2^{-1/2}W(-\alpha/2, 0)[W(-\alpha/2, -2^{1/2}\lambda)-W(-\alpha/2, 2^{1/2}\lambda)] \quad .. \ (15b)$$

where W is the Whittaker function.

The properties of the parabolic cylinder functions D have been extensively studied by a number of authors during the half century. Consequently, many properties of H_e and H_0 can be derived from the known properties of D. Power series expansions for $H_e(\alpha, \lambda)$ and $H_0(\alpha, \lambda)$ are obtained by method of Frobenius (Morse and Feshback 1953) as

$$H_e(\alpha, \lambda) = 1-\tfrac{1}{2}\alpha\lambda^2+(1/24)(\alpha^2-2)\lambda^4-(1/720)(\alpha^3-14\alpha)\lambda^6+ \quad .. \ (16a)$$

$$H_0(\alpha, \lambda) = \lambda-\tfrac{1}{6}\alpha\lambda^3+(1/120)(\alpha^3-6)\lambda^5-(1/5040)(\alpha^3-26\alpha)\lambda^7+ \quad .. \ (16b)$$

where the non-zero coefficients c_n of λ^n are related by

$$n(n-1)c_n+\alpha c_{n-2}+c_{n-4} = 0. \quad .. \quad .. \quad .. \ (17)$$

$H_e(\alpha, \lambda)$ and $H_0(\alpha, \lambda)$ may also be expressed in terms of the confluent hypergeometric functions

$$H_e(\alpha, \lambda) = e^{-i\lambda^2/2}F\left(\frac{1}{4}+\frac{i\alpha}{4}, \frac{1}{2}, i\lambda^2\right) \quad .. \quad .. \quad .. \ (18a)$$

$$H_0(\alpha, \lambda) = \lambda e^{-i\lambda^2/2}F\left(\frac{3}{4}+\frac{i\alpha}{4}, \frac{3}{2}, i\lambda^2\right). \quad .. \quad .. \quad .. \ (18b)$$

In our problem we shall need to know the zeros of the functions and their derivative. So, we shall say something about the zeros of the functions and their oscillatory behaviours. The graphs indicate (Tung 1950), that when $\alpha > 0$, the even and odd solutions appear somewhat like damped cosine and sine functions. When $\alpha < 0$ and x is small, the even functions somewhat resemble a hyperbolic cosine and the odd function a hyperbolic sine, but as x increases a point of inflection occurs at $x = \sqrt{|\alpha|}$. When $x < \sqrt{|\alpha|}$ the curve is convex to the x-axis, and when $x > \sqrt{|\alpha|}$, it is concave thereto. The functions rise to a fairly large value, depending on α, then turns gradually toward the x-axis, crossing it and thereafter oscillates about the x-axis. When α becomes large negatively, the zeros of the even and odd functions for the same α tend to coincide.

The number of zeros of $H_e(\alpha, \lambda)$ or $H_e(-\alpha, \lambda)$ in any interval cannot differ by more than one from the number of zeros of $H_0(\alpha, \lambda)$ or $H_0(-\alpha, \lambda)$. This follows from the well-known separation theorem (Ince 1927) which, in effect, states that the zeros of all solutions of the differential equation oscillate with equal rapidity. For $\alpha > 0$, it can be shown that $H_e(\alpha, \lambda)$ and $H_0(\alpha, \lambda)$ will always oscillate. A similar statement for $\alpha < 0$ also holds, provided $\lambda^2 > |\alpha|$; however, for $\lambda^2 < |\alpha|$, the solutions are non-oscillatory.

From Sturm's fundamental theorem we see that if the solutions of (14) oscillate in a given interval, they will do so more rapidly when α is increased. Hence, both $H_e(\alpha, \lambda)$ and $H_0(\alpha, \lambda)$ will oscillate more rapidly than either $H_e(-\alpha, \lambda)$ or $H_0(-\alpha, \lambda)$. We can also prove that the above statements regarding the oscillatory nature of the functions are equally true for their derivatives. In our problem we are more interested in the derivatives of the functions rather than the functions themselves.

When $\alpha = 0$, (14) can be reduced to a Bessel's differential equation. From this we can at once write the relationship between the functions $H_e(\alpha, \lambda)$ and $H_0(\alpha, \lambda)$ and the Bessel function. The relations are

$$H_e(0, \lambda) = \Gamma(3/4)(\lambda/2)^{1/2}J_{-1/4}(\lambda^2/2), \quad \text{(19a)}$$

$$H_0(0, \lambda) = \Gamma(5/4)(2\lambda)^{1/2}J_{1/4}(\lambda^2/2). \quad \text{(19b)}$$

By using the properties of the Bessel functions (Watson 1962) we can write

$$H'_e(0, \lambda) = -\Gamma(3/4)2^{-1/2}\lambda^{3/2}J_{3/4}(\lambda^2/2), \quad \text{(20a)}$$

$$H'_0(0, \lambda) = \Gamma(5/4)2^{1/2}\lambda^{3/2}J_{-3/4}(\lambda^2/2). \quad \text{(20b)}$$

From the relationships (20) and Sturm's fundamental theorem we have the following result: If $J_{3/4}(\lambda^2/2)$ or $J_{-3/4}(\lambda^2/2)$ has n zeros in a given interval, then both $H'_e(\alpha, \lambda)$ or $H'_0(\alpha, \lambda)$ and $H'_e(-\alpha, \lambda)$ or $H'_0(-\alpha, \lambda)$ have at least n zeros in the same interval. Also, the nth zero of $H'_e(\alpha, \lambda)$ or $H'_0(\alpha, \lambda)$ is less than the nth zero of $J_{3/4}(\lambda^2/2)$ or $J_{-3/4}(\lambda^2/2)$ respectively. Similarly, the nth zero of $H'_e(-\alpha, \lambda)$ or $H'_0(-\alpha, \lambda)$ is greater than the nth zero of $J_{3/4}(\lambda^2/2)$ or $J_{-3/4}(\lambda^2/2)$.

