ON THE LAMINAR MOTION OF A MAXWELL FLUID NEAR AN OSCILLATING POROUS INFINITE PLANE

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The present paper considers the problems associated with the flow of linear viscoelastic fluids of the Maxwell type near a porous oscillating plane (known as Stokes' problems). Effects of the fluid's relaxation and the mass transfer parameters on their behaviour are sought.

Notations

ρ = density of the fluid
ν = kinematic viscosity
λ_0 = stress relaxation time
t = time
u = component of velocity parallel to the plate
$\bar{u}$ = dimensionless velocity $\left(=\frac{u}{U_0}\right)$
U_0 = velocity amplitude of oscillating plate
v = component of velocity normal to the plate
v_0 = velocity of normal mass transfer through the porous plate (negative for suction and positive for blowing)
x = distance parallel to the plate
y = distance normal to the plate
η = dimensionless distance $\left(=y\sqrt{\frac{\omega}{\nu}}\right)$
R = dimensionless mass transfer parameter $\left(=\frac{v_0}{\sqrt{\omega\nu}}\right)$, positive for suction and negative for blowing
λ = dimensionless relaxation parameter $(=\lambda_0\omega)$
T = dimensionless time $(=\omega t)$
ω = frequency of wall oscillation

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Introduction

The laminar flow of a fluid near a porous oscillating plate is of interest for two reasons: Firstly, by virtue of its being one of those special cases for which the equations of motion of the fluid yield to exact analysis; secondly, and perhaps more important, the decay of the amplitude of the oscillations with distance from the plate, and the effects of the fluid's relaxation and mass transfer on this decay, gives a quantitative basis from which the effects of relaxation and mass transfer on the turbulent boundary layer can be determined.

It is appropriate here to observe that Lighthill (1954) studied the problem of the Newtonian flow near a stationary plate for an oscillating external flow; his result was extended by Stuart (1955) to include the effect of constant inward mass transfer (suction). In turn, Stuart's result was extended by Watson (1958) to encompass any arbitrary unsteady external flow. It is important to note that these workers concluded that the valid solutions to the Navier-Stokes equations were admitted only for the case of mass transfer into the wall. No such limitation appears to exist for the present problem.

Na and Sidhom (1967) have obtained some interesting results for the flow of viscoelastic fluids of the Maxwell type near an accelerating or oscillating plate. In the present paper we have considered the above problem when the plate is porous. It will be assumed that the fluid properties are uniform in time and space; that the flow is laminar, that the surface is smooth, and that the 'no slip' condition exists at the surface, i.e. the fluid in the immediate vicinity of the surface moves with the surface. The pressure gradient is assumed to be zero and the fluid motion a consequence of the shear stress only. The questions to be answered are: how does the disturbance affect the motion of the surrounding fluid and how is this motion influenced by fluid's relaxation time and mass transfer through the surface?

Basic Equation and its Solution

For a linear, isotropic, viscoelastic fluid, the stress tensor is given by (Fredrickson 1964)

$$S^{ij} = -p\, g^{ij} + p^{ij} \qquad \ldots \qquad (1)$$

where p is the static pressure, g^{ij} is the associated metric tensor, and p^{ij} is a tensor usually related to the rate of strain, e^{ij}, by the 'equation of state':

$$P' p^{ij} = 2Q' e^{ij}, \qquad \ldots \qquad (2)$$

P' and Q' are two operators defined by

$$P' = 1 + \lambda_0 \frac{d}{dt} + \lambda_1 \frac{d^2}{dt^2} + \ldots + \lambda_n \frac{d^{n+1}}{dt^{n+1}} \qquad \ldots \qquad (3)$$

and

$$Q' = \mu\left(1+S_0\frac{d}{dt}+S_1\frac{d^2}{dt^2}+\ldots+S_m\frac{d^{m+1}}{dt^{m+1}}\right). \qquad (4)$$

The quantity μ in eqn. (4) is the viscosity of material at zero rate of shear, $\lambda_0, \lambda_1, \ldots, \lambda_n, S_0, S_1, \ldots, S_m$ are considered as physical constants, and d/dt is the convected derivative.

By definition, the type of fluids known as Maxwell fluids obeys the following equation of state:

$$\left(1+\lambda_0\frac{d}{dt}\right)p^{ij} = 2\mu e^{ij}. \qquad (5)$$

Equation (5) is a special case of eqn. (2). The physical significance of λ_0 (the relaxation time) is that, if the motion suddenly stops, the shear stress will decay as $\exp\left(-\frac{t}{\lambda_0}\right)$.

