ON THE EXISTENCE OF AFFINE MOTION IN A $HR-F_n$

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Recurrent Finsler spaces have been studied by Moór (1963), Sen (1968) and Misra (*in press*). On the pattern of Takano's (1961) paper, Sinha (1969) made an attempt to study the affine motion in such spaces and derived some formulae concerning the existence of such a motion. However, he could not proceed further in the light of Takano's (1966*a*) theory. In the present paper we have developed the theory of Takano (1966*a, b*) to Finsler geometry. The notation employed here is exactly based on the previous papers, viz. Misra and Pandey (1970), Misra and Meher (1971) and Misra (*in press*).

1. Introduction

Let F_n be an n-dimensional Finsler space whose metric function $F(x^i, \dot{x}^i)$,* $i, j, k, \ldots = 1, 2, \ldots, n$, satisfies the requisite conditions. The covariant derivative (in the sense of Berwald) $\mathfrak{B}_k X^i$ of a vector $\mathbf{X}$ with respect to x^k is defined by

$$\mathfrak{B}_k X^i = \partial_k X^i - (\dot{\partial}_j X^i) G^j_k + X^j G^i_{jk}, \quad (\partial_k = \partial/\partial x^k,\ \dot{\partial}_j = \partial/\partial \dot{x}^j), \quad \ldots \quad (1.1)$$

where the functions G^i_{jk} are the connection parameters of Berwald. These functions are positively homogeneous of degree zero in the variables $\dot{x}^i$'s and are symmetric in their lower indices. Also they satisfy

$$G^i_{jk} \dot{x}^k = G^i_j, \quad G^i_{hjk} \dot{x}^h = 0, \quad \ldots \quad \ldots \quad \ldots \quad (1.2)$$

where $G^i_{hjk} \overset{\text{def}}{=} \dot{\partial}_h G^i_{jk}$. The entities G^i_{hjk} are also symmetric in their lower indices and constitute a tensor field.

The commutation formula

$$2\mathfrak{B}_{[j} \mathfrak{B}_{k]} X^i = H^i_{jkh} X^h - (\dot{\partial}_r X^i) H^r_{jk} \dagger \quad \ldots \quad \ldots \quad \ldots \quad (1.3)$$

* Henceforward all the geometric objects used here will be considered as functions of line-element $(x^i, \dot{x}^i)$ unless stated otherwise and, for simplicity, will be written without assigning the line-element.

† Square brackets denote the skew-symmetric part with respect to the indices enclosed within them.

gives rise to the curvature tensor

$$H^i_{jkh} \stackrel{\text{def}}{=} 2\left\{\partial_{[j}G^i_{k]h}+G^i_{rh[j}G^r_{k]}+G^i_{r[j}G^r_{k]h}\right\} \quad \ldots \quad (1.4)$$

together with

$$H^i_{jkh}\dot{x}^h = H^i_{jk}. \quad \ldots \quad (1.5)$$

The space whose curvature tensor **H** satisfies

$$\mathcal{B}_l H^i_{jkh} = \lambda_l H^i_{jkh} \quad \ldots \quad (1.6)$$

for some non-zero vector $\boldsymbol{\lambda}$ is called a recurrent (in the sense of Berwald) Finsler space and will be denoted by $HR-F_n$. It is seen (Misra *in press*) that the tensor-field H^i_{jk} is also recurrent in a $HR-F_n$:

$$\mathcal{B}_l H^i_{jk} = \lambda_l H^i_{jk}. \quad \ldots \quad (1.7)$$

The infinitesimal transformation

$$\bar{x}^i = x^i+\epsilon v^i, \quad \ldots \quad (1.8)$$

where ϵ is an infinitesimal constant and v^i are the components of a vector independent of the directional arguments, gives rise to the Lie differentiation. The Lie derivatives $\pounds X^i$ and $\pounds G^i_{kh}$ of a vector-field and connection parameters respectively are given by

$$\pounds X^i = v^j\mathcal{B}_j X^i - X^j\mathcal{B}_j v^i + (\dot{\partial}_j X^i)\mathcal{B}_k v^j \dot{x}^k, \quad \ldots \quad (1.9)$$

$$\pounds G^i_{kh} = \mathcal{B}_k\mathcal{B}_h v^i + v^j H^i_{jkh} + G^i_{jkh}\mathcal{B}_l v^j \dot{x}^l. \quad \ldots \quad (1.10)$$