5. Eigenvalues and Eigenfunctions

We can now write the solutions of (10) as

$$U(u) = AH_e(\alpha, \sqrt{k}\,u) + BH_0(\alpha, \sqrt{k}\,u). \quad \text{(21)}$$

To satisfy the boundary conditions (12) it is necessary that the determinant

$$\begin{vmatrix} H'_e(\alpha, \sqrt{k}\,a) & H'_0(\alpha, \sqrt{k}\,a) \\ H'_e(\alpha, \sqrt{k}\,b) & H'_0(\alpha, \sqrt{k}\,b) \end{vmatrix} = 0. \quad \text{(22)}$$

Similarly, from (11), we get

$$V(v) = A'H_e(-\alpha, \sqrt{k}\,v) + B'H_0(-\alpha, \sqrt{k}\,v) \quad \text{(23)}$$

and the boundary conditions (13) are satisfied if the determinant

$$\begin{vmatrix} H'_e(-\alpha, \sqrt{k}\,c) & H'_0(-\alpha, \sqrt{k}\,c) \\ H'_e(-\alpha, \sqrt{k}\,d) & H'_0(-\alpha, \sqrt{k}\,d) \end{vmatrix} = 0. \quad \text{(24)}$$

Equations (22) and (24) are the required characteristic equations in α and k as variables. The solutions of this simultaneous, transcendental system will determine the eigenvalues of our problem. If k is the eigenvalue of the problem, it must satisfy (22) and (24) simultaneously. In other words, we have to obtain α and k satisfying

$$\frac{H'_e(\alpha, \sqrt{k}\, a)}{H'_0(\alpha, \sqrt{k}\, a)} = \frac{H'_e(\alpha, \sqrt{k}\, b)}{H'_0(\alpha, \sqrt{k}\, b)} \quad \ldots \quad (25a)$$

$$\frac{H'_e(-\alpha, \sqrt{k}\, c)}{H'_0(-\alpha, \sqrt{k}\, c)} = \frac{H'_e(-\alpha, \sqrt{k}\, d)}{H'_0(-\alpha, \sqrt{k}\, d)}. \quad \ldots \quad (25b)$$

Suppose that k and α can be determined, and that the determinant is of rank one; that is not all elements vanish for a particular k and m. We must have then A and B be proportional to cofactors of the determinant of (22), and A', B' be proportional to cofactors of the determinant of (24). Hence, we take the solutions of (10) and (11) as

$$U(u) = H'_0(\alpha, \sqrt{k}\, a)H_e(\alpha, \sqrt{k}\, u) - H'_e(\alpha, \sqrt{k}\, a)H_0(\alpha, \sqrt{k}\, u) \quad \ldots \quad (26a)$$

$$V(v) = H'_0(-\alpha, \sqrt{k}\, c)H_e(-\alpha, \sqrt{k}\, v) - H'_e(-\alpha, \sqrt{k}\, c)H_0(-\alpha, \sqrt{k}\, v). \quad \ldots \quad (26b)$$

Therefore,

$$\psi(u, v) = [H'_0(\alpha, \sqrt{k}\, a)H_e(\alpha, \sqrt{k}\, u) - H'_e(\alpha, \sqrt{k}\, a)H_0(\alpha, \sqrt{k}\, u)] \times [H'_0(-\alpha, \sqrt{k}\, c)H_e(-\alpha, \sqrt{k}\, v) - H'_e(-\alpha, \sqrt{k}\, c)H_0(-\alpha, \sqrt{k}\, v)]. \quad (27)$$

In the case of cylinder bounded by two symmetrical confocal orthogonal parabolas one has to take proper care of the continuity along $u = 0$. To avoid discontinuities (Morse and Feshback 1953), one has to take the product $U(u)\ V(v)$ in the potential as the sum of two independent solutions

$$H_e(\alpha, \sqrt{k}\, u)H_e(-\alpha, \sqrt{k}\, v) \text{ and } H_0(\alpha, \sqrt{k}\, u)H_0(-\alpha, \sqrt{k}\, v).$$

The other product pair

$$H_e(\alpha, \sqrt{k}\, u)H_0(-\alpha, \sqrt{k}\, v) \text{ and } H_e(-\alpha, \sqrt{k}\, v)H_0(\alpha, \sqrt{k}\, u)$$

has to be discarded. But it does not occur in our case. Because, the continuity requirements are satisfied everywhere inside the cylinder since we are now concerned with a region free from branch cuts, whence all points in the region correspond to points lying in the same sheet of the Riemann surface.

6. Estimation of Eigenvalues

The problem reduces to one of solving eqns. (25a, b) for α and k. Exact solutions have been proved intractable. Rewriting (14) and putting $x_a = \sqrt{k}\ a$, $x_b = \sqrt{k}\ b$ in the boundary conditions yields

$$\begin{cases} \dfrac{d^2U}{dx^2} + (x^2+\alpha)U = 0 \\ U'(x_a) = 0 \\ U'(x_b) = 0 \end{cases} \quad \ldots \quad (28)$$

where $x = \sqrt{k}\, u$.

