TRANSIENT THERMAL STRESSES IN A SLAB UNDER CONTINUOUS SURFACE TEMPERATURE

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The thermo-elastic stresses and displacements in an infinite elastic slab due to the sudden application of a transient temperature field have been obtained. One face of the slab is assumed stress-free and the other face is rigidly constrained. To the stress-free face, a spatially varying temperature is suddenly applied while the other face is held at constant temperature. The solution of the quasi-static thermo-elastic field equation is obtained by means of 'symbolic method' in terms of differential operators. The general solutions are expressed as functions of the applied temperature and the temperature, displacement and stresses in the slab are numerically evaluated at successive times.

1. Introduction

In this paper, the effect of sudden application of transient temperature distribution on the elastic field in an infinite slab bounded by two parallel planes has been discussed. The slab is rigidly constrained and at constant temperature on one face while the other face is assumed to be stress-free and the temperature field is applied at this face. The thermo-elastic field equations in the quasi-static case are solved by means of 'symbolic method' (Lurye 1955, Lekhnitskii 1963) when the applied surface temperature is a continuous function of length and time. Numerical results for the temperature in the slab are obtained for a particular material of the slab by taking a particular surface temperature function.

Thermo-elastic stresses in a slab under similar boundary conditions have been considered by Martin (1966) and Martin and Payton (1964) when the applied surface temperature is a discontinuous step function. In this case, they made use of the theory of complex variables and the Fourier transforms. That model was generated by them to approximate the elastic field in bonding materials used in missiles and space vehicles. The steady-state thermo-elastic stresses in such a slab were considered by taking the applied temperature function as a continuous function of distance and the solution was obtained by the method of differential operators (Wadhawan and Singh 1968).

In the present case, exact numerical solution can be obtained by the method of differential operators when the surface temperature is a continuous function. The method used here is easier to apply for getting numerical results than the usual method of integral transforms and complex variables.

2. Formulation of the Problem

Using the rectangular Cartesian system of coordinates x, y, z, let the infinite slab be bounded by two parallel planes $y = 0$ and $y = h$ and be infinite in extent in x and z directions. The face $y = 0$ is maintained at zero temperature while the face $y = h$ is exposed to a continuous surface temperature which varies with x and the time t. Also, the face $y = 0$ is rigidly constrained and the face $y = h$ is held stress-free.

The temperature T in the slab satisfies the linear heat conduction equation (Carslaw and Jaeger 1959)

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2} = \frac{\rho c}{k}\frac{\partial T}{\partial t} \qquad \text{(1)}$$

where ρ, c, and k are respectively the density, specific heat and the thermal conductivity of the solid.

The thermal boundary conditions are

$$T(x, 0, t) = 0 \text{ for all } x \text{ and } t \qquad \text{(2a)}$$

$$T(x, h, t) = H(x, t) \qquad \text{(2b)}$$

where $H(x, t)$ is a prescribed function of x and t which is continuous and continuously differentiable.

The linear elastic, quasi-static plane strain field equations are as follows:

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y} = 0. \qquad \text{(3)}$$

Stress-displacement relations

$$\left.\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}-\alpha(1+\nu)T\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\frac{\partial u}{\partial x}+(1-\nu)\frac{\partial v}{\partial y}-\alpha(1+\nu)T\right] \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y}+\frac{\partial x}{\partial v}\right), \quad \tau_{xz} = \tau_{yz} = 0 \\
\tau_{zz} &= \nu(\sigma_{xx}+\sigma_{yy})-E\alpha T
\end{aligned}\right\} \qquad \text{(4)}$$

where $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz})$ and $(\tau_{xy}, \tau_{yz}, \tau_{zx})$ are the components of normal and shear stresses respectively. (u, v) are the components of the displacement in x and y direction respectively.

As usual E, ν and α are the Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion respectively.

