ON H-PROJECTIVE CONNEXIONS

by B. B. SINHA, *Mathematics Department, Banaras Hindu University, Varanasi 5*

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The holomorphically projective transformation in an almost complex space corresponds to the projective change in the Riemannian geometry. The purpose of the present paper is to define and to study the quantities with respect to the symmetric F-connexion and the half-symmetric F-connexion on the lines of the projective connexion as defined by Thomas (1934) and Sinha (1965). The notations of Yano (1965) have been used in the sequel.

1. INTRODUCTION

Let us consider a $2n$-dimensional almost complex space C_n endowed with an almost complex structure F_i^h such that

$$F_j^{\ i}F_i^{\ h} = -A_j^h. \qquad \ldots \qquad (1.1)$$

In an almost complex space an affine connexion Γ_{ji}^h is called F-connexion if the almost complex structure F_i^h is a covariant constant with respect to this connexion. With an F-connexion Γ_{ji}^h we consider a curve $\xi^h = \xi^h(t)$ satisfying the differential equations

$$\frac{d^2\xi^h}{dt^2} + \Gamma_{ji}^h \frac{d\xi^j}{dt}\frac{d\xi^i}{dt} = \alpha(t)\frac{d\xi^h}{dt} + \beta(t) F_r^h \frac{d\xi^r}{dt} \qquad \ldots \qquad (1.2)$$

where $\alpha(t)$ and $\beta(t)$ are certain functions of parameter t, such a curve is defined as a holomorphically planar curve.

Two F-connexions Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ are said to be holomorphically projectively related to each other if they have all holomorphically planar curves in common. Two symmetric F-connexions Γ_{ji}^h and $\bar{\Gamma}_{ji}^h$ are H-projectively related if and only if

$$\bar{\Gamma}_{ji}^h = \Gamma_{ji}^h + 2P_{(j}A_{i)}^h - 2P_t F_{(j}^t F_{i)}^h \qquad \ldots \qquad (1.3)$$

where P_t is a vector field.

Let C_n and $\bar{C}_n$ be two almost complex spaces. The two spaces are said to be H-projectively related if their holomorphically planar curves are same. Equation (1.3) gives H-projective transformation of the F-connexions (Yano 1965).

Contracting (1.3) with respect to h and j, we get

$$\bar{\Gamma}^h_{ht} = \Gamma^h_{ht} + (n+2)P_t. \quad \ldots \quad (1.4)$$

2. H-projective Symmetric F-connexion

Eliminating P_i from (1.3) with the help of (1.4), we have

$$\Pi^h_{ji} = \Gamma^h_{ji} - \frac{2}{n+2}\left[\Gamma^l_{l(j}A^h_{i)} - \Gamma^l_{lt}F^t_{(j}F^h_{i)}\right] \quad \ldots \quad (2.1)$$

which is independent of the choice of P_i and so it is H-projective invariant.

We have the following theorems:

Theorem 2.1—The necessary and sufficient condition that the two symmetric F-connexions be H-projectively related to each other is that the quantities Π^h_{ji} corresponding to them coincide.

Proof: When two symmetric F-connexions are H-projectively related, their quantities Π^h_{ji} are H-projective invariants.

Conversely, if the quantities Π^h_{ji} corresponding to the symmetric F-connexions $\bar{\Gamma}^h_{ji}$ and Γ^h_{ji} are same, then we have

$$\bar{\Gamma}^h_{ji} - \Gamma^h_{ji} = \frac{2}{n+2}\Big[(\bar{\Gamma}^l_{lj} - \Gamma^l_{lj})A^h_i + (\bar{\Gamma}^l_{li} - \Gamma^l_{li})A^l_j$$
$$- (\bar{\Gamma}^l_{lt} - \Gamma^l_{lt})F^t_{(j}F^h_{i)}\Big].$$

Putting $P_i = \frac{1}{n+2}(\bar{\Gamma}^l_{li} - \Gamma^l_{li})$, we have (1.3) which proves that the symmetric F-connexions are H-projectively related.

Theorem 2.2—In order that an almost complex space admits H-projective transformation, it is necessary and sufficient that there exists a coordinate system with respect to which the quantities Π^h_{ji} are independent of one of the coordinates.