For unsteady parallel flow of Maxwell fluids over an infinite porous flat plate, the momentum equation can be written as

$$\rho\left(\frac{\partial u}{\partial t}-v_0\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(p_{xy}) \qquad (6)$$

where p_{xy} is the physical component of p^{ij}, defined by

$$\left(1+\lambda_0\frac{\partial}{\partial t}\right)p_{xy} = \mu\frac{\partial u}{\partial y}. \qquad (7)$$

In eqn. (6), the y-component of velocity, which as a consequence of continuity, is constant, has been given its value at the surface—v_0 (suction). Eliminating p_{xy} between (6) and (7), the momentum equation becomes

$$\frac{\partial u}{\partial t}-v_0\frac{\partial u}{\partial y}+\lambda_0\frac{\partial^2 u}{\partial t^2}-\lambda_0 v_0\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) = \nu\frac{\partial^2 u}{\partial y^2}. \qquad (8)$$

The boundary conditions corresponding to the present problem are

$$u(0, t) = U_0 e^{i\omega t} \qquad (9)$$

$$\lim_{y\to\infty} u(y, t) = 0 \qquad (10)$$

in which only real parts have physical meaning. The initial quiescent state of the fluid may be expressed by

$$u(y, 0) = 0. \qquad (11)$$

These equations may be non-dimensionalized by the introduction of the following dimensionless variables:

$$\bar{u} = \frac{u}{U_0}$$

$$\eta = y\sqrt{\frac{\omega}{\nu}}$$

$$T = \omega t$$

and the mass transfer parameter (positive for suction and negative for blowing)

$$R = \frac{v_0}{\sqrt{\omega\nu}}.$$

The equation of motion can now be written as

$$\frac{\partial \bar{u}}{\partial T} - R\frac{\partial \bar{u}}{\partial \eta} + \lambda\frac{\partial^2 \bar{u}}{\partial T^2} - \lambda R\frac{\partial}{\partial T}\left(\frac{\partial \bar{u}}{\partial \eta}\right) = \frac{\partial^2 \bar{u}}{\partial \eta^2} \quad \ldots \quad (12)$$

where $\lambda = \lambda_0\omega$ is the dimensionless relaxation parameter.
The corresponding boundary conditions become

$$\bar{u}(0, T) = e^{iT} \quad \ldots \quad (13)$$

$$\lim_{\eta \to \infty} \bar{u}(\eta, T) = 0 \quad \ldots \quad (14)$$

the initial condition being

$$\bar{u}(\eta, 0) = 0. \quad \ldots \quad (15)$$

The solution of (12) with the above boundary conditions is

$$\bar{u} = \exp\left[-\frac{R\sqrt{2}+s_1}{2\sqrt{2}}\eta\right]\cos\left[T - \frac{\sqrt{2}\lambda R s_1 + s_2}{2\sqrt{2}s_1}\eta\right] \quad \ldots \quad (16)$$

in which only the real part has been taken and

$$s_1 = \sqrt{r + R^2 - \lambda^2 R^2 - 4\lambda}$$

$$s_2 = 2(2 + \lambda R^2)$$

$$r = \sqrt{(R^2 + \lambda^2 R^2 + 4\lambda)^2 + 16}.$$

The steady state solution is obtained by formally letting T become very large. In the limiting case of zero mass transfer ($R = 0$), the solution reduces to the result of Na and Sidhom (1967) and, when $\lambda = 0$, eqn. (16) is reduced to the Newtonian flow solution (Nicoll *et al.* 1968). In other words, the parameter λ indicates the deviation of fluids from Newtonian behaviour. Figs. (1) and (2) represent the velocity profiles for various values of λ and R at $T = \frac{\pi}{2}$.

The main features of the general solution are as expected. The fluid, initially at rest, oscillates harmonically in the x-direction.

The velocity profile $\bar{u}(\eta, T)$ has the form of damped harmonic oscillation, the amplitude of which is $\exp\left(-\frac{R\sqrt{2}+s_1}{2\sqrt{2}}\eta\right)$, in which a fluid layer at a distance y has a phase lag $\left(\frac{\lambda R}{2} + \frac{s_2}{2\sqrt{2}s_1}\right)\eta$ with respect to the motion of the wall.

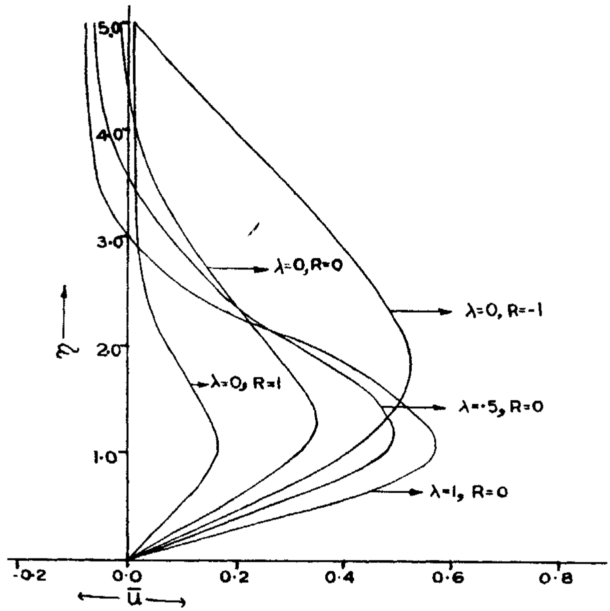

FIG. 1. Velocity profiles of Newtonian and Maxwell fluids near an oscillating porous plate for $T = \frac{\pi}{2}$.

