THERMAL STRESSES IN AN ELASTIC CONE DUE TO A MIXED THERMAL BOUNDARY CONDITION

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This paper is concerned with investigation of thermal stresses in an elastic cone due to a mixed thermal loading on its surface. Distribution of temperature on its bounding surface and in the interior is obtained by applying Wiener-Hopf technique to heat conduction equation. The solution for temperature distribution thus obtained is shown to tally with the results for semi-infinite body in the limiting case. Thermal stresses are then obtained for this temperature distribution.

1. INTRODUCTION

The stress analysis of circular cones and of solids of revolution bounded by one or two cones and of bodies bounded by two co-axial conical surfaces with a common vertex has been the object of numerous investigations in the classical linear theories of elasticity and thermo-elasticity. Solutions for a semi-infinite solid cone under a concentrated force of arbitrary orientation applied at the vertex were established by Michell (1900). Foppl (1921) studied the analogous pure torsion problem. Michell's results were extended by Neuber (1934) to the case of conical shell of varying thickness and obtained the solutions for the flexure of the shell due to a concentrated couple at the apex. Knops (1958) has rederived the results given by Neuber (1934) in a different manner.

Certain axisymmetric problems concerning solid and hollow cones subjected to a quite general axially symmetric external loading condition distributed over the conical boundary have been considered by Morgan (1954, 1956). In his first paper (Morgan 1954) the lateral loading is assumed to be proportional to an arbitrary real power of the distance from the vertex while in the second paper (Morgan 1956) piecewise continuous surface tractions have been assumed such that they vanish identically beyond a fixed distance from the apex. The procedure applied by Morgan (1956) is the generalization of the method suggested by Tranter (1948) in connection with the similar two-dimensional equilibrium problems for an elastic wedge. Morgan's scheme rests on the use of the Mellin transform together with Love's stress function for the torsionless rotational symmetry. Muki and Sternberg (1960) have obtained the solutions for the steady state thermal stresses in an elastic cone subjected to a discontinuous surface temperature, the tip of the cone (up to a given distance from the apex) is held at a constant temperature while the remainder of the boundary is maintained at another uniform temperature.

Instead of applying the Love's stress function they have treated the thermoelastic problem on the basis of the axisymmetric version of Boussenesq Papkovich's general solution to the isothermal elastostatic field equations, which was readily modified to account for the presence of temperature field and was introduced by McDowell and Sternberg (1957). In contrast to Morgan's treatment which is purely formal with the aid of Mellin transform and of the corresponding inversion theorem they have deduced the complex integral representations of the thermal stresses into real integrals involving Legendre functions of the first kind of complex degree. The validity of the solutions is examined and numerical evaluation is discussed in detail with the help of highly cumbersome, voluminous computational work.

In the present work, we seek the solutions for the thermal stresses in a solid elastic circular cone which is free from external loads, due to a mixed thermal loading on the boundary surface of the cone: the tip of the cone (up to a given distance from the apex) is held at a given temperature distribution while the remainder of the surface is thermally insulated, Mellin transform technique together with Boussinesq Papkovich's scalar potential functions are employed to obtain the complex integral representation of the temperature stress required. Weiner-Hopf technique is employed to obtain the temperature on the surface and in the interior of the cone.

2. Formulation of the Problem

Consider a medium occupying an isotropic homogeneous region of space D with boundary B. The spherical polar-coordinates (r, θ, φ) are introduced by means of the mapping

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$
$$0 \leqslant r < \infty, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi.$$

Suppose the axis of the cone in question coincides with z-axis. Let its apex be at the origin and let $\alpha(0 < r \leqslant \pi/2)$ be its semi-opening angle. Further, let us introduce the following auxiliary variables

$$p = \cos\theta, \quad q = \sin\theta$$
$$p_0 = \cos\alpha, \quad q_0 = \sin\alpha.$$

The region D is then in this case given by

$$0 < r < \infty, \quad p_0 < p < 1, \quad (0 < p_0 < 1)$$

while the boundary B corresponds to $p = p_0$.

The cone is subjected to the following thermal and elastic loadings (Fig. 1)

$$\left.\begin{aligned} &T(r, p_0) = m(r), \quad 0 < r < 1 \\ &\frac{\partial T}{\partial_\theta}(r, p_0) = 0, \quad r > 1 \\ &\left.\begin{aligned}\sigma_{\theta\theta}(r, p_0) &= 0 \\ \sigma_{r\theta}(r, p_0) &= 0\end{aligned}\right\} \quad 0 < r < \infty \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (2.1)$$

where $m(r)$ is the given function.

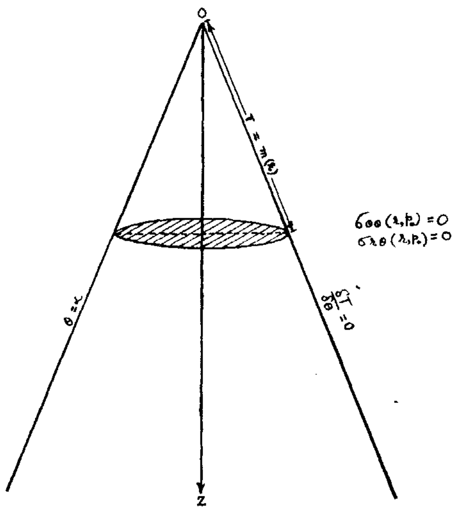

FIG. 1. Thermal loading on the surface.

