THERMAL INSTABILITY OF NON-HOMOGENEOUS FLUIDS; A MATHEMATICAL THEORY—II

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A unified linear theory of the well-known problems of thermal and thermohaline instability which also includes a more general analysis of the Rayleigh-Taylor instability is given in the subsequent pages. The configuration investigated here is that of a continuously stratified layer of viscous, incompressible fluid, statically confined between two horizontal boundaries of different, but uniform, temperature. The original stratification, which might be produced, for example, by a dissolved solute of negligible diffusivity, is assumed to be of the exponential type, namely $\rho = \rho_0 e^{-\delta z}$, where δ is a constant and z is the vertical coordinate. The results obtained are divided into three sections, 3–5. Sections 3 and 4 completely analyse the cases according as the fluid layer is subjected to a uniform underside heating with $\delta > 0$ or a uniform underside cooling with $\delta < 0$, respectively, where the assumed properties of the fluid are such that an increase in temperature produces a decrease in density. One of the principal results established in section 3 is a 'Circle Theorem' which limits the complex amplification rate of an arbitrary oscillatory mode inside a circle. In section 5, the 'Circle Theorem' is further shown to give rise to an upper as well as a lower bound for the frequency of oscillations of an arbitrary mode in the linear axisymmetric stability problem of spiral flows which is in complete accordance with the numerical calculations of Chandrasekhar (1961).

1. INTRODUCTION

The present paper is an attempt to reconstruct the stability problem of a fluid layer heated or cooled from below on the hypothesis of an original non-homogeneity present in the fluid. Viewing from another angle, it can also be looked upon as an extension of the Rayleigh-Taylor instability problem (density being a continuous function of the vertical coordinate) where the fluid is heat conducting and its upper and lower layers are at different temperatures. The motivations for undertaking the present investigation are the following: (a) to present a unified treatment of the Bénard and Rayleigh-Taylor instability problems, (b) to propose a mechanism which can directly lead to overstable solutions in the Bénard configuration and (c) to make detailed investigations of the models treated by Stern (1960), Turner and Stommel (1964), Turner (1965, 1968), Sani (1965), Veronis (1965, 1968) and others when the mass diffusivity of the chemical dissolved is negligible. The subsequent analysis is based on the rather plausible hypothesis that in reality a fluid is originally non-homogeneous. The gravitational effects of even a very small amount of this original non-homogeneity may turn out to be quite significant for the problem and consequently its neglect everywhere may not be justified. The other assumptions involved in the earlier theories (on Bénard convection), namely (i) the smallness of the initial motion, (ii) the well-known Boussinesq approximation, (iii) a linear temperature profile extending throughout the entire fluid layer at the start of the motion and (iv) the negligibility of the forces arising due to surface tension, are assumed to be valid throughout this investigation. It is further assumed that the disturbances are infinitesimal so that the linear stability theory holds good. The normal mode technique is applied in the mathematical analysis.

The results obtained in the paper are divided into three sections, namely 3, 4 and 5, and the configurations treated therein are as explained before. The stability problem thus formulated is shown to depend on four non-dimensional numbers, viz. $R_1 = \frac{g\alpha\beta d^4}{\kappa\nu}$, $R_2 = \frac{g\delta d^4}{\kappa\nu}$, $M = d\delta$ and $P = \nu/\kappa$ where g is the acceleration due to gravity, α, κ and ν are respectively the coefficients of volume expansion, thermometric conductivity and kinematic viscosity of the fluid; d is the layer depth and β and δ are respectively the maintained uniform temperature gradient and the original non-homogeneity factor. The non-dimensional numbers R_1 and P are the well-known Rayleigh and Prandtl numbers which occur in the study of the classical Bénard problem. Further, for small values of M, we have, at places, used the approximation $R_2 e^{-Mz} \simeq R_2$. This will now be referred to as the small M approximation and a result is to be regarded as independent of this approximation if it is not mentioned.

2. The Physical Problem and its Formulation

A viscous incompressible fluid of varying density is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are maintained at constant temperatures T_0 and T_1 respectively. The original non-homogeneity of the fluid is assumed to be of the exponential type, namely $\rho = \rho_0 e^{-\delta z}$. The problem is to investigate the stability of this configuration.

Let the origin be taken on the lower boundary $z = 0$ with the z-axis perpendicular to it along the vertical. The xy plane then constitutes the horizontal plane.

We now proceed to obtain the resultant density distribution which arises due to the interaction between the original and the thermal stratifications.