The appropriate boundary and initial conditions in the present case are

$$u(x, 0, t) = v(x, 0, t) = 0, \ t \geqslant 0 \quad \ldots \quad (5a)$$

$$\sigma_{yy}(x, h, t) = \tau_{xy}(x, h, t) = 0, \ t \geqslant 0 \quad \ldots \quad (5b)$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = u = v = 0 \ \text{for} \ t < 0. \quad \ldots \quad (5c)$$

Defining dimensionless quantities

$$x' = \frac{x}{h}, \ y' = \frac{y}{h}, \ t' = \frac{kt}{\rho c h^2}, \ u' = \frac{u}{h}, \ v' = \frac{v}{h}, \quad \ldots \quad (6)$$

the heat conduction equation (1) and the thermal boundary conditions given by eqns. (2*a*) and (2*b*) take the form

$$\frac{\partial^2 T}{\partial x'^2} + \frac{\partial^2 T}{\partial y'^2} = \frac{\partial T}{\partial t'} \quad \ldots \quad (7)$$

$$\left.\begin{array}{l} T(x', 0, t') = 0 \\ T(x', 1, t') = H'(x', t') \end{array}\right\}. \quad \ldots \quad (8)$$

The form of eqns. (3) to (5) remains unaltered by the introduction of these primes. In what follows, we shall omit the primes.

3. Solution of the Problem

Writing $p \equiv \frac{\partial}{\partial x}$, $q \equiv \frac{\partial}{\partial t}$ and $D_y \equiv \frac{\partial}{\partial y}$, the heat conduction equation takes the form

$$(D_y^2 + p^2 - q)T = 0. \quad \ldots \quad (9)$$

Solving eqn. (9) by using boundary conditions (8)

$$T(x, y, t) = \frac{\sin y\sqrt{p^2-q}}{\sin \sqrt{p^2-q}} H(x, t). \quad \ldots \quad (10)$$

Writing eqns. (3) and (4) in terms of differential operators, we obtain the following differential equations after some simplification:

$$(D_y^2 + p^2)^2 u = lpq \frac{\sin y\sqrt{p^2-q}}{\sin \sqrt{p^2-q}} H(x, t) \quad \ldots \quad (11a)$$

$$(D_y^2 + p^2)^2 v = lD_y q \frac{\sin y\sqrt{p^2-q}}{\sin \sqrt{p^2-q}} H(x, t) \quad \ldots \quad (11b)$$

where

$$l = \frac{\alpha(1+\nu)}{(1-\nu)}.$$

Equations (11a) and (11b) are solved simultaneously for u and v by applying the boundary conditions (5). The expressions for u and v take the following form after a long but straightforward simplification:

$$
\begin{aligned}
u = {} & \frac{y\cos yp}{\Delta_1}\Big[(2\nu-1)p-p\cos^2 p+p\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\} \\
& -\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big]H(x,t) \\
& +\frac{\sin yp}{\Delta_1}\Big[\beta\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-p^2-2(1-\nu)(1-2\nu) \\
& -\frac{\beta p\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big]H(x,t) \\
& +\frac{y\sin yp}{\Delta_1}\Big[p^2-p\sin p\cos p-p\cos\sqrt{p^2-q}\{(1-2\nu)\sin p+p\cos p\} \\
& -\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{2(\nu-1)\cos p+p\sin p\}\Big]H(x,t) \\
& +\frac{lp\sin y\sqrt{p^2-q}}{q\sin\sqrt{p^2-q}}H(x,t) \qquad \ldots \quad (12)
\end{aligned}
$$

$$
\begin{aligned}
v = {} & \frac{\cos yp}{\Delta_1}\Big[\beta\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-p^2-2(1-\nu)(1-2\nu) \\
& -\frac{\beta p\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big]H(x,t) \\
& +\frac{1}{\Delta_1}\left(y\cos yp-\frac{\beta}{p}\sin yp\right)\Big[p^2-p\sin p\cos p-p\cos\sqrt{p^2-q}\{(1-2\nu)\sin p+p\cos p\} \\
& -\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{2(\nu-1)\cos p+p\sin p\}\Big]H(x,t) \\
& -\frac{1}{\Delta_1}\left(\frac{\beta\cos yp}{p}+y\sin yp\right)\Big[(2\nu-1)p-p\cos^2 p+p\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\} \\
& -\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big]H(x,t) \\
& +\frac{l\sqrt{p^2-q}\cos y\sqrt{p^2-q}}{q\sin\sqrt{p^2-q}}H(x,t) \qquad \ldots \quad (13)
\end{aligned}
$$

where

$$
\left.
\begin{aligned}
\Delta_1 &= \frac{q\sin\sqrt{p^2-q}}{4l\sqrt{p^2-q}}(4p^2-2\beta\cos 2p-\beta^2-1) \\
\text{and}\quad \beta &= 3-4\nu.
\end{aligned}
\right\} \qquad \ldots \quad (14)
$$