By virtue of rotationally symmetric character of the problem, the temperature field is independent of the longitude φ and satisfies Laplace's equation, which govern the steady state heat conduction of a thermally homogeneous and isotropic solid. Thus

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial T}{\partial p}\right] = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (2.2)$$

must hold in the region $0 < r < \infty$.

The equation of thermoelastic displacement potential F when referred to spherical polar coordinates takes the form

$$\frac{\partial^2 F}{\partial r^2}+\frac{2}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial p}\left[(1-p^2)\frac{\partial F}{\partial p}\right] = mT \qquad \ldots \quad \ldots \quad (2.3)$$

with

$$m = \frac{2(1+\nu)}{1-2\nu}\alpha_t.$$

The components of stress tensor and displacement vector corresponding to the function F are given by the following relations:

$$\left.\begin{aligned}
\bar{\sigma}_{rr} &= \frac{\partial^2 F}{\partial r^2}\\
\frac{\bar{\sigma}_{\theta\theta}}{2\mu} &= -\frac{q}{r^2}\frac{\partial^2 F}{\partial p^2}-\frac{p}{r^2}\frac{\partial F}{\partial p}+\frac{1}{r}\frac{\partial F}{\partial r}\\
\frac{\bar{\sigma}_{\phi\phi}}{2\mu} &= \frac{p}{r^2}\frac{\partial F}{\partial p}+\frac{1}{r}\frac{\partial F}{\partial r}\\
\frac{\bar{\sigma}_{r\theta}}{2\mu} &= q\left\{\frac{1}{r^2}\frac{\partial F}{\partial p}-\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial p}\right\}\\
\bar{\sigma}_{r\phi} &= 0
\end{aligned}\right\} \quad \ldots \quad (2.4)$$

$$\left.\begin{aligned}
\bar{u}_r &= \frac{\partial F}{\partial r}\\
\bar{u}_\theta &= -\frac{q}{r}\frac{\partial F}{\partial p}\\
\bar{u}_\phi &= 0
\end{aligned}\right\}. \quad \ldots \quad (2.5)$$

For the removal of the stress 'residuals' on the surface of the cone produced by temperature distribution T, the Boussinesq-Papkovich approach is used, according to which the general solution of the displacement equation of equilibrium in the case of torsion-free rotational symmetry and absence of body forces admits the representation as

$$u_i = \phi_{,i}-(3-4\nu)\psi_i+X_{j,ji} \quad (i, j = 1, 2, 3) \quad \ldots \quad (2.6)$$

$$\phi_{,ij} = 0, \quad \psi_{,ij} = 0 \quad \ldots \quad (2.7)$$

where ν is the Poisson's ratio and the subscript comma denotes partial differentiation.

In the case of axisymmetry about z-axis, the general solution (2.6) and (2.7) of the isothermal elastostatic field equation remains complete if ϕ and ψ are restricted by

$$\left.\begin{aligned}
\phi &= \phi(r, p), \quad \psi_3 = \psi(r, p)\\
\psi_1 &= \psi_2 = 0
\end{aligned}\right\} \quad \ldots \quad (2.8)$$

whence (2.7) now reduces to

$$\left.\begin{aligned}
\nabla^2\phi(r, p) &= 0\\
\nabla^2\psi(r, p) &= 0
\end{aligned}\right\} \quad \ldots \quad (2.9)$$

in which the Laplacian operator ∇^2 is given as

$$\nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial p}+\frac{1}{r^2}\left[(1-p^2)\frac{\partial}{\partial p}\right]. \quad \ldots \quad (2.10)$$

Equation (2.6) when referred to spherical polar coordinates becomes.

$$\left.\begin{aligned}\bar{\bar{u}}_r &= \frac{\partial\phi}{\partial_r}+pr\frac{\partial\psi}{\partial_r}-(3-4\nu)p\psi\\ \bar{\bar{u}}_\theta &= -\frac{q}{r}\frac{\partial\phi}{\partial p}+pq\frac{\partial\psi}{\partial p}+(3-4\nu)q\psi\\ \bar{\bar{u}}_\phi &= 0\end{aligned}\right\} \quad \ldots \quad \ldots \quad (2.11)$$