Consider $Q(x, y, z)$ to be an arbitrary point in the fluid at which the temperature is $T(z)$. We then assume that the resultant density at $Q(x, y, z)$ can be thought of as due to the following three density fields: (a) a homogeneous fluid of density $\rho = \rho_0$, (b) an originally homogeneous fluid of density $\rho = \rho_0$

upon which a non-homogeneity of the form $\rho = \rho_0 e^{-\delta z}$ is imposed and (c) an originally homogeneous fluid of density $\rho = \rho_0$ upon which a uniform temperature gradient $(T_0 - T_1/d)$ is applied and thus making it non-homogeneous according to the linear law $\rho = \rho_0[1+\alpha(T_0-T)]$. Then on the hypothesis that the resultant density distribution at $Q(x, y, z)$ can be obtained by superposing the changes due to (b) and (c) on (a) at the point $Q(x, y, z)$, we have

$$\rho = \text{resultant density at } Q(x, y, z) = \rho_0[e^{-\delta z}+\alpha(T_0-T)]. \quad \ldots \quad (1)$$

The stationary state of the system whose stability we wish to examine is then characterized by the following solutions for the velocity, temperature, density and pressure fields respectively:

$$\left.\begin{aligned} \text{velocity} &\equiv 0 \\ T &= T_0 - \beta z \\ \rho &= \rho_0[e^{-\delta z}+\alpha(T_0-T)] \\ \text{and} \quad p &= p_0 - g\rho_0\left[\frac{1}{\delta}(1-e^{-\delta z})+\frac{\alpha\beta z^2}{2}\right] \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where p_0 is the pressure and ρ_0 is the density at the lower boundary $z = 0$ and $\beta = \dfrac{T_0 - T_1}{d}$ denotes the maintained temperature gradient.

Let the initial state described by eqns. (2) be slightly perturbed so that the perturbed state is given by

$$\left.\begin{aligned} \text{perturbed velocity} &= (u, v, w) \\ T' &= T_0 - \beta z + \theta \\ \rho' &= \rho_0\left[e^{-\delta z}+\frac{\delta\rho}{\rho_0}+\alpha(T_0-T-\theta)\right] \\ \text{and} \quad p' &= p + \delta p \end{aligned}\right\} \quad \ldots \quad \ldots \quad (3)$$

where (u, v, w), θ, $\delta\rho$ and δp are respectively the perturbations in the velocity, temperature, original density and pressure fields.

Then the linearized perturbation equations (using Boussinesq approximation) of momentum, continuity, incompressibility and heat conduction, when the disturbances are analysed in terms of normal modes by seeking solutions whose dependence on x, y and t is given by

$$\exp\,[i(k_x x + k_y y + nt)], \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

become

$$[in\rho_0+\mu k^2]u = -ik_x\delta p+\mu\frac{d^2u}{dz^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$[in\rho_0+\mu k^2]v = -ik_y\delta p+\mu\frac{d^2v}{dz^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

$$[in\rho_0+\mu k^2]w = -\frac{d\delta p}{dz}+g\alpha\rho_0\theta-g\delta\rho+\mu\frac{d^2w}{dz^2} \quad \ldots \quad (7)$$

$$i[uk_x+vk_y] = -\frac{dw}{dz} \quad \ldots \quad (8)$$

$$in\delta\rho = (w\delta)\rho_0 e^{-\delta z} \quad \ldots \quad (9)$$

and

$$in\theta-\kappa[D^2-k^2]\theta = \beta w \quad \ldots \quad (10)$$

where $k = \sqrt{(k_x^2+k_y^2)}$ is the wavenumber of the disturbance, μ the coefficient of viscosity of the fluid and n a constant which can be complex.

Eliminating u, v, δp and $\delta\rho$ from eqns. (5) to (9) and changing the resulting equations into non-dimensional form, we obtain

$$\sigma(D^2-a^2-\sigma)(D^2-a^2)w_1 = R_1a^2\sigma\theta-\frac{R_2a^2e^{-Mz_1}}{P}w_1 \quad \ldots \quad (11)$$

and

$$(D^2-a^2-P\sigma)\theta = -w_1 \quad \ldots \quad (12)$$

together with the associated boundary conditions

$$\left.\begin{array}{ll} w_1 = 0 = \theta \text{ for } z_1 = 0 \text{ and } 1 & \\ \text{and either} & \\ Dw_1 = 0 \quad \text{for } z_1 = 0 \text{ and } 1 \text{ (rigid boundary)} & \\ \text{or} & \\ D^2w_1 = 0 \quad \text{for } z_1 = 0 \text{ and } 1 \text{ (free boundary)} & \end{array}\right\} \quad \ldots \quad (13)$$

where

$$\left.\begin{array}{lll} z_1 = z/d\,; & a = kd\,; & \theta = \theta \\ D = d\,d/dz\,; & \sigma = ind^2/\nu\,; & P = \nu/\kappa \\ M = d\delta\,; & w_1 = \dfrac{\beta d^2}{\kappa}w\,; & R_1 = \dfrac{g\alpha\beta d^4}{\kappa\nu} \\ R_2 = \dfrac{g\delta d^4}{\kappa\nu} & & \end{array}\right\}. \quad \ldots \quad (14)$$

In the subsequent analysis we shall use w for w_1 and z for z_1.

