SOME ABELIAN THEOREMS FOR THE DISTRIBUTIONAL MEIJER-LAPLACE TRANSFORMATION*

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(Communicated by R. S. Varma, F.N.A.)

(*Received 20 June 1970*)

The purpose of this work is to extend classical Meijer-Laplace transform to Schwartz (1966) distributions. In this paper we have proved initial and final value theorems for Meijer-Laplace transform. [By an 'initial (final)-value theorem' we mean a theorem that relates the initial (final) value of a distribution to the final (initial) value of the transform].

§ 1. INTRODUCTION

Two-sided Laplace transformation of distributions has been defined by Schwartz (1966) and Zemanian (1966). Cooper (1966) has defined the Laplace transform of distributions:

$$F(s) = \int_0^\infty e^{-st}\phi(t)\,dt. \quad \ldots \quad (1.1)$$

Here, we take the function $\phi(t)$ that is absolutely integrable over $-\infty < t < \infty$. We shall also impose that $\phi(t)$ is a right-sided and locally integrable function, which satisfies the following conditions:

(*a*) $\phi(t) = 0$ for $-\infty < t < T$.

(*b*) There exists a real number c such that $e^{-wct}G_{pq}^{hu}\left(2ct\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\phi(t)$ is absolutely integrable over $-\infty < t < \infty$, where w is a given positive quantity.

The Meijer-Laplace transformation is an operation $\mathcal{L}$ that assigns a function $F(s)$ of the complex variable s to each locally integrable function $\phi(t)$ that satisfies conditions (*a*) and (*b*). $\mathcal{L}$ is defined by

$$\mathcal{L}\phi(t) \stackrel{\Delta}{=} F(s) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} e^{-wst}G_{pq}^{hu}\left(2st\,\middle|\,\begin{matrix}a_1, a_2, \ldots a_p\\ b_1, b_2, \ldots b_q\end{matrix}\right)\phi(t)\,dt \quad \ldots \quad (1.2)$$

where $p+q < 2(h+u)$, further only sectors with exponentially small behaviour of G-function at infinity should be allowed.

The function $F(s)$ is called Meijer-Laplace transform of $\phi(t)$, where $G_{pq}^{hu} \times \left(2st\,\middle|\,\begin{matrix}a_1, a_2, \ldots, a_p\\ b_1, b_2, \ldots, b_q\end{matrix}\right)$ is Meijer's G-function, $\left[\text{in short, we write } G_{pq}^{hu}\left(2st\,\middle|\,\begin{matrix}a_p\\ b_q\end{matrix}\right)\right]$.

* The part of this work is supported by C.S.I.R.

The fact that $\phi(t) = 0$ for $-\infty < t < T$ permits us to write:

$$\mathcal{L}\phi(t) \overset{\Delta}{=} F(s) \overset{\Delta}{=} \int_T^\infty e^{-wst} G_{pq}^{hu}\left(2st \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)\phi(t)\,dt. \qquad \ldots \qquad (1.3)$$

We shall refer to (1.3) as a right-sided Meijer-Laplace transform in order to indicate that the lower limit on the integral (1.3) is a finite number.

If we put in (1.3) $T = 0$, $w = \frac{1}{2}$, $h = 2$, $u = 0$, $p = 1$, $q = 2$, $a_1 = a$, $b_1 = a$, $b_2 = 0$ and take G-function as

$$e^{-st/2} G_{1\,2}^{2\,0}\left(\frac{st}{2} \middle| \begin{matrix} a \\ a,\ 0 \end{matrix}\right) = e^{-st}$$

we will get (1.1).

In this paper, we shall establish some Abelian Theorems for the generalized Meijer-Laplace transformation. We prove initial and final value theorems for Meijer-Laplace transformation in § 2 and § 3, respectively. [By an 'initial (final)-value theorem' we mean a theorem that relates the initial (final) value of a distribution to the final (initial) value of the transform].

Our notations and terminology are the same as used in Schwartz (1966) and Zemanian (1965, 1966), x, y, t and σ are the real variables, $s = \sigma + iw$ is a complex variable. The space S' and D_R' are the same as defined by Zemanian (1965).

§ 2. An Initial Value Theorem for Meijer-Laplace Transformation

First, we shall prove an Abelian Theorem for the ordinary Meijer-Laplace transformation.