The corresponding stress field in terms of the stress function $\phi\ (r, p)$ and $\psi\ (r, p)$ is given as

$$\left.\begin{aligned}\frac{\bar{\bar{\sigma}}_{rr}}{2\mu} &= \frac{\partial^2\phi}{\partial r^2}+pr\frac{\partial^2\psi}{\partial r^2}-2(1-\nu)p\frac{\partial\psi}{\partial r}-\frac{2q}{r}\frac{\partial\psi}{\partial p}\\ \frac{\bar{\bar{\sigma}}_{\theta\theta}}{2\mu} &= \frac{q}{r^2}\frac{\partial^2\phi}{\partial p^2}-\frac{p}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{pq^2}{r}\frac{\partial^2\psi}{\partial p^2}\\ &\qquad +(1-2\nu)p\frac{\partial\psi}{\partial r}-\frac{1}{r}\{1+(1-2\nu)q^2\}\frac{\partial\psi}{\partial p}\\ \bar{\bar{\sigma}}_{\phi\phi} &= -\frac{p}{r^2}\frac{\partial\phi}{\partial p}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{1}{r}\frac{\partial\psi}{\partial p}+(1-2\nu)\left\{p\frac{\partial\psi}{\partial r}+\frac{q}{r}\frac{\partial\psi}{\partial p}\right\}\\ \bar{\bar{\sigma}}_{r\theta} &= q\left\{\frac{1}{r_r}\frac{\partial\phi}{\partial p}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial p}-\frac{1}{p}\frac{\partial^2\psi}{\partial r\partial p}+(1-2\nu)\frac{\partial\psi}{\partial r}\right.\\ &\qquad \left.+2(1-\nu)\frac{p}{r}\frac{\partial\psi}{\partial p}\right\}\\ \bar{\bar{\sigma}}_{r\phi} &= 0, \quad \bar{\bar{\sigma}}_{\theta\phi} = 0\end{aligned}\right\} \quad \ldots \quad (2.12)$$

The resultant state of displacement and stress is obtained by adding (2.5) to (2.11) and (2.4) to (2.12). The problem under consideration thus reduces to the determination of the solutions F, ϕ, ψ, T.

3. Application of the Mellin Transform

Let $\hat{F}(p, s)$, $\hat{\phi}(p, s)$, $\hat{\psi}(p, s)$ and $\hat{T}(p, s)$ be the Mellin transforms with respect to r of $r^{-2}F(r, p)$, $r^{-2}\phi(r, p)$, $r^{-1}\psi(r, p)$ and $T(r, p)$; s being the transform parameter.

Thus

$$\left.\begin{aligned}\hat{F}(p, s) &= \int_0^\infty F(r, p)r^{s-3}\, dr\\ \hat{\phi}(p, s) &= \int_0^\infty \phi(r, p)r^{s-3}\, dr\\ \hat{\psi}(p, s) &= \int_0^\infty \psi(r, p)r^{s-2}\, dr\\ \hat{T}(p, s) &= \int_0^\infty T(r, p)r^{s-1}\, dr\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

Further, following Muki and Sternberg (1960) we assume that the order of each of

$$r^{-2}F,\ r^{-1}\frac{\partial F}{\partial r},\ r^{-1}\frac{\partial \phi}{\partial r},\ r^{-1}\psi,\ \frac{\partial \psi}{\partial r} \text{ is } r^{-s^*} \text{ as } r \to \infty\ (s^* > 0). \quad \ldots \quad (3.2)$$

In addition, we assume that the six functions appearing in (3.2) remain bounded as $r \to 0$. Then (3.1) and integration by parts imply the identities

$$\left.\begin{aligned}
&\int_0^\infty \frac{\partial F}{\partial r} r^{s-2}\,dr = -(s-2)\hat{F},\ \int_0^\infty \frac{\partial^2 F}{\partial r^2} r^{s-1}\,dr = (s-1)(s-2)\hat{F} \\
&\int_0^\infty \frac{\partial \phi}{\partial r} r^{s-2}\,dr = -(s-2)\hat{\phi},\ \int_0^\infty \frac{\partial^2 \phi}{\partial r^2} r^{s-1}\,dr = (s-1)(s-2)\hat{\phi} \\
&\int_0^\infty \frac{\partial \psi}{\partial r} r^{s-1}\,dr = -(s-1)\hat{\psi},\ \int_0^\infty \frac{\partial^2 \psi}{\partial r^2} r^{s}\,dr = s(s-1)\hat{\psi}
\end{aligned}\right\}. \quad (3.3)$$

4. Solution of the Conduction Equation of Heat

Applying Mellin transform to eqn. (2.2) we obtain in view of (3.3) the following Legendre equation

$$\frac{d}{dp}\left[(1-p^2)\frac{d\hat{T}}{dp}\right] + s(s-1)\hat{T} = 0. \quad \ldots \quad \ldots \quad \ldots \quad (4.1)$$

The general solution of this equation is given by

$$\hat{T}(p_1 s) = A(s)P_{s-1}(p) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4.2)$$

where $A(s)$ is arbitrary function of the transform parameter 's' and is determined from the boundary conditions.

5. Reduction to Weiner-Hopf Equation

We rewrite the boundary conditions (2.1) as

$$\left.\begin{aligned} T(r, p_0) &= m_+(r),\ 0 < r < 1 \\ &= m_-(r),\quad r > 1 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5.1)$$

$$\left.\begin{aligned} \frac{\partial T}{\partial \theta}(r, p_0) &= n_+(r),\ 0 < r < 1 \\ &= 0,\quad r > 1 \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (5.2)$$

In the above equation $m_+(r)$ is of course the given function $m(r)$ but $m_-(r)$ and $n_+(r)$ are unknown functions of r for $r \to 1$ and $0 < r < 1$ respectively. We assume the following order properties of the functions appearing in (5.1) and (5.2):

$$m_+(r) = O(1), \text{ as } r \to 0^+,\ r \to 1^-$$

$$m_-(r) = \begin{cases} O(1), & r \to 1^+ \\ O(r^{-1}), & r \to \infty \end{cases},\quad n_+(r) = \begin{cases} O(1), & r \to 0^+ \\ O(1-r)^{-\frac{1}{3}}, & r \to 1^- \end{cases}$$