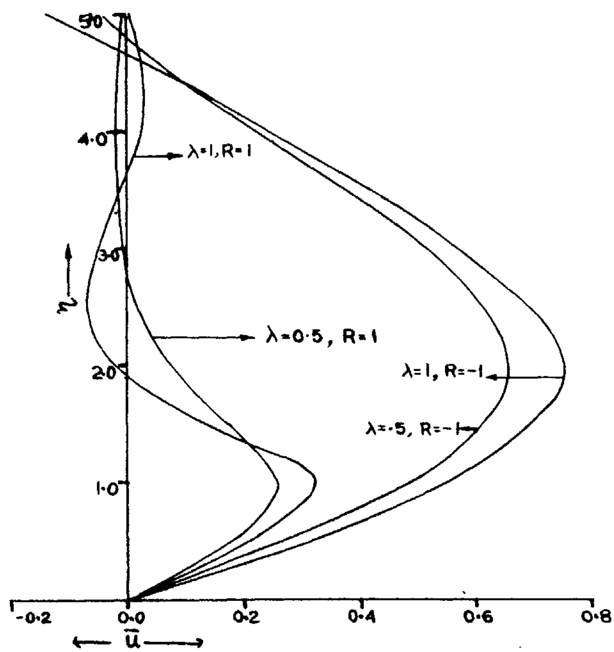

FIG. 2. Velocity profiles of a Maxwell fluid near an oscillating porous plate for $T = \frac{\pi}{2}$.

Tables I and II show that the amplitude of velocity and the phase lag increase with the increase of λ, but the former decreases with the increase of R and the latter increases with R. Furthermore, since the shortest distance between two fluid layers oscillating in phase, often referred to as the 'depth of penetration' and sometimes considered as the wavelength of motion, is defined as

$$l = \frac{2\pi\sqrt{\frac{\nu}{\omega}}}{B}. \qquad (17)$$

TABLE I

Values of $A = \frac{R\sqrt{2}+s_1}{2\sqrt{2}}$ *for various values of* λ *and* R

	Blowing			Suction	
λ \ R	−2·0	−1·0	0·0	1·0	2·0
0·5	0·0643	0·1987	0·5558	1·1987	2·0643
1·0	0·0399	0·1336	0·4554	1·1336	2·0399
1·5	0·0258	0·0944	0·3889	1·0944	2·0258

TABLE II

Values of $B = \frac{\sqrt{2}\lambda R s_1+s_2}{2\sqrt{2}s_1}$ *for various values of* λ *and* R

	Blowing			Suction	
λ \ R	−2·0	−1·0	0·0	1·0	2·0
0·5	0·4395	0·6447	0·8997	1·1447	1·4395
1·0	0·4428	0·6839	1·0981	1·6839	2·4428
1·5	0·4502	0·7225	1·2858	2·2225	3·4502

From Table II and the expression (17) we find that the wavelength l decreases as λ increases and the mass transfer parameter R will further decrease it. This influence of λ and R is also observed in Figs. 1 and 2. Finally, the skin friction on the plate can be written as

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$= \mu U_0\sqrt{\frac{\omega}{\nu}}\left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0}.$$

Hence

$$\frac{\tau_w}{\mu U_0\sqrt{\frac{\omega}{\nu}}} = -\left(\frac{R\sqrt{2}+s_1}{2\sqrt{2}}\right)\cos T + \left(\frac{\lambda R}{2}+\frac{s_2}{2\sqrt{2}s_1}\right)\sin T$$

$$= -R_1 \cos(T+\phi) \qquad \ldots \ldots \ldots \ldots \ldots \quad (18)$$

where

$$R_1^2 = \left(\frac{R\sqrt{2}+s_1}{2\sqrt{2}}\right)^2 + \left(\frac{\lambda R}{2}+\frac{s_2}{2\sqrt{2}s_1}\right)^2$$

$$\tan\phi = \frac{\left(\frac{\lambda R}{2}+\frac{s_2}{2\sqrt{2}s_1}\right)}{\left(\frac{R\sqrt{2}+s_1}{2\sqrt{2}}\right)}.$$

Tables III and IV show that both the phase lead with respect to the plate motion and the amplitude of skin friction increases with the increase of λ, but the former decreases with the increase of R and the latter increases with R. An increase in skin friction has also been predicted by Beard and Walters (1964) for the case of boundary layer flow of viscoelastic fluids near a stagnation point.

TABLE III

Values of R_1^2 for various values of λ and R

	Blowing			Suction	
λ \ R	−2·0	−1·0	0·0	1·0	2·0
0·5	0·1970	0·4550	1·1183	2·7471	6·3331
1·0	0·1977	0·4855	1·4131	4·1160	10·1283
1·5	0·2032	0·5309	1·8044	6·1372	16·0076

TABLE IV

Values of $\tan\phi$ for various values of λ and R

	Blowing			Suction	
λ \ R	−2·0	−1·0	0·0	1·0	2·0
0·5	6·8336	3·2446	1·6187	0·9549	0·6972
1·0	11·0970	5·1075	2·4113	1·4854	1·1975
1·5	17·4496	7·6324	3·3062	2·0307	1·7031

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