Solution of (28) comprises a typical eigenvalue problem. From the theory of differential equation we know there exists infinitely many discrete sets of values of α. Several well-known methods are available for estimating these values. Thus Rayleigh's minimum principle for determining an upper-bound of α, the integral equation method for determining a lower bound, the perturbation method for determining intermediate values and so forth. However, actual application of any of these methods rapidly increases in complexity as the order of the mode increases. Thus Rayleigh's method can be used to estimate higher eigenvalues if we properly restrict the comparison functions used, but in turn this restriction involves knowledge of lower eigenfunctions, which in general, are much more difficult to determine than the eigenvalues themselves.

If we write

$$M[U] \equiv -(f_1 U')' + f_0 U$$

$$N[U] \equiv g_0 U$$

where f_1, f_0 and g_0 are functions of x, M is a self-adjoint operator, then the eigenvalue problem is

$$M[U] = \lambda N[U]$$

$$U'(x_a) = U'(x_b) = 0.$$

Rayleigh's quotient is defined as

$$R[y] = \frac{\int_{x_a}^{x_b} y\, M[y]\, dx}{\int_{x_a}^{x_b} y\, N[y]\, dx}$$

where y is a comparison function which has the following properties:

(*a*) It is twice differentiable.

(*b*) It satisfies the boundary conditions.

(*c*) $\int_{x_a}^{x_b} y\, M[y]\, dx > 0$, $\int_{x_a}^{x_b} y\, N[y]\, dx > 0$

(*d*) $\int_{x_a}^{x_b} y\, N[U_i(x)]\, dx = 0$, $i = 1, 2, \ldots s$

where U_i is the ith eigenfunction of the problem. Then

$$\lambda_{s+1} \leqslant R[y].$$

In other words the minimum of $R[y]$ is the $(s+1)$th eigenvalue of the problem.

We observe that the determination of λ_{s+1} depends upon the determination of s lower eigenfunctions. To overcome this difficulty, Courant devised a method in which the minimum property of λ_{s+1} can be determined independently of the lower modes. We quote the following theorem (Courant and Hilbert 1953):

Theorem—Let $\omega_1, \omega_2, \ldots, \omega_s$ be s linearly independent integrable functions and let the comparison function y be so chosen that they are orthogonal to these s chosen functions, i.e.

$$\int_{x_a}^{x_b} y\omega_\sigma \, dx = 0, \quad \sigma = 1, 2, \ldots s.$$

The maximum of Rayleigh's quotient for these y has a value which also depends on the chosen functions $\omega_1, \omega_2, \ldots, \omega_s$. Call this M. Then

$$\min_y R[y] = M(\omega_1, \omega_2, \ldots, \omega_s).$$

Then it can be proved that

$$\lambda_{s+1} = \max_{\omega} M(\omega_1, \omega_2, \ldots, \omega_s).$$

This is known as maximum-minimum principle.

Applying this principle to (10) and boundary condition (12) we have

$$\left(\frac{i\pi}{b-a}\right)^2 - k^2b^2 < m_i < \left(\frac{i\pi}{b-a}\right)^2 - k^2a^2, \quad i = 0, 1, 2, 3. \qquad .. \quad (29)$$

Similarly, (11) and boundary condition (13) give

$$k^2c^2 - \left(\frac{j\pi}{d-c}\right)^2 \leqslant m_j < k^2d^2 - \left(\frac{j\pi}{d-c}\right)^2, \quad j = 0, 1, 2, 3. \qquad .. \quad (30)$$

Equations (29) and (30) give the approximate range of (m_{ij}, k_{ij}) numbered according to the following scheme:

Equation (29) defines a family of curves, plotting m as a function of k. The curve nearest k-axis is labelled $i = 0$, the next $i = 1$, and so on. The equation (30) produces another set of curves on the (m, k) plane, the lowest of which is called $j = 0$, the next $j = 1$, and so on. Whenever the two families of curves intersect defines an allowed value of m, designated m_{ij}, and an allowed value of k, designated k_{ij}.

If we put $a = c$, $b = d$, then the curves of (30) are the images of the curves of (29) in the k-axis, i.e. k-axis becomes the axis of symmetry. From the image symmetry of these two groups of characteristic curves, the following properties can be easily derived:

1. The ith curve of the first group intersects with the ith curve of the second group on the axis of symmetry, i.e. the k-axis. These modes are characterized by the fact that $m = 0$.

2. $$|m_{ij}| = |m_{ji}|, \quad k_{ij} = k_{ji}.$$

In other words the two different modes of oscillations U_{ij} and $U_{ji}(i \neq j)$ are degenerated modes which have the same cut-off frequencies. The values of (m_{ij}, k_{ij}) are plotted in Fig. 4. As k becomes larger and larger the area of uncertainty becomes proportionally larger, consequently little information can be obtained.

However, when k is small and i and j are very large the above inequalities do give a good estimate of m and k. Thus when $k = 0$

$$m_i = \left(\frac{i\pi}{b-a}\right)^2 \quad \ldots \quad (31)$$

$$m_j = -\left(\frac{j\pi}{d-c}\right)^2. \quad \ldots \quad (32)$$

These results are exact.