Theorem 2.1—Assume that the function $\phi(t)$ satisfies the following conditions:

(i) $\phi(t) \to 0$ as $t \to \infty$

(ii) $\frac{\phi(t)}{t^\gamma}$ is absolutely continuous on $0 \leqslant t < \infty$ and if there exists a complex number α and a real number $\gamma > -1$ such that

$$\lim_{t \to 0+} \frac{\phi(t)}{t^\gamma} = \alpha \qquad \ldots \qquad (2.1)$$

then

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\gamma+1} F(\sigma)}{G(2/w)} = \alpha \qquad \ldots \qquad (2.2)$$

where

$$G\left(\frac{2}{w}\right) \equiv G_{p+1,\,q}^{h,\,u+1}\left(\frac{2}{w} \middle| \begin{matrix} -\gamma, a_1, \ldots, a_p \\ b_1, \ldots, \quad \ldots b_q \end{matrix}\right)$$

provided $p+q < 2\,(h+u)$, $\operatorname{Re} b_j + \gamma + 1 > 0$, $j = (1, 2, \ldots h)$ and

$$F(\sigma) \overset{\Delta}{=} \mathcal{L}\phi(t) \overset{\Delta}{=} \int_0^\infty e^{-w\sigma t} G_{pq}^{hu}\left(2\sigma t \middle| \begin{matrix} a_p \\ b_p \end{matrix}\right)\phi(t)\,dt.$$

Note that condition (ii) assures the existence of limit (i).

PROOF: First note that for $\gamma > -1$, and

$$\int_0^\infty t^\gamma e^{-w\sigma t} G_{pq}^{hu}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)dt = (w\sigma)^{-\gamma-1} G_{p+1,\,q}^{h,\,u+1}\left(\frac{2}{w}\left|\begin{matrix}-\gamma, a_1, \ldots, a_p\\ b_1, \ldots, \ldots b_q\end{matrix}\right.\right)$$

by Erdélyi (1954, p. 419, (5)).

Thus by using this integral and assuming that $\sigma > 0$ and with $\sigma t = z$, and $\sigma y = k$ $(0 < y < \infty)$, we may write

$$\left|(w\sigma)^{\gamma+1}F(\sigma) - \alpha G\left(\frac{2}{w}\right)\right| = (w\sigma)^{\gamma+1}\left|\int_0^\infty e^{-w\sigma t} G_{pq}^{hu}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\phi(t)\,dt\right.$$

$$\left. -\alpha\int_0^\infty t^\gamma e^{-w\sigma t} G_{pq}^{hu}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)dt\right| \quad \ldots \quad \ldots \quad \ldots \quad (2.3)$$

$$\leqslant (w)^{\gamma+1}\int_0^k\left|z^\gamma e^{-wz} G_{pq}^{hu}\left(2z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right|dz \sup_{0<t<y}\left|\frac{\phi(t)}{t^\gamma} - \alpha\right|$$

$$+ (w\sigma)^{\gamma+1}\int_y^\infty\left|t^\gamma e^{-w\sigma t} G_{pq}^{hu}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right|dt \sup_{y<t<\infty}\left|\frac{\phi(t)}{t^\gamma} - \alpha\right|. \quad \ldots \quad (2.4)$$

Since as $z \to 0+$, $G_{pq}^{hu}\left(z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right) = O(|z|^\beta)$, where $\beta = \min$ (Re $b_1, b_2 \ldots b_h$) (Erdélyi 1953, p. 212, (8)) and according to Erdélyi (1953, p. 212, (11)), we know that G function vanishes exponentially as $z \to \infty$, if $h+u > \frac{p}{2}+\frac{q}{2}$, and $|\arg z| < \left(h+u-\frac{p}{2}-\frac{q}{2}\right)\pi$ (in case of Meijer-Laplace transform, we take $u = 0$). Since $\frac{\phi(t)}{t^\gamma} \to \alpha$ as $t \to 0+$ we have for $\epsilon > 0$ there exists $\delta > 0$ such that $\left|\frac{\phi(t)}{t^\gamma} - \alpha\right| < \epsilon$ for all $0 < t < \delta$, and $y < \delta$.