The Mellin transform of (5.1) and (5.2) yields

$$\left.\begin{aligned}\hat{T}(p_0, s) &= M_+(s)+M_-(s)\\ \frac{\partial \hat{T}}{\partial \theta}(p_0, s) &= N_+(s)\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

where

$$\left.\begin{aligned}M_+(s) &= \int_0^1 m_+(r)r^{s-1}\,dr, \quad M_-(s) = \int_1^0 m_-(r)r^{s-1}\,dr\\ N_+(s) &= \int_0^1 n_+(r)r^{s-1}\,dr\end{aligned}\right\}. \quad \ldots \quad (5.4)$$

From eqns. (4.2) and (5.3) we obtain

$$\left.\begin{aligned}A(s)P_{s-1}(p_0) &= M_+(s)+M_-(s)\\ A(s)P^1_{s-1}(p_0) &= N_+(s)\end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (5.5)$$

Eliminating $A(s)$ between the two equations we obtain

$$K(s)[M_+(s)+M_-(s)] = N_+(s) \quad \ldots \quad \ldots \quad \ldots \quad (5.6)$$

where

$$K(s) = \frac{P^1_{s-1}(p_0)}{P_{s-1}(p_0)}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.7)$$

Now in (5.6), $M_+(s)$ and $K(s)$ are known, whereas $M_-(s)$ and $N_+(s)$ are to be determined. If it is shown that (5.6) holds in the strip $0 < \text{Re}(s) < 1$ that $M_+(s)$ and $N_+(s)$ are regular in $\text{Re}(s) > 0$ and that $M_-(s)$ is regular in $\text{Re}(s) < 1$. The determination of $M_-(s)$ and $N_+(s)$ is then possible by the Wiener-Hopf technique if $K(s)$ is factorized in the form

$$K(s) = \frac{K_-(s)}{K_+(s)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5.8)$$

in which $K_+(s)$ and $K_-(s)$ are regular and zeroless in $\text{Re}(s) > 0$ and in $\text{Re}(s) < 1$ respectively. From (5.7) and the asymptotic forms of the Legendre function it follows that $|K(s)| = O(|s|^p)$ where $p = 2$ unless $\alpha = \pi/2$ in which case $p = 1$. Thus it is possible (Noble 1958, p. 42) to perform the factorization in (5.8). Hence (5.6) can be rearranged in the form

$$K_+(s)N_+(s)-K_-(s)M_-(s)-K_-(s)M_+(s) = 0. \quad \ldots \quad \ldots \quad (5.9)$$

Next (Noble 1958, p. 13) if $|K_-(s)\ M_+(s)| = O(|s|^p)$; $p > 0$ as $s \to \infty$ in the strip $0 < \text{Re}(s) < 1$, then the function $K_-(s)\ M_+(s) = f(s)$, say, can be decomposed into the sum $f_+(s)+f_-(s)$ in which $f_+(s)$ and $f_-(s)$ are regular in $\text{Re}(s) > 0$ and $\text{Re}(s) < 1$ respectively. Then (5.9) becomes

$$K_+(s)N_+(s)-f_+(s) = K_-(s)M_-(s)+f_-(s). \quad \ldots \quad \ldots \quad (5.10)$$

Now the left-hand side of (5.10) is regular in $\text{Re}(s) > 0$, the right side in $\text{Re}(s) < 1$. Since the two sides are equal in the strip $0 < \text{Re}(s) < 1$, they are analytic continuations of each other into the whole 's' plane.

Thus an entire function $J(s)$ is defined, the representations of which are the two sides of (5.10) in right and left half planes. Thus

$$J(s) = K_+(s)N_+(s) - f_+(s) = K_-(s)M_-(s) + f_-(s). \quad .. \quad (5.11)$$

Now suppose that it can be shown that

$$|K_+(s)N_+(s) - f_+(s)| < |s|^p \text{ as } s \to \infty, \operatorname{Re}(s) > 1$$
$$|K_-(s)M_-(s) + f_-(s)| < |s|^q \text{ as } s \to \infty, \operatorname{Re}(s) < 1. \quad .. \quad (5.12)$$

Then by the extended form of Liouvilles theorem $J(s)$ is a polynomial $P(s)$ of degree less than that or equal to the integral part of min (p, q), i.e.

$$K_+(s)N_+(s) - f_+(s) = P(s)$$
$$K_-(s)M_-(s) + f_-(s) = P(s)$$

$$\left. \begin{aligned} N_+(s) &= \frac{P(s)+f_+(s)}{K_+(s)} \\ M_-(s) &= \frac{P(s)-f_-(s)}{K_-(s)} \end{aligned} \right\}. \quad .. \quad .. \quad .. \quad (5.13)$$

Making an appeal to the inversion theorem of Mellin transform the functions $m_-(r)$ and $n_+(r)$ are given by

$$\left. \begin{aligned} n_+(r) &= \frac{1}{2\pi i} \int_\gamma N_+(s) r^{-s}\, ds \\ m_-(r) &= \frac{1}{2\pi i} \int_\gamma M_-(s) r^{-s}\, ds \end{aligned} \right\}. \quad .. \quad .. \quad .. \quad (5.14)$$

Having determined $N_+(s)$ and $M_-(s)$ we can now find $A(s)$ from (5.5) and then the temperature field is obtained by inverting the relation (4.2).