Thus, for a given a, M, R_1, R_2 and P, eqns. (11), (12) and (13) present an eigenvalue problem for σ and the system is unstable, neutral or stable for these values of the parameters according as the real part of σ, namely σ_r, is positive, zero or negative respectively.

It is to be noted that eqns. (11) and (12) can be obtained by putting $K_s = 0$ in eqns. (1), (2), (3) and (4) of Stern's paper (1960) but still we prefer to deduce them here on account of the totally different motivations of the two problems. The approach presented here is precisely on lines of unifying the classical Bénard and Rayleigh-Taylor instability problems and consequently has its own points of interest.

3. On the Marginal State

The configuration in this case is characterized by $\beta > 0$ and $\delta > 0$. Then multiplying eqn. (11) throughout by w^* (the complex conjugate of w), integrating the resulting equation over the vertical range of z by making use of the boundary conditions (13) and replacing for $\int_0^1 \theta w^* dz$ from eqn. (12), we get from the real part of the final equation

$$\sigma_r \int_0^1 [|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz+(\sigma_r^2-\sigma_i^2)\int_0^1 [|Dw|^2+a^2|w|^2]dz$$
$$= \sigma_r R_1 a^2 \int_0^1 [|D\theta|^2+a^2|\theta|^2]dz+R_1a^2P(\sigma_r^2+\sigma_i^2)\int_0^1 |\theta|^2 dz-\frac{R_2a^2}{P}\int_0^1 e^{-Mz}|w|^2dz. \quad \ldots \quad (15)$$

Equation (15) shows that the marginal state is definitely of oscillatory character, because

$$\sigma_r = 0 \text{ implies } \sigma_i \text{ is real and} \neq 0 \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

(σ_i being the imaginary part of σ).

This confirms that principle of exchange of stabilities is not valid whether the boundaries are rigid or free and it is the overstable oscillations which will manifest at the marginal state. Hence, the introduction of an original non-homogeneity in the fluid alters the character of the marginal state (compare Pellew and Southwell 1940) in the Bénard configuration. It is further noted here that the analysis of Veronis (1965) also indicates a similar possibility but his investigations are restrictive in the sense that it deals only with the case of free bounding surfaces.

3.1. *An Exact Solution: Case of Free Boundaries*

The governing differential equations in the present case can be obtained by putting $\sigma = i\sigma_i$ in eqns. (11) and (12). Further, these are to be considered with the boundary conditions for free bounding surfaces. It can then be easily shown that the value of the critical Rayleigh number together with the corresponding frequency and wavenumber are given by

$$R_{1c} = \frac{R_2 P}{1+P}+\frac{27\pi^4}{4} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$\sigma_{ic}^2 = \frac{R_2}{3P(1+P)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

and

$$a_c^2 = \frac{\pi^2}{2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

which correspond to the solution

$$w = \sin \pi z. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

The above exact solution has also been obtained by Veronis (1965). One observes here that, for a given R_2, it is necessary for the destabilizing temperature gradient, expressed in terms of the Rayleigh number R_1, to exceed the minimum value for ordinary Bénard convection by $\frac{P}{1+P} R_2$. Further, in situations where the original stratification is very stable, i.e. $R_2 \gg \frac{27\pi^4}{4}$, the destabilizing temperature gradient need provide an effect on density that is only $\frac{P}{P+1}$ of R_2. In other words, it signifies that even when the total density field is gravitationally stable, the system can become unstable through overstable motions. These conclusions are valid, of course, in the framework of linear stability theory (Veronis 1965).

3.2. *Solution of the Characteristic Value Problem: Case of Rigid Boundaries*

Here we propose to solve the above characteristic value problem for the case of rigid boundaries by a method which, though differing from the variational methods developed in Appendix A, leads to the same secular determinant. The present method of deriving the secular determinant is applicable even if a variational principle does not underline the problem. But the existence of a variational principle ensures that by keeping more and more terms in the Fourier expansion for F and solving the secular determinant for R_1, we approach the true characteristic value monotonically from above.