If $\beta+\gamma > -1$, the first integral in the right-hand side of (2.4) is convergent. Therefore, given an $\epsilon > 0$, the first term on the right-hand side of (2.4), which is independent of σ, can be made less than $\epsilon/2$, by choosing y small enough. Moreover, $\left|e^{-wz}z^\gamma G_{pq}^{hu}\left(2z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right|$ is bounded on $0 < z < \infty$, provided $\beta+\gamma$ is positive. Hence, with y fixed, the second term on the right-hand side of (2.4) is less than $\epsilon/2$ for all sufficiently large σ. This proves (2.2).

In extending these results to distributions, we shall need the following Lemma.

Lemma 1—Meijer-Laplace transformation of distributions has been defined by Misra (1970). Let $\phi(t)$ be a Meijer-Laplace transformable distribution (Misra 1970) with its support in $y \leqslant t < \infty$, where $y > 0$. Let the

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real number c be such that $e^{-wct}\phi(t)$ is in S'. Then for

$$c+1 < \sigma < \infty$$

$$|F(\sigma)| \leqslant M(\sigma)^r e^{-w(\sigma-c)\tau}$$

where M is a constant and τ is any real number satisfying $0 < \tau < y$.

PROOF: For $\sigma > c$

$$F(\sigma) = < \phi(t)e^{-wct}, \lambda(t)e^{-w(\sigma-c)t}G^{hu}_{pq}\left(2\sigma t \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) >$$

The support of $\lambda(t)$ is contained in $x \leqslant t < \infty$, where x is a fixed number satisfying $0 < x < y$. Moreover, $F(\sigma)$ is of slow growth as $\sigma \to \infty$. A distribution of slow growth satisfies a boundedness condition of the type given by Zemanian (1966, p. 109). Thus, we have

$$\left|F(\sigma)\right| \leqslant C \sup_{x \leqslant t < \infty} \left|(1+t^2)^r D_t^r \left[\lambda(t)e^{-w(\sigma-c)t}G^{hu}_{pq}\left(2\sigma t \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)\right]\right|$$

$$= C \sup_{x \leqslant t < \infty} \left|(1+t^2)^r \sum_{n=0}^{r} \binom{r}{n} \overset{r-n}{\lambda(t)}\, D_t^n \left[e^{-w(\sigma-c)t}G^{hu}_{pq}\left(2\sigma t \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)\right]\right|$$

$$= C \sup_{x \leqslant t < \infty} \left|(1+t^2)^r \sum_{n=0}^{r} \binom{r}{n} \overset{r-n}{\lambda(t)} \sum_{k=0}^{n} \binom{n}{k} [-w(\sigma-c)]^{n-k}\right.$$

$$\left. . e^{-w(\sigma-c)(t-x)} e^{-w(\sigma-c)x} . (2\sigma)^k . D_z^k G^{hu}_{pq}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)\right| \quad \ldots \quad (2.5)$$

where

$$D_t = \frac{d}{dt}, k = 0, 1, 2, \ldots, n, \text{ and } n = 0, 1, 2, \ldots, r \text{ and } D_z^k = \left(\frac{\partial}{\partial z}\right)^k.$$

The constant C and the integer r depend only on ϕ. Moreover, if τ is a fixed number such that $0 < \tau \leqslant x$, then

$$[w(\sigma-c)]^{n-k}\ e^{-w(\sigma-c)x} \leqslant B\ e^{-w(\sigma-c)\tau},\ \sigma > c$$

where B is a constant independent of k. Furthermore, with $\sigma > c+1$

$$e^{-w(\sigma-c)(t-x)} \leqslant e^{-w(t-x)};\ t \geqslant x$$

and

$D_z^k G^{hu}_{pq}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = 0(1)$; $z \to 0+$ for $k = 0, 1, 2, \ldots$; provided that $(\beta - k)$ is positive; where $\beta = \min(\text{Re } b_1, b_2, \ldots b_h)$ and $D_z^k G^{hu}_{pq}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right) = 0(1)$; $z \to \infty$.

This can be seen by differentiation formula (Meijer 1952, p. 375) and asymptotic behaviour of G-function (Erdélyi 1953, p. 212 eqns. (8), (11)). Since $D_z^k G^{hu}_{pq}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)$ is continuous on $0 < z < \infty$, it now follows that it is also bounded there. When $x > 0$, $D_z^k G^{hu}_{pq}\left(z \middle| \begin{matrix} a_p \\ b_q \end{matrix}\right)$ will also be bounded on $x \leqslant z < \infty$.