As an application of the above results we consider the elastic half space $0 \leqslant \theta \leqslant \pi/2$ on whose free surface $\theta = \alpha = \pi/2$, the following conditions are prescribed:

$$\frac{\partial T}{\partial \theta}(r, 0) = 0, \qquad r > 0$$
$$T(r, 0) = T_0, \quad 0 < r < 1. \quad .. \quad .. \quad .. \quad (5.15)$$

The solution of this problem is well known (Nowacki 1962, p. 119) but its solution by the above method serves as a check on the method.

From relation (5.4) we see that

$$M_+(s) = T_0 \int_0^1 r^{s-1}\, dr = \frac{T_0}{s}, \operatorname{Re}(s) > 0 \quad .. \quad .. \quad (5.16)$$

when $\alpha = \pi/2$ we have

$$K(s) = \frac{P^1_{s-1}(0)}{P_{s-1}(0)} = \frac{2\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}. \quad .. \quad .. \quad (5.17)$$

From this we deduce that $K(s)$ is regular and zeroless in $0 < \text{Re}\,(s) < 1$. We now choose

$$K_+(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)},\; K_-(s) = \frac{2\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \text{ so that}$$

$K_+(s)$ and $K_-(s)$ are regular and zeroless in Re $(s) > 0$ and Re $(s) < 1$ respectively. Also $K_+(s) = 0\ (|s|^{-\frac{1}{2}})$ and $K_-(s) = 0\ (s^{\frac{1}{2}})$ as $|s| \to \infty$ in the appropriate half planes.

Hence (5.9) becomes

$$K_+(s)N_+(s) - K_-(s)M_-(s) + f(s) = 0 \quad \ldots \quad \ldots \quad \ldots (5.18)$$

where

$$f(s) = \frac{2T_0\Gamma\left(\frac{2-s}{2}\right)}{s\Gamma\left(\frac{1-s}{2}\right)}.$$

We now split up $f(s)$ into $f_+(s)+f_-(s)$ such that $f_+(s)$ is regular in Re $(s) > 0$ and $f_-(s)$ in Re $(s) < 1$. To this end we make use of Cauchy's theorem and thus get

$$\left.\begin{aligned} f_+(s) &= \frac{2T_0}{2\pi i}\int_{\gamma_1} \frac{\Gamma\left(\frac{2-l}{2}\right)}{l\Gamma\left(\frac{1-l}{2}\right)}\frac{dl}{l-s} \\ f_-(s) &= \frac{2T_0}{2\pi i}\int_{\gamma_2} \frac{\Gamma\left(\frac{2-l}{2}\right)}{l\Gamma\left(\frac{1-l}{2}\right)}\frac{dl}{l-s} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots (5.19)$$

On integration we get

$$f_+(s) = \frac{2T_0}{\sqrt{\pi}s},\; f_-(s) = \frac{2T_0}{s}\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right]. \quad \ldots \quad \ldots (5.20)$$

Substituting these values of $f_+(s)$ and $f_-(s)$ in the Weiner-Hopf equation (5.10) we obtain

$$K_+(s)N_+(s) + \frac{2T_0}{\sqrt{\pi}s} = K_-(s)M_-(s) - 2T_0\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right]. \quad \ldots (5.21)$$

From the order properties established above in section 5 we see that the left side is $O(|s|^{-1})$ and the right side is $O(|s|^{\frac{1}{2}})$ in their half planes. Hence the polynomial $J(s)$ is identically zero and we obtain

$$\left.\begin{aligned} N_+(s) &= \frac{2T_0}{\sqrt{\pi}SK_+(s)} = \frac{T_0}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{2+s}{2}\right)} \\ M_-(s) &= \frac{2T_0}{SK_-(s)}\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right] = \frac{T_0}{s}\left[1 - \frac{\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{2-s}{2}\right)}\right] \end{aligned}\right\}. \tag{5.22}$$

Now

$$\begin{aligned} n_+(r) &= \int_\gamma N_+(s)r^{-s-1}\,ds, \quad 0 < r < 1 \\ &= \frac{T_0}{\sqrt{\pi}}\frac{1}{2\pi i}\int_\gamma \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{2+s}{2}\right)} r^{-s-1}\,dr, \quad 0 < r < 1. \end{aligned} \tag{5.23}$$

The integrand has simple poles at $s = -(2n+1)$, $n = 1, 2, 3$ and, therefore, the sum of the residues at these poles are

$$2\sum \frac{\lfloor n+\frac{1}{2}}{n!\,\pi} r^{2n}.$$

Hence

$$n_+(r) = \frac{2T_0}{\pi}(1-r^2)^{-\frac{1}{2}}, \quad 0 < r < 1. \tag{5.24}$$

Again

$$\begin{aligned} m_-(r) &= T_0\int_\gamma M_-(s)r^{-s}\frac{ds}{s} \\ &= \frac{T_0}{2\pi i}\int_\gamma \left[1 - \frac{\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{2-s}{2}\right)}\right]\frac{r^{-s}}{s}\,ds, \quad r > 1. \end{aligned} \tag{5.25}$$