Proof: When the components of the H-projective symmetric F-connexion are independent of ξ^i, the components Γ^h_{ji} have the form

$$\Gamma^h_{ji} = f^h_{ji}(\xi^2\xi^3 \ldots \xi^n) + 2\left[A^h_{(j}P_{i)} - 2P_tF^t_{(j}F^h_{i)}\right] \quad \ldots \quad (2.2)$$

where P_i are functions of $\xi^1 \ldots \xi^n$.

Conversely, if the components Γ^h_{ji} of the symmetric F-connexion have the form (2.2), then from (2.1) it is clear that the quantities Π^h_{ji} are independent of the variable ξ^1.

Corollary 2.1—In order that a complex space admits an H-projective transformation, it is necessary and sufficient that there exists a coordinate system to which the components Γ^h_{ji} of the symmetric F-connexion have the form (2.2).

In an almost complex space with symmetric F-connexion if the infinitesimal point transformation

$$\bar{\xi}^h = \xi^h + v^h(\xi)\, dt \qquad \qquad (2.3)$$

maps any holomorphically planar curve into another one, the vector is defined as H-projective vector corresponding to the F-connexion.

Theorem 2.3—For a contravariant analytic H-projective vector corresponding to a symmetric F-connexion, $\underset{v}{L}\Pi^h_{ji} = 0$.

PROOF: If v^h is an H-projective vector of the symmetric F-connexion Γ^h_{ji}, we have

$$\underset{v}{L}\Gamma^h_{ji} = 2\rho_{(j}A^h_{i)} - \rho_t F^t_{(j}F^h_{i)} \qquad \qquad (2.4)$$

for a certain vector field ρ_t. Contracting h and j in (2.4), we obtain

$$\rho_i = \frac{1}{n+2}\, \underset{v}{L}\Gamma^l_{li}. \qquad \qquad (2.5)$$

Taking Lie-derivative of (2.1) and making use of (2.4), (2.5) and the fact that v^h is a contravariant analytic vector field for which $\underset{v}{L}F_j{}^i = 0$, we have

$$\underset{v}{L}\Pi^h_{ji} = 0 \qquad \qquad (2.6)$$

which proves the statement.

Theorem 2.4—If v^h is a contravariant analytic H-projective vector, the Lie-derivative of the quantities Π^h_{ji} relative to a symmetric F-connexion with respect to (2.3) is pure in h and j.

PROOF: For a vector field v^h

$$\underset{v}{L}\nabla_j F^h_i - \nabla_j \underset{v}{L}F^h_i = F^t_i \underset{v}{L}\Pi^h_{jt} - F^h_r \underset{v}{L}\Pi^r_{ji}$$

which implies, because of $\nabla_j F_i{}^h = 0$, $\underset{v}{L}F_j{}^h = 0$ and (1.1)

$$\underset{v}{L}\Pi^h_{js} = -F^h_r F^i_s \underset{v}{L}\Pi^r_{ji}.$$

Hence $\underset{v}{L}\Pi^h_{ji}$ is pure in h and j.

Theorem 2.5—In order that an almost complex space admits a contravariant analytic H-projective vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the quantities Π^h_{ji} are a homogeneous function of degree -1 of the coordinate system.

PROOF: If we choose a coordinate system with respect to which $v^h = \xi^h$ the equation $\underset{v}{L}\Pi^h_{ji} = 0$ takes the form

$$\underset{v}{L}\Pi^h_{ji} = \xi^k \partial_k \Pi^h_{ji} + \Pi^h_{ji} = 0, \quad \partial_k = \partial/\partial\xi^k \qquad \qquad (2.7)$$

which proves the statement.

3. *H*-projective Half-symmetric *F*-connexion

An F-connexion Γ^h_{ji} is half-symmetric if its torsion tensor S^h_{ji} satisfies

$$O^{tk}_{ji} O^{sh}_{kr} S^r_{ts} = 0 \quad \ldots \quad (3.1)$$

where

$$2O^{tk}_{ji} = A^t_j A F^k_i - F^t_j F^k_i.$$

Two half-symmetric F-connexions Γ^h_{ji} and $\bar{\Gamma}^h_{ji}$ are H-projectively related if

$$\bar{\Gamma}^h_{ji} = \Gamma^h_{ji} + 2P_{(j}A^h_{i)} - 2P_t F^t_{(j} F^h_{i)} + Q_j A^h_i + Q_t F^t_j F^h_i \quad \ldots \quad (3.2)$$

where P_i and Q_i are certain vector fields (Yano 1965).