In view of the symmetry of this problem with respect to the bounding planes, we shall find it convenient to translate the origin of z to be midway between the two planes. The fluid will then be confined between $z = \pm 1/2$ and we shall have to seek the solutions of equations

$$(D^2-a^2-i\sigma_i)(D^2-a^2)w = \frac{iR_2a^2}{P\sigma_i}w+F \quad \quad (21)$$

and

$$(D^2-a^2-iP\sigma_i)F = -R_1a^2w \quad \quad (22)$$

which satisfy the boundary conditions

$$F = 0 = w = Dw \text{ for } z = \pm\tfrac{1}{2} \quad \quad (23)$$

where

$$F = R_1a^2\theta. \quad \quad (24)$$

To solve the above problem, we expand F in cosine series in the form

$$F = \sum_{m=0}^{\infty} A_m \cos[(2m+1)\pi z] \quad \quad (25)$$

and express w in the manner

$$w = \sum_{m=0}^{\infty} A_m w_m. \qquad (26)$$

The w_m satisfies the differential equation

$$(D^2-a^2-i\sigma_i)(D^2-a^2)w_m - \frac{iR_2a^2}{P\sigma_i} w_m = \cos[(2m+1)\pi z] \qquad (27)$$

with

$$w_m = 0 = Dw_m \text{ for } z = \pm\tfrac{1}{2}. \qquad (28)$$

Now putting for F and w in eqn. (22) in accordance with eqns. (25)–(28) and equating the Fourier coefficients of both sides, one gets, after a lengthy but essentially straightforward calculation, the following secular equation

$$\left\| \frac{1}{2}\left(\frac{C_{2n+1}}{R_1a^2} - \gamma_{2n+1}\right)\delta_{nm} - (n/m) \right\| = 0 \qquad (29)$$

where

$$(n/m) = (-1)^{m+n+1}2(2n+1)(2m+1)\pi^2\gamma_{2n+1}\gamma_{2m+1}\Delta(q_1^2-q_2^2) \qquad (30)$$

$$C_n = n^2\pi^2+a^2+i\sigma_i P \qquad (31)$$

$$\frac{1}{\gamma_n} = [n^2\pi^2+a^2+i\sigma_i][n^2\pi^2+a^2] - \frac{iR_2a^2}{P\sigma_i} \qquad (32)$$

$$\Delta = 1/(q_1 \tanh q_1/2 - q_2 \tanh q_2/2) \qquad (33)$$

and q_1^2 and q_2^2 are the roots of the quadratic

$$(q^2-a^2)(q^2-a^2-i\sigma_i) - \frac{iR_2a^2}{P\sigma_i} = 0. \qquad (34)$$

A first approximation to R_1 will be given by setting the (0, 0) element of the secular matrix equal to zero and ignoring all others. This corresponds to the choice of $\cos \pi z$ as a trial function for F. Since, in the above process, we solved the fourth order differential eqn. (27) relating F and w, i.e. in effect solved 'two-thirds' of the problem exactly, we expect that the first approximation will give good results.

The solution for R_1 in the first approximation is given by

$$R_1 = \frac{\pi^2+a^2+iP\sigma_i}{a^2\gamma_1[1-4\pi^2\gamma_1\Delta(q_1^2-q_2^2)]}. \qquad (35)$$

The results of some calculations based on eqn. (35) are given in Table I.

TABLE I

R_2	a	σ_i	R_1
0	3·12	0	1715
60	3·1	2·9	1753
1320	3·4	22·3	2309
3340	4·2	17	3381

Rayleigh numbers for values of R_2, a and σ_i when both the bounding surfaces are rigid and $P = 1$.

3.3. *Bounds for the Growth Rate: A Circle Theorem*

Here, we shall show that for an arbitrary oscillatory mode ($\sigma_i \neq 0$) the complex rate of growth must lie inside a circle whose centre is the origin and radius $\sqrt{(R_2/P)}$.

To show this we write eqn. (11) as

$$[D^2-a^2]^2w-\sigma[D^2-a^2]w = R_1a^2\theta-\frac{R_2a^2\sigma^*e^{-Mz}}{P|\sigma|^2}w \quad \ldots \quad \ldots \quad (36)$$

where σ^* is the complex conjugate of σ.

Now multiplying eqn. (36) throughout by w^*, integrating over the range of z by making use of (13) and replacing for $\int_0^1 \theta w^* dz$ from (12), we obtain from the imaginary part of the resulting equation

$$\int_0^1 |Dw|^2dz+a^2\int_0^1\left[1-\frac{R_2e^{-Mz}}{P|\sigma|^2}\right]|w|^2dz = -R_1a^2P\int_0^1|\theta|^2dz < 0. \quad (37)$$

Hence, from (37) we must have

$$\sigma_r^2+\sigma_i^2 < R_2/P. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Thus, the growth rate of an arbitrary oscillatory mode (whether stable, neutral or unstable) must lie inside the circle given by (38). It can be readily checked that oscillations given by (18) for neutral modes do lie inside the above circle.