The stresses σ_{xx}, σ_{yy} and τ_{xy} can now be determined by substituting the values of u, v and T in the stress-displacement relations (4). The expressions for the stresses are as follows:

$$\frac{(1+\nu)(1-2\nu)}{E}\sigma_{xx} = \frac{1}{\Delta_1}[(1-2\nu)yp\cos yp+\nu(\beta-1)\sin yp]$$
$$\times\left[(2\nu-1)p-p\cos^2 p+p\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}\right.$$
$$\left.-\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\right]H(x,t)$$
$$+\frac{1}{\Delta_1}(1-2\nu)p\sin yp\left[\beta\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-p^2-2(1-\nu)(1-2\nu)\right.$$
$$\left.-\frac{\beta p\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\right]H(x,t)$$
$$+\frac{1}{\Delta_1}[(1-2\nu)yp\sin yp+\nu(1-\beta)\cos yp]\left[p^2-p\sin p\cos p\right.$$
$$\left.-p\cos\sqrt{p^2-q}\{(1-2\nu)\sin p+p\cos p\}-\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{2(\nu-1)\cos p+p\sin p\}\right]$$
$$\times H(x,t)+l(1-2\nu)\frac{(p^2-q)\sin y\sqrt{p^2-q}}{q\sin\sqrt{p^2-q}}H(x,t) \qquad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$\frac{(1+\nu)(1-2\nu)}{E}\sigma_{yy} = \frac{1}{\Delta_1}[(2\nu-1)yp\cos yp-(1-\nu)(1-\beta)\sin yp]$$
$$\times\left[(2\nu-1)p-p\cos^2 p+p\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}\right.$$
$$\left.-\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\right]H(x,t)$$
$$+\frac{1}{\Delta_1}(2\nu-1)p\sin yp\left[\beta\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-p^2-2(1-\nu)(1-2\nu)\right.$$
$$\left.-\frac{\beta p\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\right]H(x,t)$$
$$+\frac{1}{\Delta_1}[(1-\nu)(1-\beta)\cos yp+(2\nu-1)yp\sin yp]\left[p^2-p\sin p\cos p\right.$$
$$\left.-p\cos\sqrt{p^2-q}\{(1-2\nu)\sin p+p\cos p\}-\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{2(\nu-1)\cos p+p\sin p\}\right]$$
$$\times H(x,t)+\frac{(2\nu-1)lp^2\sin y\sqrt{p^2-q}}{q\sin\sqrt{p^2-q}}H(x,t) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$
\begin{aligned}
\frac{2(1+\nu)}{E}\tau_{xy} &= \frac{1}{\Delta_1}(\cos yp-\beta\cos yp-2yp\sin yp)\Big[(2\nu-1)p-p\cos^2 p \\
&\quad +p\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big] \\
&\quad \times H(x,t)+\frac{2p\cos yp}{\Delta_1}\Big[\beta\cos\sqrt{p^2-q}\{p\sin p+2(1-\nu)\cos p\}-p^2-2(1-\nu)(1-2\nu) \\
&\quad -\frac{\beta p\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{p\cos p+(2\nu-1)\sin p\}\Big]H(x,t) \\
&\quad +\frac{1}{\Delta_1}[2yp\cos yp+(1-\beta)\sin yp]\Big[p^2-p\sin p\cos p \\
&\quad -p\cos\sqrt{p^2-q}\{(1-2\nu)\sin p+p\cos p\} \\
&\quad -\frac{p^2\sin\sqrt{p^2-q}}{\sqrt{p^2-q}}\{2(\nu-1)\cos p+p\sin p\}\Big]H(x,t) \\
&\quad +\frac{2lp\sqrt{p^2-q}\cos y\sqrt{p^2-q}}{q\sin\sqrt{p^2-q}}H(x,t). \qquad (17)
\end{aligned}
$$

A similar expression for σ_{zz} can be easily obtained. It is easy to verify that the boundary conditions $u = v = 0$ at $y = 0$ and $\tau_{xy} = \sigma_{yy} = 0$ at $y = 1$ are satisfied.