Here $r > 1$, so we must close the path in the right half plane and in this region the integrand in the first integral of (5.25) is regular and hence we have

$$\begin{aligned} m_-(r) &= \frac{T_0}{2\pi i\sqrt{\pi}}\int_\gamma \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{2-s}{2}\right)}\frac{r^{-s}}{s}\,ds \\ &= \frac{2T_0}{\pi}\sum \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(2n+1)}\cdot\frac{r^{-2n-1}}{n!} \\ &= \frac{2T_0}{\pi}\sin^{-1}\frac{1}{r}, \quad r > 1. \end{aligned} \tag{5.26}$$

Hence the temperature distribution on the free surface of the half space is

$$
\begin{aligned}
T &= T_0,\ 0 < r < 1 \\
&= -\frac{2T_0}{\pi}\sin^{-1}\frac{1}{r}, \quad r > 1
\end{aligned}
$$

$$
T(r, 0) = T_0\left[\eta(1-r) - \frac{2}{\pi}\eta(r-1)\sin^{-1}\frac{1}{r}\right] \quad \ldots \quad \ldots (5.27)
$$

which tallies with the result obtained in Nowacki (1962, p. 120).

6. Solution of Thermoelastic Problem

Mellin transform of eqns. (2.3) yields

$$
\frac{d}{dp}\left[(1-p^2)\frac{d\hat{F}}{dp}\right] + (s-2)(s-3)\hat{F} = m\hat{T} \quad \ldots \quad \ldots (6.1)
$$

where

$$
\left.\begin{aligned}
\hat{F}(p, s) &= \int_0^\infty F(r, p) r^{s-3}\, dr \\
\hat{T}(p, s) &= \int_0^\infty T(r, p) r^{s-1}\, dr
\end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots (6.2)
$$

Taking into account the relation (4.2) we obtain a particular solution of (6.1) as

$$
\hat{F} = -\frac{mA(s)P_{s-1}(p)}{2(2s-3)P_{s-1}(p_0)} . \quad \ldots \quad \ldots \quad \ldots (6.3)
$$

Again applying the Mellin transform to (2.9) and bearing in mind (3.3) we obtain in this manner the Legendre equations

$$
\left.\begin{aligned}
\frac{d}{dp}\left[(1-p^2)\frac{d\hat{\phi}}{dp}\right] + (s-2)(s-3)\hat{\phi} = 0 \\
\frac{d}{dp}\left[(1-p^2)\frac{d\hat{\psi}}{dp}\right] + (s-2)(s-1)\hat{\psi} = 0
\end{aligned}\right\} . \quad \ldots \quad \ldots (6.4)
$$

Taking into account (6.3) and keeping in view that the thermal displacements and stresses sought must be regular along the axis of the cone $\alpha = 0$ for $p = 1$ we take the solutions of (6.4) in the form

$$
\left.\begin{aligned}
\hat{\phi}(p, s) &= B(s)P_{s-3}(p) \\
\hat{\psi}(p, s) &= C(s)P_{s-2}(p)
\end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots (6.5)
$$

The arbitrary functions $B(s)$ and $C(s)$ appearing in (6.5) are to be determined from the transformed boundary conditions (2.1), i.e.

$$
\hat{\sigma}_{\theta\theta}(p_0, s) = 0,\ \hat{\sigma}_{r\theta}(p_0, s) = 0.
$$

The final displacement and the stress field in the spherical polar coordinates are obtained by adding

(2.5) to (2.11) and (2.4) to (2.12). Thus

$$\left.\begin{aligned} u_r &= \frac{\partial\phi}{\partial r}+\frac{\partial F}{\partial r}+pr\frac{\partial\psi}{\partial r}-(3-4\nu)p\psi \\ u_\theta &= -\frac{q}{r}\frac{\partial F}{\partial p}-\frac{q}{r}\frac{\partial\phi}{\partial p}-pq\frac{\partial\psi}{\partial p}+(3-4\nu)q\psi \\ u_\phi &= 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad (6.6)$$