It is known (Yano 1957) that the vanishing of $\pounds G^i_{kh}$ is a necessary and sufficient condition for the transformation (1.8) to be an affine motion and further it implies the vanishing of $\pounds H^i_{jkh}$. Sinha (1969) has shown that a $HR-F_n$ admitting an affine motion (to be denoted by $AHR-F_n$) not only satisfies

$$\text{(i) } \pounds G^i_{kh} = 0, \quad \text{(ii) } \pounds H^i_{jkh} = 0 \quad \ldots \quad (1.11)$$

but also

$$\pounds \lambda_l = 0. \quad \ldots \quad (1.12)$$

2. Cases when £H Vanishes

Noting (1.6) the Lie derivative of the curvature tensor **H** in a $HR-F_n$ may be written by means of the formula (1.9):

$$\pounds H^i_{jkh} = LH^i_{jkh} - H^r_{jkh}\mathcal{B}_r v^i + H^i_{rkh}\mathcal{B}_j v^r + H^i_{jrh}\mathcal{B}_k v^r + H^i_{jkr}\mathcal{B}_h v^r + (\dot{\partial}_r H^i_{jkh})\mathcal{B}_s v^r \dot{x}^s, \quad \ldots \quad (2.1)$$

where we have put

$$L = \lambda_l v^l. \quad \ldots \quad (2.2)$$

Next, applying the commutation formula (1.3) for the curvature tensor **H** and noting (1.6), we observe that the relation

$$2(\mathfrak{B}_{[l}\lambda_{m]})H^i_{jkh} = H^i_{lmr}H^r_{jkh} - H^r_{lmj}H^i_{rkh} - H^r_{lmk}H^i_{jrh} - H^r_{lmh}H^i_{jkr} - H^r_{lms}\dot{x}^s\dot{\partial}_r H^i_{jkh} \quad \ldots \quad (2.3)$$

holds in a $HR-F_n$. The entities $2\mathfrak{B}_{[l}\lambda_{m]}$ constitute a skew-symmetric covariant tensor-field which, in general, is a non-vanishing one. This tensor plays an important role in our discussion and will be discussed in detail in the next section. Thus, for simplicity, it is desirable to put

$$A_{lm} \overset{\text{def}}{=} 2\mathfrak{B}_{[l}\lambda_{m]}. \quad \ldots \quad (2.4)$$

Now we propose a problem whether there exists a skew-symmetric tensor-field f^{jk} satisfying

$$\mathfrak{B}_h v^i = f^{jk}H^i_{jkh} \quad \ldots \quad (2.5)$$

in a $HR-F_n$. Thus, if we accept this proposition, transvecting (2.3) by f^{lm} and noting (2.5) we derive the relation

$$f^{lm}A_{lm}H^i_{jkh} = H^r_{jkh}\mathfrak{B}_r v^i - H^i_{rkh}\mathfrak{B}_j v^r - H^i_{jrh}\mathfrak{B}_k v^r - H^i_{jkr}\mathfrak{B}_h v^r - (\dot{\partial}_r H^i_{jkh})\mathfrak{B}_s v^r \dot{x}^s. \quad (2.6)$$

Comparing (2.1) and (2.6) we thus find that the proposition has led us to consider

$$\pounds H^i_{jkh} = (L - f^{lm}A_{lm})H^i_{jkh}. \quad \ldots \quad (2.7)$$

Therefore, in a $HR-F_n$ the vanishing of Lie derivative of its curvature tensor is implied by

$$L = f^{lm}A_{lm}. \quad \ldots \quad (2.8)$$

These conclusions give rise to the following :

Theorem 2.1—There exists a skew-symmetric tensor-field f^{jk} satisfying (2.5) and (2.8) if and only if the curvature tensor of a $HR-F_n$ is a Lie-invariant.