One of the implications of the above circle theorem is that for small values of R_2, oscillatory modes are not expected to manifest. In fact for $R_2 = 0$, the above circle theorem is violated and oscillatory modes cannot exist in that situation. This fact, as one knows, is in perfect accordance with the results of classical Bénard problem (for which $R_2 = 0$). Further, since the radius of this bounding circle is independent of R_1, it is probable that a more rigid limitation on $\sigma_r^2+\sigma_i^2$ might be found.

3.4. *Non-oscillatory Modes: A Sufficient Condition of Stability: Case of Free Boundaries*

For non-oscillatory modes, $\sigma_i = 0$ so that $\sigma = \sigma_r$ only. Equations (11) and (12) can then be combined as

$$[(D^2-a^2)^3-\sigma_r(1+P)(D^2-a^2)^2+P\sigma_r^2(D^2-a^2)]w$$
$$= -R_1a^2w\frac{R_2a^2}{P\sigma_r}(a^2+P\sigma_r)(we^{-Mz})-\frac{R_2a^2}{P\sigma_r}D^2[we^{-Mz}]. \quad \ldots \quad (39)$$

Multiplying (39) by w^* throughout, integrating the resulting equation over the range of z by making use of the boundary conditions (13) and using the result

$$\text{Real}\int_0^1 w^*D^2[e^{-Mz}w]dz = -\int_0^1|Dw|^2e^{-Mz}dz+\frac{M^2}{2}\int_0^1 e^{-Mz}|w|^2\,dz \quad \ldots \quad (40)$$

(which can be readily obtained by integration by parts and using (13)), we have from the real part of the final equation

$$-\int_0^1[|D^3w|^2+3a^2|D^2w|^2+3a^4|Dw|^2+a^6|w|^2]dz-\sigma_r(1+P)\int_0^1[|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz$$
$$-P\sigma_r^2\int_0^1[|Dw|^2+a^2|w|^2]dz = \frac{R_2a^2}{P\sigma_r}\int_0^1|w|^2\left[a^2-\frac{M^2}{2}\right]e^{-Mz}dz$$
$$+\frac{R_2a^2}{P\sigma_r}\int_0^1|Dw|^2e^{-Mz}dz+a^2\int_0^1|w|^2[R_2e^{-Mz}-R_1]dz. \quad (41)$$

Equation (41) clearly shows that if

$$\left.\begin{aligned} R_2e^{-M} &> R_1 \\ \text{and} \qquad a^2 &> \frac{M^2}{2} \end{aligned}\right\} \quad (42)$$

then we must have $\sigma_r < 0$. Thus, (42) gives a sufficient condition of stability for non-oscillatory modes. It indicates the fact that, even if the fluid is stably stratified in its initial state, one can be sure about the complete stability of the system with respect to those non-oscillatory perturbations only whose wavenumbers (in units of d) exceed $M/\sqrt{2}$. Thus, one gets a feeling that the violation of conditions (42) might give rise to instability even though the total density field is gravitationally stable (as in the case of oscillatory modes). We, therefore, continue the discussion further under the small M approximation. In that case, one can similarly show that the above sufficient condition of stability, namely (42), reduces to $R_2 > R_1$ only. This implies that if the fluid is stably stratified in its initial state then the system remains stable to all infinitesimal non-oscillatory perturbations when the boundaries are free and the small M approximation holds good.

In fact, under this approximation the characteristic value problem can be exactly solved in the present situation and the secular equation is given by

$$[\lambda^2P]\sigma_r^3+[\lambda^4(1+P)]\sigma_r^2+[\lambda^6-(R_1-R_2)a^2]\sigma_r+\frac{R_2a^2\lambda^2}{P} = 0 \quad (43)$$

where

$$\lambda^2 = n^2\pi^2+a^2, \quad (44)$$

n being an integer.

Equations (43) and (44) show that for a wavenumber satisfying

$$\frac{(\pi^2+a^2)^3}{a^2} > R_1-R_2 \quad (45)$$

we have $\sigma_r < 0$ and hence stability. Further, since the minimum value of $\frac{(\pi^2+a^2)^3}{a^2}$ is $\frac{27\pi^4}{4}$ which occurs at $a^2 = \frac{\pi^2}{2}$, it follows that so long as

$$R_1 < R_2+\frac{27\pi^4}{4} \quad (46)$$

all the non-oscillatory modes are damped. Thus for instability with respect to non-oscillatory modes we necessarily have

$$\frac{(\pi^2+a^2)^3}{a^2} < R_1-R_2. \qquad (47)$$

Our next attempt is to find out a sufficient condition of instability for these modes.