Equations (10) and (12) to (17) determine the general solution to the given thermoelastic boundary value problem. In order to get the numerical results, we have only to prescribe the surface temperature function $H(x, t)$.

4. Numerical Example

As an illustrative example, set $H(x, t) = e^{-b|x|} . e^{-ct} . f(t)$ where b, c are positive constants and $f(t)$ is a polynomial in t of the form

$$f(t) = \sum_{n=0}^{n} a_n t^n.$$

Since x occurs only in $e^{-b|x|}$ in $H(x, t)$, p is replaced by $(-b)$ in the expressions for temperature, displacements and the stresses, and $e^{-b|x|}$ is written outside the expressions in the operator form.

Thus, we shall have expressions of the type

$$e^{-b|x|}F(q)[e^{-ct}f(t)]$$

which is equal to

$$e^{-b|x|}e^{-ct}F(q-c)f(t).$$

$F(q-c)$ is expanded as an infinite series in powers of q

i.e. $$F(q-c) = \sum_{0}^{\infty} \alpha_n q^n$$

where $q^n \equiv \frac{\partial^n}{\partial t^n}$, and α_n are functions of y only. Hence the solution reduces to the form

$$e^{-b|x|}e^{-ct}\left(\sum_{n=0}^{\infty} \alpha_n q^n\right) f(t). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Since $f(t)$ is a polynomial in t, we shall obtain the exact values of the elastic field even if the coefficients α_n in the infinite series $\sum_0^{\infty} \alpha_n q^n$ may be increasing in magnitude.

Clearly, the restriction on b is that it is not a root of the equation

$$4b^2 - 2\beta \cos 2b - \beta^2 - 1 = 0.$$

The expression for the temperature is

$$T = e^{-b|x|}e^{-ct}\frac{\sin y\sqrt{a^2-q}}{\sin\sqrt{a^2-q}} f(t) \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where

$$a^2 = b^2 + c.$$

Similar expressions for the displacements and stresses can be obtained.

For numerical calculations, we take $f(t) = t^2$, $\nu = 0{\cdot}25$, $b = 2\pi$ and $c = 0{\cdot}1$.

At $y = \frac{1}{2}$, the expressions (19) for T yields

$$T = e^{-2\pi|x|}e^{-(0\cdot1)t}[-0{\cdot}498933 + 0{\cdot}0103297q + 0{\cdot}101911q^2 + \ldots] t^2. \qquad (20)$$

Thus the values for the temperature for various values of x and t can be calculated (Table I).

TABLE I

Values of T at $y = \frac{1}{2}$ when $H(x, t) = e^{-b|x|} . e^{-ct} . f(t), f(t) = t^2, b = 2\pi, c = 0{\cdot}1$

t	x					
	0	0·1	0·2	0·3	0·4	0·5
1	−0·050615	−0·027002	−0·014406	−0·007685	−0·004099	−0·002187
2	−0·292130	−0·155845	−0·083149	−0·044360	−0·023662	−0·021623

The expressions for u and v at $y = 1$ are

$$\frac{u}{l} = e^{-2\pi|x|}e^{-(0\cdot1)t}[3{\cdot}707386 + 37{\cdot}440001q + 371{\cdot}724035q^2 + \ldots] t^2 \qquad \ldots \quad (21)$$

$$\frac{v}{l} = e^{-2\pi|x|}e^{-(0\cdot1)t}[15{\cdot}929118 + 159{\cdot}201952q + 1592{\cdot}636562q^2 + \ldots] t^2. \qquad (22)$$

Similar expressions can be obtained for the stresses at any value of y in $0 \leqslant y \leqslant 1$.

Since the function $H(x, t)$ is symmetrical in x, the values when x is negative are the same when x is positive.

It may be remarked that if the temperature function $H(x, t)$ is purely an exponential function of the form

$\Sigma(e^{-b_n|x|} . e^{-c_n t})$, the results are immediately determined.

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