PROOF: The condition is clearly necessary and follows immediately from (2.7) and (2.8). Conversely when

$$\pounds H^i_{jkh} = 0 \quad \ldots \quad (2.9)$$

holds in a $HR-F_n$ we have (2.8) together with

$$LH^i_{jkh} = f^{lm}A_{lm}H^i_{jkh}. \quad \ldots \quad (2.10)$$

For (2.9), eqn. (2.1) determines

$$LH^i_{jkh} = H^r_{jkh}\mathfrak{B}_r v^i - H^i_{rkh}\mathfrak{B}_j v^r - H^i_{jrh}\mathfrak{B}_k v^r - H^i_{jkr}\mathfrak{B}_h v^r - (\dot{\partial}_r H^i_{jkh})\mathfrak{B}_s v^r \dot{x}^s. \quad (2.11)$$

Moreover $A_{lm}H^i_{jkh}$ is given by (2.3) so that $f^{lm}A_{lm}H^i_{jkh}$ may be derived from (2.3) by transvecting it with f^{lm}:

$$f^{lm}A_{lm}H^i_{jkh} = H^r_{jkh}f^{lm}H^i_{lmr} - H^i_{rkh}f^{lm}H^r_{lmj} - H^i_{jrh}f^{lm}H^r_{lmk} - H^i_{jkr}f^{lm}H^r_{lmh} - (\dot{\partial}_r H^i_{jkh})f^{lm}H^r_{lms}\dot{x}^s. \quad \ldots \quad (2.12)$$

Finally, substituting from (2.11) and (2.12) in (2.10), we obtain

$$H^r_{jkh}(\mathcal{B}_r v^i - f^{lm} H^i_{lmr}) - H^i_{rkh}(\mathcal{B}_j v^r - f^{lm} H^r_{lmj}) - H^i_{jrh}(\mathcal{B}_k v^r - f^{lm} H^r_{lmk})$$
$$- H^i_{jkr}(\mathcal{B}_h v^r - f^{lm} H^r_{lmh}) - (\dot{\partial}_r H^i_{jkh})(\mathcal{B}_s v^r - f^{lm} H^r_{lms})\dot{x}^s = 0$$

from which we have (2.5). This proves the sufficiency of the condition.

The vectors $\boldsymbol{\lambda}$ and $\mathbf{v}$ being non-zero imply that the scalar function L is, in general, non-zero. Particularly if the recurrence vector $\boldsymbol{\lambda}$ is a gradient one and independent of the directional arguments, i.e. if there exists a scalar function λ such that

$$\lambda_m = \partial_m \lambda = \mathcal{B}_m \lambda \quad .. \quad .. \quad .. \quad .. \quad (2.13)$$

we may easily conclude from (2.4) that $A_{lm} = 0$. Consequently eqn. (2.7) reduces to

$$\pounds H^i_{jkh} = L H^i_{jkh}. \quad .. \quad .. \quad .. \quad .. \quad (2.14)$$

Thus we may propose the following :

Theorem 2.2—If any two of the following conditions hold in a $HR-F_n$ the remaining also holds there: (i) the recurrence vector $\boldsymbol{\lambda}$ satisfies (2.13), (ii) the curvature tensor $\mathbf{H}$ is Lie-invariant and (iii) the function L vanishes.

PROOF: When the first two conditions hold it is easy to see that the third also follows from (2.13) and (2.14). Similarly when the first and last conditions hold the second may also be seen to hold. But when the last two conditions hold the first one does not directly follow from the above equations. To establish this we proceed as follows.

In view of the previous theorem it follows from (2.8) that the conditions (ii) and (iii) imply that

$$A_{lm} = 0 \quad .. \quad .. \quad .. \quad .. \quad (2.15)$$

holds. Moreover, for $L = 0$, it follows from (2.2) that the recurrence vector $\boldsymbol{\lambda}$ is orthogonal to the vector $\mathbf{v}$. Therefore, the vector $\boldsymbol{\lambda}$ is independent of the directional arguments as is the case with $\mathbf{v}$. Consequently it follows from (2.15) that $\boldsymbol{\lambda}$ is a gradient vector and so we have (2.13). This establishes the theorem.