$$\left.\begin{aligned} \frac{\sigma_{rr}}{2\mu} &= \frac{\partial^2 F}{\partial r^2}+\frac{\partial^2\phi}{\partial r^2}+\phi r\,.\,\frac{\partial^2\psi}{\partial r^2}-2(1-v)p\frac{\partial\psi}{\partial r}-2\frac{\nu q^2}{r}\frac{\partial\psi}{\partial p} \\ \frac{\sigma_{\theta\theta}}{2\mu} &= \frac{q^2}{r^2}\frac{\partial^2 F}{\partial p^2}-\frac{p}{r^2}\frac{\partial F}{\partial p}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{q^2}{r^2}\frac{\partial^2\phi}{\partial p^2} \\ &\quad -\frac{p}{r^2}\frac{\partial\phi}{\partial p}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{pq^2}{r}\frac{\partial^2\psi}{\partial p^2}+(1-2\nu)p\frac{\partial\psi}{\partial r} \\ &\quad -\{1+(1-2\nu)q^2\}\frac{1}{r}\frac{\partial\psi}{\partial r} \\ \frac{\sigma_{\phi\phi}}{2\mu} &= \frac{-p}{r^2}\frac{\delta F}{\partial\phi}+\frac{1}{r}\frac{\partial F}{\partial r}-\frac{p}{r^2}\frac{\partial\phi}{\partial p}+\frac{1}{r}\frac{\partial\phi}{\partial r} \\ &\quad -\frac{1}{r}\frac{\partial\psi}{\partial p}+(1-2\nu)\left\{p\frac{\partial\psi}{\partial r}+\frac{q^2}{r}\frac{\partial\psi}{\partial p}\right\} \\ \frac{\sigma_{r\theta}}{2\mu q} &= \frac{1}{r^2}\frac{\partial F}{\partial p}-\frac{1}{r}\frac{\partial^2 F}{\partial r\partial p}+\frac{1}{r^2}\frac{\partial\phi}{\partial p}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial p}-p\frac{\partial^2\psi}{\partial r\partial p} \\ &\quad +(1-2\nu)\frac{\partial\psi}{\partial r}+2(1-\nu)\frac{p}{r}\frac{\partial\psi}{\partial p} \\ \sigma_{r\phi} &= \sigma_{\theta\phi} = 0. \end{aligned}\right\}. \quad \ldots \quad (6.7)$$

Next let $\hat{u}_r(p, s)$, $\hat{u}_\theta(p, s)$ be the Mellin transforms of $r^{-1}u_r(r, p)$, $r^{-1}u_\theta$ (r, p) and denote by $\hat{\sigma}_{rr}(p, s)$, $\hat{\sigma}_{\theta\theta}(p, s)$, etc., the transforms of the corresponding components of stress. Explicitly

$$\left.\begin{aligned} \hat{U}_r(p, s) &= \int_0^\infty u_r(r, p)r^{s-2}\,dr, \ldots \text{etc.} \\ \hat{\sigma}_{rr}(p, s) &= \int_0^\infty \sigma_{rr}(r, p)r^{s-2}\,dr, \ldots \text{etc.} \end{aligned}\right\}. \quad \ldots \quad \ldots \quad (6.8)$$

Then equations (6.6) and (6.7) with the help of (3.1), (3.3), (6.3), (6.4) and (6.8) reduce to

$$\left.\begin{aligned} \hat{U}_r &= -(s-2)\hat{F}-(s-2)\hat{\phi}-p\{s+2(1-2\nu)\}\psi \\ \hat{U}_\theta &= -q\left\{\frac{d\hat{\phi}}{dp}+p\frac{d\psi}{dp}-(3-4\nu)\hat{\psi}\right\} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (6.9)$$

$$\left.\begin{aligned}
\hat{\sigma}_{rr} &= (s-1)(s-2)\hat{F}+(s-1)(s-2)\hat{\phi}+(s-1)(s+2-2\nu)p\hat{\psi} \\
&\quad -2\nu q^2\frac{d\hat{\psi}}{dp} \\
\frac{\hat{\sigma}_{\theta\theta}}{2\mu} &= -s(4s+7)\hat{F}+p\frac{d\hat{F}}{dp}-(s-2)^2\hat{\phi}+p\frac{d\hat{\phi}}{dp} \\
&\quad -(s-1)(s-1-2\nu)p\hat{\psi}+\{1-(3-2\nu)q^2\}\frac{d\hat{\psi}}{dp} \\
\frac{\hat{\sigma}_{\phi\phi}}{2\mu} &= -(s-2)\hat{F}-p\frac{d\hat{F}}{dp}-(s-2)\hat{\phi}-p\frac{d\hat{\phi}}{dp}-(1-2\nu)(s-1)p\hat{\psi} \\
&\quad -\{1-(1-2\nu)q^2\}\frac{d\hat{\psi}}{dp} \\
\frac{\hat{\sigma}_{r\theta}}{2\mu} &= q\left\{(s-1)\frac{d\hat{F}}{dp}+(s-1)\frac{d\hat{\phi}}{dp}-(1-2\nu)(s-1)\hat{\psi}\right. \\
&\quad \left.+(s+1-2\nu)p\frac{d\hat{\psi}}{dp}\right\}
\end{aligned}\right\} \quad (6.10)$$

Finally the boundary conditions (2.1) in the transformed domain become

$$\hat{\sigma}_{\theta\theta}(p,\, s) = 0,\ \hat{\sigma}_{r\theta}(p_0,\, s) = 0. \quad \text{.. .. ..} \quad (6.11)$$

Substituting (6.5) into the second and the last of (6.10) and invoking (6.11) subsequently we get a pair of simultaneous equations involving $A(s)$ and $B(s)$.