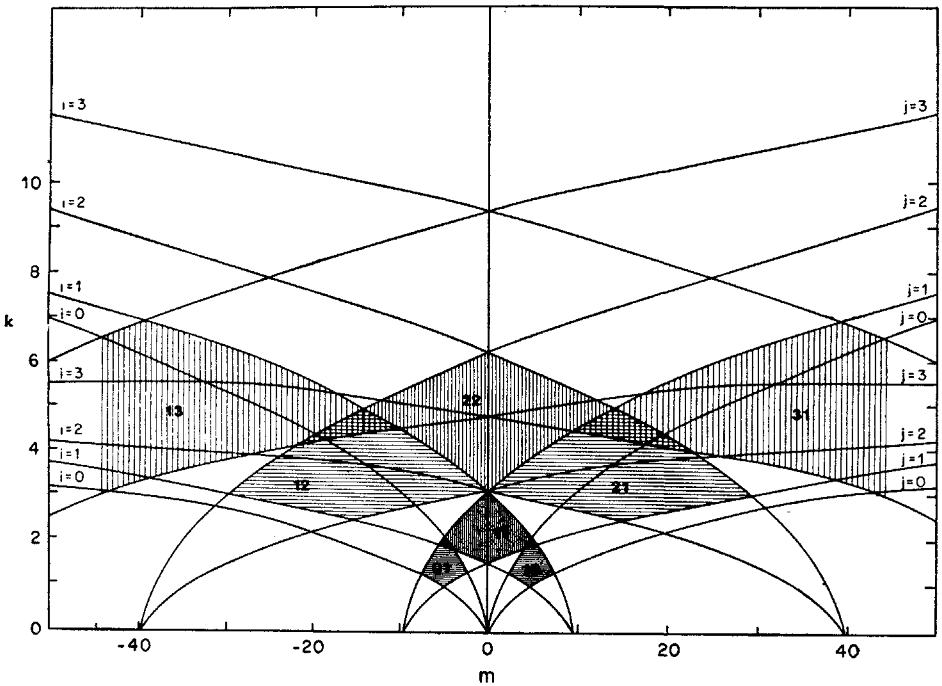

FIG. 4. Estimation of m and k by maximum-minimum principle.

To obtain some quantitative estimates, let us take

$$m_i = \left(\frac{i\pi}{b-a}\right)^2 - k^2\left(\frac{b^2+a^2}{2}\right) \quad \ldots \quad (33)$$

$$m_j = k^2\left(\frac{c^2+d^2}{2}\right) - \left(\frac{j\pi}{d-c}\right)^2. \quad \ldots \quad (34)$$

The solution of (33) and (34) gives

$$m_{ij} = \frac{1}{a^2+b^2+c^2+d^2}\left[\left(\frac{i\pi}{b-a}\right)^2(c^2+d^2) - \left(\frac{j\pi}{d-c}\right)^2(a^2+b^2)\right] \quad \ldots \quad (35)$$

$$k_{ij} = \pi\left[\frac{2\left(\frac{i}{b-a}\right)^2 + 2\left(\frac{j}{d-c}\right)^2}{a^2+b^2+c^2+d^2}\right]^{1/2}. \quad \ldots \quad (36)$$

In particular when the boundaries are symmetrical

$$b-a = d-c = l$$

$$a = c,$$

$$b = d.$$

Equations (35) and (36) reduce to

$$m_{ij} = \frac{\pi^2}{2l^2}(i^2-j^2), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

$$k_{ij} = \frac{\pi}{l}\left(\frac{i^2+j^2}{a^2+b^2}\right)^{1/2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Then when $i = j$ we have $m_{ij} = 0$. Further $|m_{ij}| = |m_{ji}|$ and $k_{ij} = k_{ji}$.

7. Physical Interpretation of Equations (35) and (36)

The physical interpretation of eqn. (36) is obvious. We notice from our Fig. 3 that when b and a, and d and c become large, while $b-a$ and $d-c$ remain small, thus

$$b-a \ll b \doteq a$$

and

$$d-c \ll d \doteq c$$

the four sides of the cylinder are practically straight and the cross-section approaches a rectangle. The linear lengths of the sides are

$\int_a^b \sqrt{u^2+v^2}\, du$ along $v = c$ and $v = d$, and $\int_c^d \sqrt{u^2+v^2}\, dv$ along $u = a$ and $u = b$.

Since $a \doteq b$, $c \doteq d$, the variation of u and v are small in the region considered. It is, therefore, reasonable to assume that

$$\sqrt{u^2+v^2} \doteq \sqrt{\frac{a^2+b^2}{2}+\frac{c^2+d^2}{2}} = \sqrt{\frac{a^2+b^2+c^2+d^2}{2}}.$$

Here the lengths of the sides of the approximate rectangle become

$$\sqrt{\frac{a^2+b^2+c^2+d^2}{2}}\,(b-a) \text{ and } \sqrt{\frac{a^2+b^2+c^2+d^2}{2}}\,(d-c).$$

The values of k of such a rectangular cylinder are well known (Benjamin and Ursell 1954), namely

$$k_{ij} = \pi\left[\frac{2i^2}{(a^2+b^2+c^2+d^2)(b-a)^2}+\frac{2j^2}{(a^2+b^2+c^2+d^2)(d-c)^2}\right]^{1/2}$$

which is exactly (36).