Now multiplication of (2.1) by A_{lm} and an application of (2.3) give

$$A_{lm}\pounds H^i_{jkh} = H^r_{jkh}(L H^i_{lmr} - A_{lm}\mathcal{B}_r v^i) - H^i_{rkh}(L H^r_{lmj} - A_{lm}\mathcal{B}_j v^r)$$
$$- H^i_{jrh}(L H^r_{lmk} - A_{lm}\mathcal{B}_k v^r) - H^i_{jkr}(L H^r_{lmh} - A_{lm}\mathcal{B}_h v^r)$$
$$- (\dot{\partial}_r H^i_{jkh})(L H^r_{lms} - A_{lm}\mathcal{B}_s v^r)\dot{x}^s.$$

From this relation it follows that the vanishing of the Lie derivative of the curvature tensor $\mathbf{H}$ is implied by the relation

$$L H^i_{jkh} = A_{jk}\mathcal{B}_h v^i. \quad .. \quad .. \quad .. \quad .. \quad (2.16)$$

Thus we may conclude the following :

Theorem 2.3—The curvature tensor of a $HR-F_n$ which admits (2.16) is a Lie-invariant.

Applying (1.9) for the Lie derivative of the recurrence vector $\boldsymbol{\lambda}$ we get, in view of (2.2) and (2.4),

$$\pounds\lambda_m = v^l A_{lm} + \mathcal{B}_m L + (\dot\partial_l \lambda_m)\mathcal{B}_r v^l \dot x^r.$$

It has been seen (Sinha 1969) that $\pounds\lambda_m$ vanishes in an $AHR-F_n$. Thus we have

$$v^l A_{lm} + \mathcal{B}_m L + (\dot\partial_l \lambda_m)\mathcal{B}_r v^l \dot x^r = 0. \qquad \ldots \quad \ldots \quad \ldots (2.17)$$

If we introduce a vector η such that

$$\eta^m\{\mathcal{B}_m L + (\dot\partial_l \lambda_m)\mathcal{B}_r v^l \dot x^r\} = L \qquad \ldots \quad \ldots \quad \ldots (2.18)$$

we have, from (2.17), the relation

$$v^l \eta^m A_{lm} = -L. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (2.19)$$

Multiplying (2.16) by $v^j \eta^k$ and using (2.19) we then obtain

$$\mathcal{B}_h v^i = -H^i_{jkh} v^j \eta^k = H^i_{jkh} v^k \eta^j. \qquad \ldots \quad \ldots \quad \ldots (2.20)$$

Thus we summarize the following:

Theorem 2.4—If a $HR-F_n$ admits an affine motion there exists a vector $\boldsymbol{\eta}$ satisfying (2.20).

3. Some Properties of A_{lm}

In view of (1.11) (i) it may be seen from the commutation formula

$$(\pounds\mathcal{B}_k - \mathcal{B}_k\pounds)X^i = X^j \pounds G^i_{jk} - (\dot\partial_j X^i)\pounds G^j_{kh}\dot x^h \qquad \ldots \quad \ldots (3.1)$$

that the Lie differentiation commutes over the covariant differentiation. Consequently in an $AHR-F_n$, where (1.12) also holds, we have

$$\pounds\mathcal{B}_l \lambda_m = \mathcal{B}_l \pounds\lambda_m = 0$$

and hence

$$\pounds A_{lm} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots (3.2)$$

Thus as in the case of Takano (1966*a*) we have the following:

Theorem 3.1—The tensor-field A_{lm} is a Lie-invariant in an $AHR-F_n$ space.

Transvecting (2.3) by $\dot x^h$ and using (1.5), (2.4) and the homogeneity properties of the curvature tensor $\mathbf{H}$ we get*

$$A_{lm} H^i_{jk} = H^i_{lmr} H^r_{jk} - H^r_{lmj} H^i_{rk} - H^r_{lmk} H^i_{jr} - H^r_{lm} H^i_{jkr}. \qquad \ldots (3.3)$$

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* The tensor-fields H^i_{jk} and H^i_{jkh} are also related by $H^i_{jkh} = \dot\partial_h H^i_{jk}$ and therefore $\dot x^h \dot\partial_r H^i_{jkh} = \dot x^h \dot\partial_r \dot\partial_h H^i_{jk} = \dot x^h \dot\partial_h H^i_{jkr} = 0$ for H^i_{jkh} being homogeneous of degree zero in $\dot x^i$'s.