These simultaneous equations can further be simplified with the aid of the recursion relations:

$$\left.\begin{aligned}
q^2P'_s(p) &= s[P_{s-1}(p)-pP_s(p)] \\
(2s+1)pP_s(p) &= (s+1)P_{s+1}(p)+sP_{s-1}(p)
\end{aligned}\right\} \quad \text{.. ..} \quad (6.12)$$

where prime denotes differentiation with respect to the argument. Setting

$$\begin{aligned}
\Delta(p_0,\, s) &= (s-1)^2(s-2)^2p_0[p_{s-2}(p_0)]^2+(s-1)(s-2)q_0^{\,2}p_{s-2}(p_0)p'_{s-2}(p_0) \\
&\quad +p_0\{2(\nu-1)+(s-1)(s-2)q_0^{\,2}\}[p'_{s-2}(p_0)]^2 \quad \text{.. .. ..} \quad (6.13)
\end{aligned}$$

and making repeated use of (6.12) we write the final formulae for $\hat{\phi}$, $\hat{\psi}$ in the form

$$\left.\begin{aligned}
\hat{\phi} &= \frac{RG_1}{\Delta}P_{s-3}(p) \\
\hat{\psi} &= \frac{RG_2}{\Delta}P_{s-2}(p)
\end{aligned}\right\} \quad \text{.. .. .. ..} \quad (6.14)$$

where

$$\left.\begin{aligned}
G_1 &= P'_{s-1}(p_0)[P'_{s-2}(p_0)\{2(1-\nu)(1+(s-2)q^2)\}+(s-1)(s-2)(s-2\nu) \\
&\quad \times p_0P_{s-2}(p_0)]-s(4s+7)P_{s-1}(p_0)[(s+1-2\nu)p_0P'_{s-2}(p_0) \\
&\quad -(s-1)(1-2\nu)P_{s-2}(p_0)] \\
G_2 &= (s-1)[s(4s+1)P_{s-1}(p_0)P'_{s-3}(p_0)-(s-2)^2P'_{s-1}(p_0)P_{s-3}(p_0)
\end{aligned}\right\} \quad (6.15)$$

and

$$R = -m\frac{A(s)}{2(2s-3)P_{s-1}(p_0)}.$$

Putting (6.14) and (6.15) into (6.10) and using (6.12) we get the expressions for the transformed stress components in the form:

$$\left.\begin{aligned}
\frac{\hat{\sigma}_{rr}}{2\mu R} &= \Big[(s-1)(s-2)\Big\{P_{s-1}(p)+\frac{G_1}{\Delta}P_{s-3}(p)\Big\}+\frac{G_2}{\Delta} \\
&\qquad \{(s-1)(s+2-2\nu)pP_{s-2}(p)-2\nu q^2P'_{s-2}(p)\}\Big] \\
\frac{\hat{\sigma}_{\theta\theta}}{2\mu R} &= \Big[-s(4s+7)P_{s-1}(p)+pP'_{s-1}(p)-\frac{G_1}{\Delta}\{(s-2)^2P_{s-3}(p)-pP'_{s-3}(p)\} \\
&\qquad -\frac{G_2}{\Delta}\{(s-1)(s-1-2\nu)pP_{s-2}(p)+(1-(3-2\nu)q^2)P'_{s-2}(p)\}\Big] \\
\frac{\hat{\sigma}_{\phi\phi}}{2\mu R} &= -\Big[(s-2)P_{s-1}(p)+pP'_{s-1}(p)+\frac{G_1}{\Delta}\{(s-2)P_{s-3}(p)-pP'_{s-3}(p)\} \\
&\qquad +\frac{G_2}{\Delta}\{(s-1)(1-2\nu)pP_{s-2}(p)+(1-(1-2\nu)q^2)P'_{s-2}(p)\}\Big] \\
\frac{\hat{\sigma}_{r\theta}}{2\mu R} &= \Big[(s-1)P'_{s-1}(p)+(s-1)\frac{G_1}{\Delta}P'_{s-3}(p)-\frac{G_2}{\Delta}\{(s-1)(1-2\nu)P_{s-2}(p) \\
&\qquad -(s+1-2\nu)pP_{s-2}(p)\}\Big]
\end{aligned}\right\}. \qquad (6.16)$$

For brevity the analogous formulae for the displacement components $\hat{u}_r$, $\hat{u}_\theta$ are omitted.

In order to pass from the transformed stress functions and stresses to their antecedents in the physical domain, we make an appeal to the inversion theorem for the Mellin transform.

By virtue of (3.1) and (6.8) we arrive at

$$\left.\begin{aligned}
F(r,p) &= \frac{r^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{F}(p,s)r^{-s}ds \\
\phi(r,p) &= \frac{r^2}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{\phi}(p,s)r^{-s}ds \\
\psi(r,p) &= \frac{r}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{\psi}(p,s)r^{-s}ds
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots (6.17)$$

and

$$\sigma_{rr}(r,p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{\sigma}_{rr}(p,s)r^{-s}ds \ldots \text{etc.} \quad \ldots \quad \ldots (6.18)$$

The integrands in (6.17) and (6.18) are fully accounted for through (6.14) and (6.15).

This completes the formal solution in complex integral form of the thermal stress problem under consideration.

In conclusion, we merely mention that the solution to the thermoelastic problem of cone established in this investigation when $\alpha = \pi/2$ is reducible to the corresponding result for the semi-infinite elastic solid obtained in Nowacki (1962, pp. 119–21).

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