8. Exact Determination of Eigenvalues

We now sketch a theoretical method for determining the eigenvalues exactly. In what follows, we shall consider the symmetrical case by putting $a = c$ and $b = d$. Referring to (26a, b) we rewrite U and V as follows:

$$U(\alpha, \sqrt{k}\, u) = H_e(\alpha, \sqrt{k}\, u) - \frac{H'_e(\alpha, \sqrt{k}\, a)}{H'_0(\alpha, \sqrt{k}\, a)} H_0(\alpha, \sqrt{k}\, u), \quad \ldots \quad \ldots (39a)$$

$$V(-\alpha, \sqrt{k}\, v) = H_e(-\alpha, \sqrt{k}\, v) - \frac{H'_e(-\alpha, \sqrt{k}\, a)}{H'_0(-\alpha, \sqrt{k}\, a)} H_0(-\alpha, \sqrt{k}\, v). \quad \ldots (39b)$$

We see at once that the boundary conditions at $u = a$ and $v = a$ are satisfied for all values of α and k. To satisfy the other boundary conditions at $u = b$ and $v = b$ we must have

$$U'(\alpha, \sqrt{k}\, b) = 0$$

which gives

$$H'_e(\alpha, \sqrt{k}\, b)H'_0(\alpha, \sqrt{k}\, a) - H'_e(\alpha, \sqrt{k}\, a)H'_0(\alpha, \sqrt{k}\, b) = 0 \quad \ldots (40a)$$

and

$$V'(-\alpha, \sqrt{k}\, b) = 0$$

yielding

$$H'_e(-\alpha, \sqrt{k}\, b)H'_0(-\alpha, \sqrt{k}\, a) - H'_e(-\alpha, \sqrt{k}\, a)H'_0(-\alpha, \sqrt{k}\, b) = 0. \quad \ldots (40b)$$

It is useful to write the power series expansion in k for the left-hand sides of (40a) and (40b). Differentiation of (16a, b) and after some algebraic manipulations we obtain the power series expansion as

$$\begin{aligned} &H'_e(\alpha, \sqrt{k}\, b)H'_0(\alpha, \sqrt{k}\, a) - H'_e(\alpha, \sqrt{k}\, a)H'_0(\alpha, \sqrt{k}\, b) \\ &= (b-a)k^{3/2}\Big[-\alpha + \frac{k}{6}\{\alpha^2(b-a)^2 - 2(a^2+b^2-ab)\} - \frac{k^2}{120}\{\alpha^3(a^4+b^4 \\ &\quad +6a^2b^2-4a^3b-4ab^3) - 2\alpha(7a^4+7b^4+2a^2b^2-8a^3b-8ab^3)\} + \ldots\Big] \qquad (41a) \end{aligned}$$

and

$$\begin{aligned} &H'_e(-\alpha, \sqrt{k}\, b)H'_0(-\alpha, \sqrt{k}\, a) - H'_e(-\alpha, \sqrt{k}\, a)H'_0(-\alpha, \sqrt{k}\, b) \\ &= (b-a)k^{3/2}\Big[\alpha + \frac{k}{6}\{\alpha^2(b-a)^2 - 2(a^2+b^2-ab)\} - \frac{k^2}{120}\{-\alpha^3(a^4+b^4 \\ &\quad +6a^2b^2-4a^3b-4ab^3) + 2\alpha(7a^4+7b^4+2a^2b^2-8a^3b-8ab^3)\} + \ldots\Big]. \qquad (41b) \end{aligned}$$

The solutions of the simultaneous, transcendental system of eqns. (40a) and (40b) are obtained using the method outlined in Morse and Feshback (1953), as described below.

The equation $U'(\alpha, \sqrt{k}\, b) = 0$ defines a family of curves, plotting $(\sqrt{k}\, b)$ as a function of α. The curve nearest the α-axis is labelled $i = 0$, the next $i = 1$, and so on. Similarly the equation $V'(-\alpha, \sqrt{k}\, b) = 0$ produces another set of curves on the $(\alpha, \sqrt{k}\, b)$ plane, the lowest of which we call $j = 0$, the next $j = 1$, and so on. Wherever the two families of curves intersect defines an allowed value of α designated α_{ij}, and an allowed value of $(\sqrt{k}\, b)$ designated γ_{ij} as shown in Fig. 5 and numbered according to the scheme used before. As before $\gamma_{ij} = \gamma_{ji}$ and $\alpha_{ij} = -\alpha_{ji}$. The numerical calculation has been carried out for $b/a = 2$. The corresponding eigenfunctions are given as

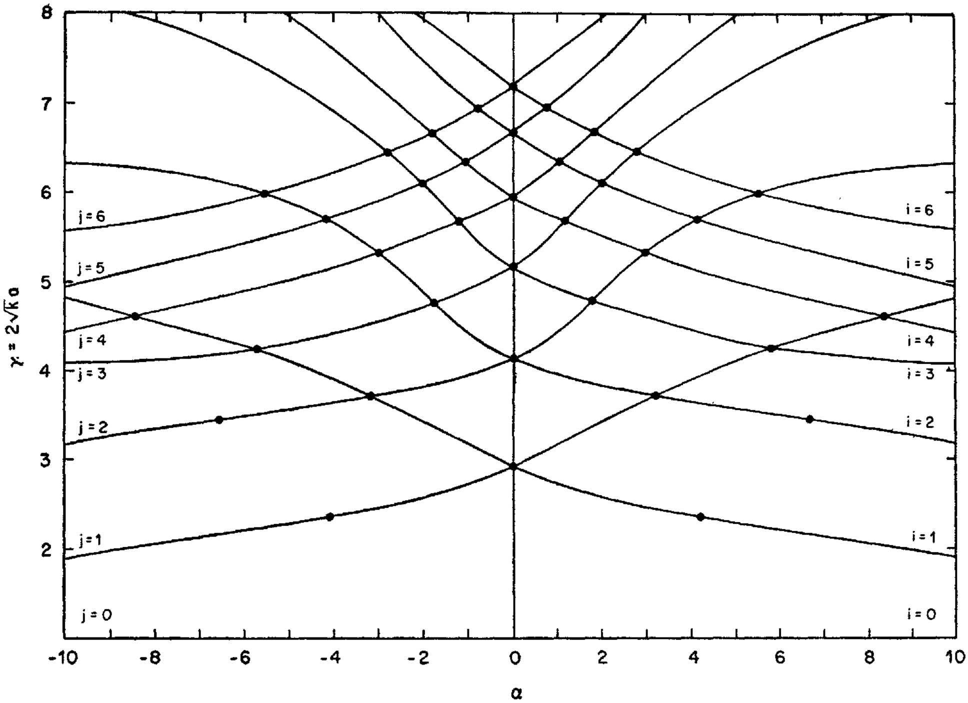

FIG. 5. Solutions of $U'(\alpha, \gamma) = 0$ and $V'(-\alpha, \gamma) = 0$.