3.5. *A Sufficient Condition of Instability for Non-oscillatory Modes: Case of Free Boundaries*

For instability we have shown that (47) must necessarily hold good. An examination of (43), in conditions in which (47) is valid, shows that there are positive real roots for σ_r, and hence instability, if

$$\lambda^6[(1+P)\{2\lambda^6(1+P)^2+9P<(R_1-R_2)a^2-\lambda^6>\}+27a^2PR_2]^2$$
$$\leqslant 4[3P\{(R_1-R_2)a^2-\lambda^6\}+\lambda^6(1+P)^2]^3. \qquad (48)$$

This follows from the theory of equations.

Thus, inequalities (47) and (48) provide sufficient conditions of instability in the present situation. The above conditions also emphasize the fact that non-oscillatory modes of the system are more stable in character than the oscillatory modes, a fact noticed by Veronis (1965) also in his analysis.

4. Analysis of the Gravitationally Opposite Case

We shall now investigate the stability of a configuration which is gravitationally opposite to that treated in section 3. In other words, here we are interested in the stability problem wherein an exponentially unstably stratified (i.e. $\delta < 0$) layer of an incompressible viscous fluid, statically confined between two horizontal planes, is cooled from below (i.e. $T_0 < T_1$).

The governing differential equations for the present problem can be obtained from eqns. (11) and (12) of section 2 by replacing w, R_1, R_2 and M by $-w$, $-R_1$, $-R_2$ and $-M$ respectively (this follows from (14), since β and M are negative for the present problem).

These are given by

$$\sigma(D^2-a^2-\sigma)(D^2-a^2)w = R_1a^2\sigma\theta+\frac{R_2a^2e^{Mz}}{P}w \qquad (49)$$

and

$$(D^2-a^2-P\sigma)\theta = w \qquad (50)$$

where R_1, R_2 and M are now positive.

4.1. *On the Existence of Neutral Modes*

We shall show that neutral modes do not exist here. For this, multiplying eqn. (49) by w^* throughout, integrating the resulting equation over the range of z by making use of (13) and replacing for $\int_0^1 \theta w^* \, dz$ from (50), we have from the real and imaginary parts of the final equation

$$\sigma_r \int_0^1 [|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz+(\sigma_r^2-\sigma_i^2)\int_0^1 [|Dw|^2+a^2|w|^2]dz$$
$$= -R_1a^2\sigma_r \int_0^1 [|D\theta|^2+a^2|\theta|^2]dz - R_1a^2P(\sigma_r^2+\sigma_i^2)\int_0^1 |\theta|^2dz+\frac{R_2a^2}{P}\int_0^1 e^{Mz}|w|^2dz \quad \ldots \quad (51)$$

and

$$\sigma_i \int_0^1 [|D^2w|^2+2a^2|Dw|^2+a^4|w|^2]dz+2\sigma_r\sigma_i \int_0^1 [|Dw|^2+a^2|w|^2]dz$$
$$= -R_1a^2\sigma_i \int_0^1 [|D\theta|^2+a^2|\theta|^2]dz. \quad \ldots \quad (52)$$

Now assume that neutral modes exist so that $\sigma_r = 0$ is allowed by the equations. Then (52) implies that $\sigma_i = 0$ while (51) implies $\sigma_i \neq 0$. Hence, the starting assumption, namely $\sigma_r = 0$, is incorrect and consequently neutral modes cannot exist; in other words, an arbitrary mode is either damped or amplified. This totally differs from the results of Stern (1960). The equations of Stern (1960) allows stationary solutions and possibly overstable solutions also at the marginal state. Stern has solved the problem on the assumption of the principle of exchange of stabilities while the present equations do not allow any marginal state solution. This discrepancy between the results of Stern and the present results is attributed to the neglect of mass diffusivity in the present situation and the point will be further illuminated in the treatment of non-oscillatory and oscillatory modes. Further, it is also clear that the present problem differs markedly in character from that treated in section 3.

4.2. *Oscillatory Modes*

For an oscillatory mode $\sigma_i \neq 0$ and eqn. (52) clearly shows that σ_r is negative. Thus, the oscillatory modes are stable irrespective of the presence of the original unstable stratification and the nature of the bounding surfaces. It is noted here that Pellew and Southwell's result, namely all oscillations must decay when $R_2 = 0$ and the layer is cooled from below, is recovered from here as a corollary. The stability of the oscillatory modes is thus solely governed by the character of the applied temperature gradient. One further observes here that oscillatory modes were the more destabilizing ones in section 3.