Now, the right-hand side of (2.5) is dominated by

$$CB(\sigma)^r e^{-w(\sigma-c)\tau} \sup_{x \leqslant t < \infty} \left| (1+t^2)^r \sum_{n=0}^{r} \binom{r}{n} \lambda(t)^{r-n} \sum_{k=0}^{n} \binom{n}{k} e^{-w(t-x)} D_z^k G_{pq}^{hu}\left(z \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right) \right|.$$

But the function of t inside the last Sup symbol is bounded for all $t \geqslant x$. Consequently, the last expression is less than $M(\sigma)^r\, e^{-w(\sigma-c)\tau}$, where M is a sufficiently large constant.

Theorem 2.2—Let $\phi(t)$ be a Meijer-Laplace transformable distribution having its support in $0 \leqslant t < \infty$ and assume that over some neighbourhood of the origin $\phi(t)$ is a regular distribution corresponding to a (Lebesgue) integrable function $h(t)$. Also assume that there exists a complex number α, such that

$$\lim_{t \to 0+} \frac{h(t)}{t^r} = \alpha$$

then

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\gamma+1}}{G(2/w)} = \alpha \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2.6)$$

where $G(2/w)$ is defined in Theorem 2.1
and

$$F(\sigma) \stackrel{\Delta}{=} \mathcal{L}\phi(t).$$

PROOF: Let the neighbourhood over which $\phi(t)$ is a regular distribution be $-\infty < t < T (T > 0)$ and let $0 < y < T$. Then $\phi(t)$ can be decomposed into

$$\phi(t) = \phi_1(t) + \phi_2(t)$$

where the supports of $\phi_1(t)$ and $\phi_2(t)$ are contained in the respective intervals $0 \leqslant t \leqslant y$ and $y \leqslant t < \infty$. Thus, $\phi_1(t)$ is a regular distribution that corresponds to $h(t)$ for $0 < t < y$. Let

$$F_1(\sigma) \stackrel{\Delta}{=} \mathcal{L}\phi_1(t) \text{ and } F_2(\sigma) \stackrel{\Delta}{=} \mathcal{L}\phi_2(t).$$

It follows from Lemma 1 that

$$(w\sigma)^{\gamma+1} F_2(\sigma) < M(w)^{\gamma+1} (\sigma)^{\gamma+r+1} e^{-w(\sigma-c)\tau} \to 0, \quad \text{as } \sigma \to \infty. \quad \ldots \quad (2.7)$$

Thus, by our convention, $\lim\limits_{t \to 0+} \dfrac{\phi_2(t)}{t^\gamma} = 0.$

Since the ordinary Meijer-Laplace transform of a (Lebesgue) integrable function is identical to distributional Meijer-Laplace transform of the corresponding regular distribution, we have now from Theorem 2.1 that

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\gamma+1} F_1(\sigma)}{G(2/w)} = \alpha. \qquad \ldots \quad \ldots \quad \ldots \quad (2.8)$$

Now, $F(\sigma) = F_1(\sigma) + F_2(\sigma)$ and thus (2.7) and (2.8) prove (2.6).

§ 3. A Final Value Theorem for the Meijer-Laplace Transformation

Theorem 3.1—Let $\phi(t)$ be a measurable function on $0 < t < \infty$, satisfying the conditions (*a*) and (*b*) and if there exist a complex number α and a real number γ $(\gamma > -1)$ such that

$$\lim_{t\to\infty} \frac{\phi(t)}{t^\gamma} = \alpha \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

then

$$\lim_{\sigma\to 0+} \frac{(w\sigma)^{\gamma+1}F(\sigma)}{G(2/w)} = \alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

where $G(2/w)$ and $F(\sigma)$ are the same as defined in Theorem 2.1.