$$\psi_{ij}(u, v) = U(\alpha_{ij}, \gamma_{ij}\, u/b)\, V(-\alpha_{ij}, \gamma_{ij}\, v/b), \quad i, j = 0, 1, 2, 3, \; \ldots \qquad (42)$$

where

$$U(\alpha_{ij}, \gamma_{ij}\, u/b) = H_e(\alpha_{ij}, \gamma_{ij}\, u/b)\, H'_0(\alpha_{ij}, \gamma_{ij}\, a/b)$$
$$-H'_e(\alpha_{ij}, \gamma_{ij}\, a/b)\, H_0(\alpha_{ij}, \gamma_{ij}\, u/b) \qquad (43a)$$

and

$$V(-\alpha_{ij}, \gamma_{ij}\, v/b) = H_e(-\alpha_{ij}, \gamma_{ij}\, v/b)\, H'_0(-\alpha_{ij}, \gamma_{ij}\, a/b)$$
$$-H'_e(-\alpha_{ij}, \gamma_{ij}\, a/b)\, H_0(-\alpha_{ij}, \gamma_{ij}\, v/b). \qquad (43b)$$

Some of the values of α_{ij}, γ_{ij}, $k_{ij}(=\gamma_{ij}^2/b^2)$ and $m_{ij}(=\alpha_{ij}k_{ij})$ are given in Table I.

9. ORTHOGONALITY OF THE EIGENFUNCTIONS

For any two eigenfunctions ψ_{mn}, ψ_{rs} with $\gamma_{mn} \neq \gamma_{rs}$ we have the orthogonality theorem

$$\int_R \int \psi_{mn} \psi_{rs}\, dR = 0 \qquad (44)$$

TABLE I

Some of the values of α, γ, k and m

i	j	α_{ij}	γ_{ij}	$k_{ij} = \gamma_{ij}^2/4a^2$	$m_{ij} = \alpha_{ij}k_{ij}$
0	0	0·00	0·00	0·00	0·00
1	1	0·00	2·92	$2{\cdot}13/a^2$	0·00
2	1	3·15	3·72	$3{\cdot}46/a^2$	$10{\cdot}90/a^2$
2	2	0·00	4·15	$4{\cdot}30/a^2$	0·00
3	1	5·80	4·25	$4{\cdot}51/a^2$	$26{\cdot}16/a^2$
3	2	1·80	4·77	$5{\cdot}69/a^2$	$10{\cdot}24/a^2$
3	3	0·00	5·17	$6{\cdot}68/a^2$	0·00
4	1	8·25	4·60	$5{\cdot}29/a^2$	$43{\cdot}64/a^2$
4	2	2·90	5·32	$7{\cdot}10/a^2$	$20{\cdot}59/a^2$
4	3	1·15	5·68	$8{\cdot}06/a^2$	$9{\cdot}27/a^2$
4	4	0·00	5·95	$8{\cdot}85/a^2$	0·00
5	2	4·15	5·70	$8{\cdot}12/a^2$	$33{\cdot}70/a^2$
5	3	2·05	6·10	$9{\cdot}30/a^2$	$19{\cdot}06/a^2$
5	4	1·10	6·35	$10{\cdot}08/a^2$	$11{\cdot}09/a^2$
5	5	0·00	6·66	$11{\cdot}09/a^2$	0·00
6	2	5·55	5·96	$8{\cdot}88/a^2$	$49{\cdot}28/a^2$
6	3	2·78	6·45	$10{\cdot}40/a^2$	$28{\cdot}91/a^2$
6	4	1·85	6·68	$11{\cdot}13/a^2$	$20{\cdot}59/a^2$
6	5	0·80	6·95	$12{\cdot}08/a^2$	$9{\cdot}66/a^2$
6	6	0·00	7·18	$12{\cdot}89/a^2$	0·00

as a result of Green's theorem. Let us set up a Lagrangian identity for the separated Laplace equation in two variables u, v. We arrive at the equation

$$\int_a^b \int_a^b (u^2+v^2)\psi_{mn}\psi_{rs}\,du\,dv = \frac{b^4}{\gamma_{mn}^4-\gamma_{rs}^4}\left\{\int_a^b \left[\psi_{mn}\frac{\partial\psi_{rs}}{\partial u} - \psi_{rs}\frac{\partial\psi_{mn}}{\partial u}\right]_a^b dv + \int_a^b \left[\psi_{mn}\frac{\partial\psi_{rs}}{\partial v} - \psi_{rs}\frac{\partial\psi_{mn}}{\partial v}\right]_a^b du\right\} \quad .. \quad (45)$$

where ψ_{mn} and ψ_{rs} are any two eigenfunctions given by (42), belonging to the eigenvalues γ_{mn} and γ_{rs} respectively. When $\gamma_{mn} \neq \gamma_{rs}$, the line integrals in (45) vanish because of the boundary conditions (40*a*, *b*) and we get

$$\int_a^b \int_a^b (u^2+v^2)\psi_{mn}\psi_{rs}\,du\,dv = 0. \quad .. \quad .. \quad .. \quad (46)$$