Contracting h, j and h, i in (3.2), we have

$$(n+2)P_i = \bar{\Gamma}^r_{ri} - \Gamma^r_{ri} \quad \ldots \quad (3.3)$$

and

$$nQ_i = 2\bar{\Gamma}^r_{[ir]} - 2\Gamma^r_{[ir]} \quad \ldots \quad (3.4)$$

respectively. Eliminating P_i and Q_i from (3.2), (3.3) and (3.4), we obtain

$$\overset{\circ}{\Pi}{}^h_{ji} = \Gamma^h_{ji} - \frac{2}{n+2}\left[\Gamma^r_{r(j}A^h_{i)} - \Gamma^r_{rt}F^t_{(j}F^h_{i)}\right] - \frac{2}{n}\left[\Gamma^r_{[jr]}A^h_i + \Gamma^r_{[tr]}F^t_j F^h_i\right] \quad \ldots \quad (3.5)$$

which is invariant of the H-projective transformation (3.2).

We have the following theorems:

Theorem 3.1—The necessary and sufficient condition that two half-symmetric F-connexions are H-projectively related is that the quantities $\overset{\circ}{\Pi}{}^h_{ji}$ corresponding to them coincide.

Theorem 3.2—In order that an almost complex space admits an H-projective transformation, it is necessary and sufficient that there exists a coordinate system with respect to which the quantities are independent of all of the coordinates.

Corollary 3.1—In order that an almost complex space admits an H-projective transformation, it is necessary and sufficient that there exists a coordinate system to which the components of half-symmetric F-connexion have the form

$$\Gamma^h_{ji} = f^h_{ji} + 2P_{(j}A^h_{i)} - 2P_t F^t_{(j}F^h_{i)} + Q_j A^h_i + Q_t F^t_j F^h_i \quad \ldots \quad (3.6)$$

where f^h_{ji} is a constant.

When a vector field v^h has the property

$$\underset{v}{L}\Gamma^h_{ji} = 2\rho_{(j}A^h_{i)} - 2\rho_t F^t_{(j}F^h_{i)} + \sigma_j A^h_i + \sigma_t F^t_j F^h_i \quad \ldots \quad (3.7)$$

it is known as H-projective vector field with respect to the half-symmetric F-connexion. ρ_i and σ_i are associated vectors (Yano 1965).

Contracting h, j and h, i in (3.7) and after some simplification, we have respectively

$$\underset{v}{L}\Gamma^{r}_{ri} = (n+2)\rho_i \quad \ldots \quad (3.8)$$

and

$$2\underset{v}{L}\Gamma^{r}_{[ir]} = n\sigma_i. \quad \ldots \quad (3.9)$$

Substituting the value of ρ and σ from (3.8) and (3.9) in (3.7), we obtain

$$\underset{v}{L}\Gamma^{h}_{ji} = \frac{2}{n+2}\left[\underset{v}{L}\Gamma^{r}_{r(j}A^{h}_{i)} - \underset{v}{L}\Gamma^{r}_{rt}F^{t}_{(j}F^{h}_{i)}\right] + \frac{2}{n}\left[\underset{v}{L}\Gamma^{r}_{[jr]}A^{h}_{i} + \underset{v}{L}\Gamma^{r}_{[tr]}F^{t}_{j}F^{h}_{i}\right]. \quad \ldots \quad (3.10)$$

We can easily obtain the following theorems:

Theorem 3.3—For a contravariant analytic H-projective vector field v^h with respect to a half-symmetric F-connexion $\underset{v}{L}\overset{\circ}{\Pi}{}^{h}_{ji} = 0$.

Theorem 3.4—If v^h is a contravariant analytic H-projective vector field $\underset{v}{L}\overset{\circ}{\Pi}{}^{h}_{ji}$ is pure in h and j.

Theorem 3.5—In order that an almost complex space admits a contravariant analytic H-projective vector field, it is necessary and sufficient that there exists a coordinate system with respect to the quantities $\overset{\circ}{\Pi}{}^{h}_{ji}$ are a homogeneous function of degree -1 of the coordinate system.

The proofs of the above theorems follow the pattern of those of the theorems of § 2.

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