Proof: Note that (3.1) indicates that $\phi(t)$ is a function of slow growth and its Meijer-Laplace transform converges for $s > 0$ and $\beta+1 > 0$. Assuming σ and y being the real positive quantities, we proceed as in the proof of Theorem 2.1 to obtain

$$\begin{aligned}
|(w\sigma)^{\gamma+1}F(\sigma)-\alpha G(2/w)| &= (w\sigma)^{\gamma+1}\left|\int_0^\infty e^{-w\sigma t}G^{hu}_{pq}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)[\phi(t)-\alpha t^\gamma]\right| \\
&\leqslant (w)^{\gamma+1}\int_0^{\sigma y}\left|(\sigma t)^\gamma e^{-w\sigma t}G^{hu}_{pq}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right| d(\sigma t) \sup_{0<t<y}\left|\frac{\phi(t)}{t^\gamma}\alpha\right| \\
&\quad + (w)^{\gamma+1}\int_y^\infty\left|(\sigma t)^\gamma e^{-w\sigma t}G^{hu}_{pq}\left(2\sigma t\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right| d(\sigma t) \sup_{y<t<\infty}\left|\frac{\phi(t)}{t^\gamma}-\alpha\right| \\
&= I_1+I_2.
\end{aligned}$$

We know that

$$\left|z^\gamma e^{-wz}G^{hu}_{pq}\left(2z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right| < Az^{\beta+\gamma}$$

where A is a large constant and $\beta+\gamma$ is positive, $\beta = \min$ (Re, b_1, b_2, b_h) (Erdélyi 1953, p. 212, (8)]

$$I_1 \leqslant A(w\sigma)^{\gamma+1}(\sigma)^\beta\int_0^y t^{\beta+\gamma}dt \sup_{0<t<y}\left|\frac{\phi(t)}{t^\gamma}-\alpha\right|$$

$$I_2 \leqslant (w)^{\gamma+1}\int_0^\infty\left|z^\gamma e^{-wz}G^{hu}_{pq}\left(2z\left|\begin{matrix}a_p\\ b_q\end{matrix}\right.\right)\right| dz \sup_{y<t<\infty}\left|\frac{\phi(t)}{t^\gamma}-\alpha\right|.$$

Both the integrals of I_1 and I_2 are convergent provided $\beta+\gamma > -1$ and I_2 is independent of σ. Therefore, given an $\epsilon > 0$ the I_2 can be made less than $\epsilon/2$ by choosing y sufficiently large. Then, there will exist a $K' > 0$, such that I_1 is also made less than $\epsilon/2$ for $0 < \sigma < K'$. This proves (3.2).

Theorem 3.2—Let $\phi(t)$ be a Meijer-Laplace transformable distribution in D'_R and assume that, over some semi-infinite interval $\tau < t < \infty$, $\phi(t)$ is a regular distribution corresponding to a locally integrable function $h(t)$ that satisfies the condition (*b*). Also, assume that there exist a complex number α

and real number γ ($\gamma > -1$), such that

$$\lim_{t \to \infty} \frac{h(t)}{t^{\gamma}} = \alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

then

$$\lim_{\sigma \to 0+} \frac{(w\sigma)^{\gamma+1} F(\sigma)}{G(2/w)} = \alpha. \quad \ldots \quad \ldots \quad \ldots \quad \ldots (3.4)$$

PROOF: $\phi(t)$ can be decomposed into

$$\phi(t) = \phi_1(t) + \phi_2(t)$$

where $\phi_2(t)$ has its support contained in $y \leqslant t < \infty$ ($y > \tau$) and $\phi_1(t)$ has a bounded support that does not extend to the right of $t = y$. $F_1(s) \stackrel{\Delta}{=} \mathcal{L}\phi_1(t)$ is then an entire function of s (by taking $p+q < 2(h+u)$, in G-function), so that

$$\lim_{\sigma \to 0+} (w\sigma)^{\gamma+1} F_1(\sigma) = 0.$$

Therefore, with

$$F_2(s) \stackrel{\Delta}{=} \mathcal{L}\phi_2(t) \stackrel{\Delta}{=} \mathcal{L}[h(t) \mid +(t-y)]$$

we have

$$\lim_{\sigma \to 0+} (w\sigma)^{\gamma+1} F(\sigma) = \lim_{\sigma \to 0+} (w\sigma)^{\gamma+1} F_2(\sigma).$$

The right-hand side equals $\alpha F(2/w)$, according to Theorem 3.1. This proves 3.4).

ACKNOWLEDGEMENT

The author is grateful to Professor K. M. Saksena and Professor A. H. Zemanian for their support and encouragement.

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