But this deduction fails if ψ_{mn}, ψ_{rs} belong to the same degenerate state, in other words if $\gamma_{mn} = \gamma_{rs}$ the corresponding expression for $\gamma_{mn} = \gamma_{rs}$ is obtained in a similar manner to that used for Bessel functions (Sneddon 1961) by putting $\gamma_{rs} = \gamma_{mn}+\epsilon$, where ϵ is small using Taylor's theorem and then letting ϵ tend to zero. The result is the same in (46), and in particular

$$\int_a^b \int_a^b (u^2+v^2)\psi_{mn}\psi_{nm}\,du\,dv = 0 \quad .. \quad .. \quad .. \quad (47)$$

for $\gamma_{mn} = \gamma_{nm}$.

10. Natural Frequencies and the Stability Equation

Because $\left(\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}\right)$ satisfies the boundary conditions (Benjamin and Ursell 1954), its expansion may be obtained from the expansion of ζ by term-by-term differentiation (Weinstein 1949). It follows from (1), (3) and (4) that the required expansions are

$$\phi(x, y, z, t) = \sum_{\substack{m=0\\ m\neq n=0}}^{\infty}\sum_{n=0}^{\infty} \frac{da_{mn}t}{dt}\cdot\psi_{mn}(u, v)\cosh\{\gamma_{mn}^2(z+h)/b^2\}+\frac{da_{00}t}{dt} \qquad (48)$$

where $a_{00}(t)$ is independent of (x, y, z).

$$\zeta(x, y, t) = -\sum_{\substack{m=0\\ m\neq n=0}}^{\infty}\sum_{n=0}^{\infty} a_{mn}(t)(\gamma_{mn}^2/b^2)\psi_{mn}(u, v)\sinh(\gamma_{mn}^2 h/b^2). \qquad .. \quad (49)$$

In writing (48) we make use of the result that

$$\alpha_{00} = 0,\ \gamma_{00} = 0 \text{ and } \psi_{00} = 1.$$

Since the total volume of fluid remains constant, $a_{00}(t)$ is constant, and the origin of ζ is so adjusted that $a_{00}(t) = 0$. The eigenfunctions which appear in (48) and (49) are given by (42). Substitution of (48) and (49) into (5) then shows that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{mn}(u, v)\cosh(\gamma_{mn}^2 h/b^2)\left[\frac{d^2a_{mn}}{dt^2}+\frac{\gamma_{mn}^2}{b^2}\tanh(\gamma_{mn}^2 h/b^2)\left\{\frac{\gamma}{\rho}\left(\frac{\gamma_{mn}^2}{b^2}\right)^2+g+\ddot{F}\right\}a_{mn}\right] = 0,\ m = n \neq 0 \qquad .. \quad .. \quad .. \quad (50)$$

and since the functions $\psi_{mn}(u, v)$ are linearly independent, the coefficients $a_{mn}(t)$ satisfy

$$\frac{d^2a_{mn}}{dt^2}+(\gamma_{mn}^2/b^2)\tanh(\gamma_{mn}^2 h/b^2)\left\{\frac{\gamma}{\rho}(\gamma_{mn}^2/b^2)^2+g+\ddot{F}\right\}a_{mm} = 0 \qquad .. \quad (51)$$

which can easily be reduced (Benjamin and Ursell 1954) to the standard form of Mathieu's equation adopted by McLachlan (1947).

The stability of the resulting surface wave system can be determined by considering the behaviour of the solutions of (51). The stability conditions have been discussed for periodic excitation of a cylinder of arbitrary cross-section by Benjamin and Ursell (1954) and for arbitrary excitation of a general cylinder by Lomen and Fontenot (1964). We shall not discuss the stability conditions here. To interested readers we refer to the work of Benjamin and Ursell (1954) and of Lomen and Fontenot (1964). If F is put equal to zero, (51) becomes an equation of simple harmonic motion relating to free vibrations of the liquid. Then the natural frequencies (1/period) of these vibrations are given by

$$\frac{\omega_{mn}}{2\pi} = \frac{1}{2\pi}(\gamma_{mn}/b)\left[\left\{g+\frac{\gamma}{\rho}(\gamma_{mn}^2/b^2)^2\right\}\tanh(\gamma_{mn}^2 h/b^2)\right]^{\frac{1}{2}}. \quad .. \quad (52)$$

Let us introduce a non-dimensional tank parameter δ defined by the relation

$$\delta = h/b^2. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (53)$$

Using this parameter eqn. (52) can be rewritten as

$$\frac{\omega_{mn}}{2\pi} = \frac{1}{2\pi}(\gamma_{mn}/b)\left[\left\{g+\frac{\gamma}{\rho}(\gamma_{mn}^2/b^2)^2\right\}\tanh(\gamma_{mn}^2\delta)\right]^{\frac{1}{2}}. \quad .. \quad (54)$$

For the purpose of simplicity we define ϵ_{mn} by the relation

$$\epsilon_{mn} = \omega_{mn}h^{\frac{1}{2}}\left[g+\frac{\gamma}{\rho}(\gamma_{mn}^2/b^2)\right]^{-\frac{1}{2}}. \quad .. \quad .. \quad .. \quad (55)$$

Substitution of the value of ω_{mn} from equation (54) into equation (55) yields

$$\epsilon_{mn} = [\gamma_{mn}^2\delta\tanh(\gamma_{mn}^2\delta)]^{\frac{1}{2}}. \quad .. \quad .. \quad .. \quad (56)$$

The values of ϵ_{mn} are plotted as a function of the tank parameter δ in Figs. 6, 7 and 8 for various values of γ_{mn}.