The covariant differentiation of this relation, in view of (1.6), (1.7) and (3.3) itself, simplifies to

$$(\mathcal{B}_s A_{lm}) H^i_{jk} = \lambda_s A_{lm} H^i_{jk}.$$

For a non-flat space it follows from this relation that

$$\mathcal{B}_s A_{lm} = \lambda_s A_{lm} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

holds in a $HR-F_n$. Thus we have the following:

Theorem 3.2—The tensor-field A_{lm} is also recurrent in a $HR-F_n$.

4. A Condition for a $HR-F_n$ to be an $AHR-F_n$

The vanishing of Lie derivative of the connection parameters G^i_{jk} is a necessary and sufficient condition for the transformation (1.8) to become an affine motion and subsequently gives rise to eqns. (1.11) (ii) and (1.12) in an $AHR-F_n$. Thus the Lie-invariance of curvature tensor **H** and recurrence vector $\boldsymbol{\lambda}$ are the necessary (but not sufficient) conditions for a $HR-F_n$ to admit an affine motion.

To study the converse problem, viz. whether the equations (1.11) (ii) and (1.12) are capable of inducing an affine motion in a $HR-F_n$, we revert to the discussion of the second section. It has been seen there that a $HR-F_n$ which admits the equation (2.16) obviously admits (1.11) (ii) (Theorem 2.3). Now in the following we shall discuss the possibilities for a $HR-F_n$ admitting (1.12) and (2.16) to become an $AHR-F_n$.

Differentiating (2.16) covariantly with respect to x^l and noting (3.3) we obtain, in a $HR-F_n$,

$$(\mathcal{B}_l L) H^i_{jkh} = A_{jk} \mathcal{B}_l \mathcal{B}_h v^i. \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

The transvection of this equation by $v^j \eta^k$, in view of (2.19), determines

$$\mathcal{B}_l \mathcal{B}_h v^i = -\frac{1}{L} (\mathcal{B}_l L) H^i_{jkh} v^j \eta^k. \quad \ldots \quad \ldots \quad \ldots \quad (4.2a)$$

As a $HR-F_n$ admitting (1.12) and (2.16) also admits (2.20) the equation (4.2*a*) reduces to

$$\mathcal{B}_l \mathcal{B}_h v^i = \frac{1}{L} (\mathcal{B}_l L) \mathcal{B}_h v^i. \quad \ldots \quad \ldots \quad \ldots \quad (4.2b)$$

Thus, in a $HR-F_n$ admitting (1.12) and (2.16), the covariant derivatives of the vector **v** are given by (2.20) and (4.2*b*).

Moreover, multiplication of (2.16) by v^j and an application of (2.17) determines, for $L \neq 0$,

$$v^j H^i_{jkh} = -\frac{1}{L} \{ \mathcal{B}_k L + (\dot{\partial}_r \lambda_k) \mathcal{B}_s v^r \dot{x}^s \} \mathcal{B}_h v^i. \quad \ldots \quad \ldots \quad (4.3)$$

Thus substituting from (4.2*b*) and (4.3) the Lie derivative of G^i_{lh} simplifies to

$$\pounds G^i_{lh} = \left\{ G^i_{jlh} - \frac{1}{L} (\dot{\partial}_j \lambda_l) \mathcal{B}_h v^i \right\} \mathcal{B}_k v^j \dot{x}^k. \quad \ldots \quad \ldots \quad (4.4)$$

The vanishing of the expression inside the curly brackets in the second member of (4.4) gives rise to the existence of an affine motion. Thus we have the following:

Theorem 4.1—A $HR-F_n$ admitting (1.12) and (2.16) will be an $AHR-F_n$ if

$$LG^i_{jlh} = (\dot{\partial}_j \lambda_l) \mathfrak{B}_h v^i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.5)$$

holds.

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