Further, one can show from eqns. (51) and (52) (details given in Appendix B) that for these modes

$$\left.\begin{aligned} &|\sigma_r|/a^2 > 1/2 && \text{for } P \geqslant 1 \\ &|\sigma_r|/a^2 > 1/2P && \text{for } \tfrac{1}{2} \leqslant P < 1 \\ \text{and} \quad &|\sigma_r|/a^2 > 1 && \text{for } P < 1/2 \end{aligned}\right\} \quad \ldots \quad (53)$$

4.3. *Non-oscillatory Modes: Case of Free Boundaries*

Here we shall analyse the problem under the small M approximation. In that case the characteristic value problem can be exactly solved and we obtain the secular equation as

$$[P\lambda^2]\sigma_r^3+[\lambda^4(1+P)]\sigma_r^2+[(R_1-R_2)a^2+\lambda^6]\sigma_r-\frac{R_2a^2}{P}\lambda^6=0 \quad .. \quad (54)$$

where both R_1 and R_2 are positive.

Equation (54) has clearly one positive root for σ_r and this implies instability. Thus, the presence of an unstable original density stratification makes the non-oscillatory modes of the system unstable, whatever be the stabilizing temperature gradient. In the model analysed by Stern (1960), the non-oscillatory modes of the system are amplified only if the original unstable stratification exceeds certain critical value (assuming the validity of the principle of exchange of stabilities) but this is not so in the present case and in this respect the present result is particularly striking. The reason for this lies in the smallness of the coefficients of mass diffusion. Actually, as noted by Stern, the mass diffusivity and the heat diffusivity respectively play a stabilizing and destabilizing role for marginal non-oscillatory modes and consequently it is expected that the neglect of mass diffusion will make an arbitrary non-oscillatory mode amplify. Further, while in section 3 we obtained a sufficient condition of stability as well as instability for non-oscillatory modes, we do not have such conditions here. On the other hand, these modes are the destabilizing ones for the present problem.

We further state without proof (which is a bit lengthy) that under the condition that the eigenfunction w is real and $|\sigma_r|/a^2 < 1/2$, one can prove the instability of non-oscillatory modes for rigid boundaries without using the small M approximation.

5. Spiral Flows: Bounds of the Frequency of Oscillations

The governing differential equations and the boundary conditions of the linear axisymmetric stability problem of spiral flows are given by

$$[(D_1^2-a_1^2)-i(\sigma_1+Ra_1)][D_1^2-a_1^2]u-12iRa_1u=v \quad .. \quad .. \quad (55)$$

and

$$[(D_1^2-a_1^2)-i(\sigma_1+Ra_1)]v=-\overline{T}a_1^2u \quad .. \quad .. \quad .. \quad .. \quad (56)$$

with

$$u=Du=v=0 \text{ for } \zeta=\pm 1/2 \quad .. \quad .. \quad .. \quad (57)$$

where the symbols used in the above equations have the same meaning as given by Chandrasekhar (1961) except that the suffix 1 is attached with D, a and σ, in order to avoid any confusion with the symbols of the present paper.

We now consider the above system of equations for marginal modes. For such modes σ_1 is purely real and then the above equations are mathematically equivalent to the system of equations consisting of (21), (22) and (23) with the following identification:

$$\left.\begin{aligned} & w = u\,;\ \sigma_i = \sigma_1 + Ra_1\,;\ P = 1 \\ & F = v\,;\ \frac{R_2 a^2}{P\sigma_i} = 12Ra_1\,; \\ & a = a_1\,;\ R_1 = \bar{T} \end{aligned}\right\} \quad \ldots \quad (58)$$

Now, for eqns. (21), (22) and (23) the circle theorem gives

$$\sigma_i^2 < R_2. \quad \ldots \quad (59)$$

(It is to be noted here that the condition of oscillatory modes which is necessary for the validity of the circle theorem is automatically satisfied because the marginal modes are definitely oscillatory.)

Inequality (59) gives, for eqns. (55), (56) and (57),

$$-Ra_1 < \sigma_1 < \frac{12R}{a_1} - Ra_1. \quad \ldots \quad (60)$$

The above gives the bounds for the frequency of oscillations of the marginal modes in the stability problem of spiral flows. It is easily seen that these bounds are in complete accordance with the numerical calculations of Chandrasekhar (1961).