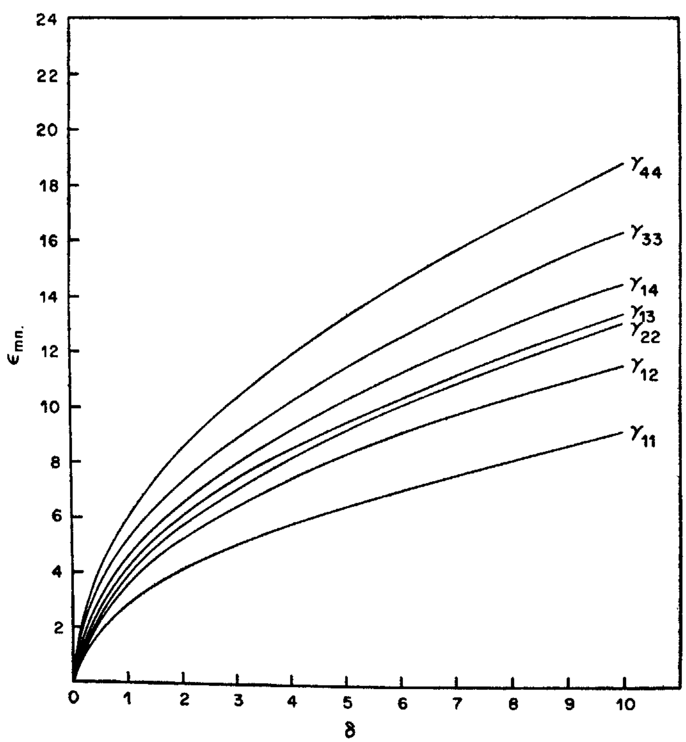

FIG. 6. Plot of natural frequencies versus the tank parameter δ.

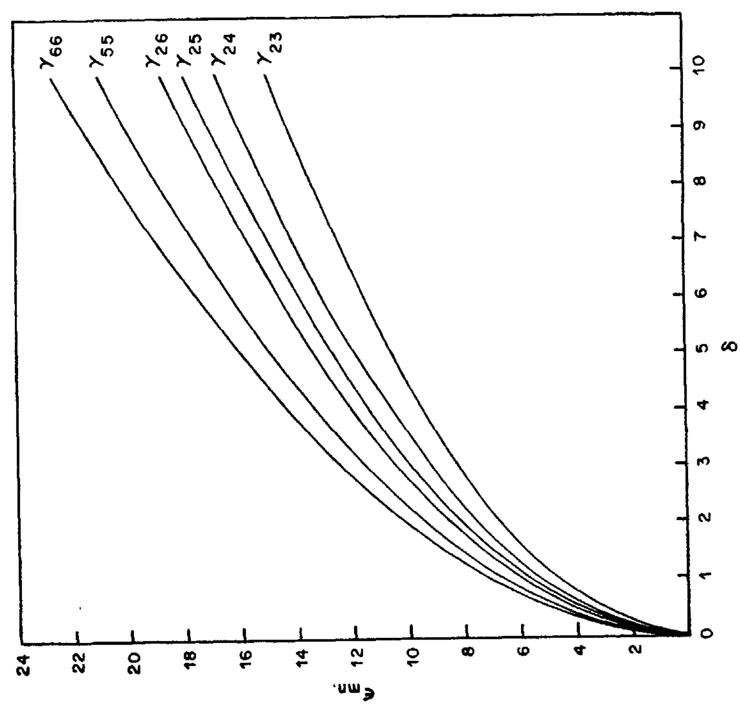

FIG. 7. Plot of natural frequencies versus the tank parameter δ.

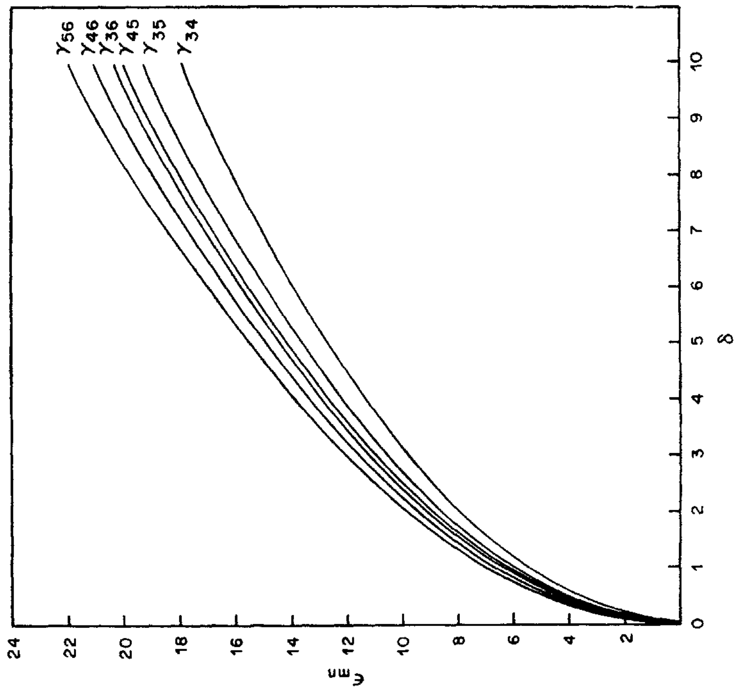

FIG. 8. Plot of natural frequencies versus the tank parameter δ.

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