In fact, inequality (60) holds good for non-marginal modes of the system also (that is, when $\sigma_1 = \sigma_{1r} + i\sigma_{1i}$ is complex.). To prove this, we multiply eqn. (55) by u^* (u^* being the complex conjugate of u) throughout, integrate the resulting equation over the range of ζ by making use of eqn. (57), replace for $\int_{-1/2}^{1/2} vu^* d\zeta$ from eqn. (56) and obtain by separating in the imaginary part of the final equation

$$(\sigma_{1r} + Ra_1)\int_{-1/2}^{1/2} |D_1 u|^2 d\zeta + a^2\left(\sigma_{1r} + Ra_1 - \frac{12R}{a_1}\right)\int_{-1/2}^{1/2} |u|^2 d\zeta + \frac{(\sigma_{1r} + Ra_1)}{\bar{T} a_1^2}\int_{-1/2}^{1/2} |v|^2 d\zeta = 0. \quad \ldots \quad (61)$$

Equation (61) clearly shows that

$$-Ra_1 < \sigma_{1r} < \frac{12R}{a_1} - Ra_1.$$

Consequently, non-oscillatory modes cannot exist for wavenumbers exceeding $\surd 12$. In other words, principle of exchange of stabilities appears rather unlikely.

6. Concluding Remarks

We shall end up with the discussions here with a few remarks on the relationship between the present work and the experiments performed on thermal and thermohaline convection. The results of Turner and Stommel

(1964), Turner (1968), Shirtcliffe (1969) and others bear ample evidence of the fact that thermohaline instability sets in as overstability, a result which is so correctly predicted by the present work. It appears that the present work can accurately describe, both with respect to qualitative as well as quantitative points of view, the phenomenon of thermohaline convection where the mass diffusivity of the solute is negligible in comparison to its heat diffusivity. Further, in the experiments on thermal convection, namely the experiments of Bénard (1900), Schmidt and Milverton (1935), Silveston (1958) and others, although there is, in general, an agreement between the experimental and the theoretically calculated critical Rayleigh number but still this can hardly be called a very good agreement. This lack of agreement is possibly due to the neglect of the gravitational effects of original non-homogeneity in the theoretical investigations, which cannot always be justified. A recalculation of the critical Rayleigh number by including the effects of this original non-homogeneity may put the above work on firm grounds.

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Appendix A: Variational Principles

Let us consider the marginal state of the system. In that case, eqns. (11), (12) together with the boundary conditions (13) can be conveniently combined to yield

$$a^2R_1 = \frac{\int_0^1 [(DF)^2+a^2F^2+iP\sigma_i F^2]dz}{\int_0^1 [(D^2w)^2+2a^2(Dw)^2+a^4w^2]dz+i\sigma_i\int_0^1 [(Dw)^2+a^2w^2]dz-\frac{iR_2a^2}{P\sigma_i}\int_0^1 w^2e^{-Mz}dz} \quad \text{.. (62)}$$

where

$$F = R_1a^2\theta \quad \text{.. .. .. .. (63)}$$

one can show that R_1 given by (62) has a stationary property when the quantities on the right-hand side are evaluated in terms of the true characteristic functions w.

Similarly, one can show that R_1 given by

$$R_1 = \frac{i\sigma_i\int_0^1 [(D^2w)^2+2a^2(Dw)^2+a^4w^2]dz-\sigma_i^2\int_0^1 [(Dw)^2+a^2w^2]dz+\frac{R_2a^2}{P}\int_0^1 e^{-Mz}w^2dz}{i\sigma_i a^2\int_0^1 [(D\theta)^2+a^2\theta^2+iP\sigma_i\theta^2]} \quad \text{.. (64)}$$

(which can again be obtained from (11), (12) and (13)) has a stationary property when the quantities on the right-hand side are evaluated in terms of the true characteristic function.

The above variational principles provide useful means of evaluating the critical Rayleigh number when the surfaces bounding the fluid are rigid.

Appendix B: Deduction of (53)

One can put (52) as

$$2(a^2-|\sigma_r|)\int_0^1 |D^2w|^2dz-a^2|\sigma_r|(a^2-2|\sigma_r|)\int_0^1 a^2|w|^2dz < 0. \quad \text{.. (65)}$$

Further, (51) and (52) can be combined as

$$-|\sigma_r|(2a^2-|\sigma_r|)\int_0^1 |Dw|^2dz-a^2|\sigma_r|(a^2-|\sigma_r|)\int_0^1 |w|^2dz-R_1a^2|\sigma_r|(a^2-P|\sigma_r|)\int_0^1 |\theta|^2dz$$
$$-\frac{\sigma_i^2}{a^2}(a^2-2|\sigma_r|P)\int_0^1 [|Dw|^2+a^2|w|^2]dz > 0. \quad \text{.. .. .. (66)}$$

From (65) and (66) result (53) follows.