ON ALMOST HERMITE SPACES

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In this paper we have obtained some results concerning almost Hermite, almost Kaehler and almost Tachibana spaces.

1. INTRODUCTION

Let us consider a $2n$-dimensional space M_{2n} of differentiability class C^{r+1}. Let there be defined in M_{2n} a vector-valued linear function F such that

$$\bar{\bar{X}}+X=0 \qquad \text{(1.1)}$$

for an arbitrary vector field X, where

$$\bar{X}\overset{\text{def}}{=}F(X). \qquad \text{(1.2)}$$

Then F is said to give an almost complex structure to M_{2n} and M_{2n} is called an 'almost complex space'.

Agreement 1.1—All the equations which follow, hold for arbitrary vector fields X, Y, Z, . . . , etc.

Let the almost complex space M_{2n} be also endowed with the Hermitian metric tensor g:

$$g(\bar{X}, \bar{Y})=g(X, Y). \qquad \text{(1.3)}$$

Then M_{2n} is called an almost Hermite space.

Let us put

$$'F(X, Y)\overset{\text{def}}{=}g(\bar{X}, Y). \qquad \text{(1.4)}$$

Then from (1.1), (1.2), (1.3) and (1.4), we have

$$'F(\bar{X}, \bar{Y})=-g(X, \bar{Y})=g(\bar{X}, Y)='F(X, Y) \qquad \text{(1.5a)}$$

$$-'F(\bar{X}, Y)=+g(X, Y)='F(X, \bar{Y}) \qquad \text{(1.5b)}$$

$$'F(X, Y)+'F(Y, X)=0. \qquad \text{(1.6)}$$

Suppose D is a Riemannian connexion in M_{2n}.

$$D_XY-D_YX=[X, Y] \qquad \text{(1.7)}$$

$$X.\{g(Y, Z)\}=g(D_XY, Z)+g(Y, D_XZ). \qquad \text{(1.8)}$$

If in addition to (1.1), (1.3) and (1.6), we also have

$$(D_XF)(Y)=0 \qquad \text{(1.9)}$$

M_{2n} is called Kaehler space. An almost Hermite space, for which

$$(D_X F)(Y)+(D_Y F)(X)=0 \quad \ldots \quad (1.10)$$

is satisfied, is said to be an almost Tachibana space. An almost Hermite space, for which

$$(D_X{}'F)(Y, Z)+(D_Y{}'F)(Z, X)+(D_Z{}'F)(X, Y)=0 \quad \ldots \quad (1.11)$$

is satisfied, is said to be an almost Kaehler space. An almost Hermite space, for which

$$\operatorname{div} F=0 \quad \ldots \quad (1.12)$$

is satisfied, is said to be an almost semi-Kaehler space.

For an almost Hermite space, we have (Mishra 1969)

$$(D_X{}'F)(Y, \bar{Z})=(D_X{}'F)(\bar{Y}, Z) \quad \ldots \quad (1.13a)$$

$$(D_X{}'F)(\bar{Y}, \bar{Z})=-(D_X{}'F)(Y, Z). \quad \ldots \quad (1.13b)$$

For an almost Kaehler space, we have (Mishra 1969)

$$(D_{\bar{X}}{}'F)(\bar{Y}, Z)+(D_{\bar{Y}}{}'F)(Z, \bar{X})+(D_{\bar{Z}}{}'F)(\bar{X}, Y)=0. \quad \ldots \quad (1.14)$$

For an almost Tachibana space, we have (Mishra 1969)

$$(D_{\bar{X}}{}'F)(Y, Z)+(D_{\bar{Y}}{}'F)(X, Z)=0. \quad \ldots \quad (1.15)$$

Nijenhuis tensor N is given by (Mishra and Ram Hit 1971)

$$N(X, Y) \stackrel{\text{def}}{=} [\bar{X}, \bar{Y}]-[X, Y]-\overline{[\bar{X}, Y]}-\overline{[X, \bar{Y}]} \quad \ldots \quad (1.16a)$$

where

$$[X, Y]=D_X Y-D_Y X. \quad \ldots \quad (1.16b)$$

If we put

$$'N(X, Y, Z) \stackrel{\text{def}}{=} g(N(X, Y), Z) \quad \ldots \quad (1.17)$$

$$'M(X, Y, Z) \stackrel{\text{def}}{=} (D_{\bar{X}}{}'F)(Y, Z)+(D_X{}'F)(\bar{Y}, Z) \quad \ldots \quad (1.18)$$

then (Yano 1965)

$$'N(X, Y, Z)={}'M(X, Y, Z)-{}'M(Y, X, Z). \quad \ldots \quad (1.19)$$

From (1.19) we have

$$'M(X, Y, Z)=\tfrac{1}{2}('N(X, Y, Z)+'N(Z, X, Y)+'N(Z, Y, X)). \quad \ldots \quad (1.20)$$

From (1.19) and (1.20) we can see that the vanishing of the Nijenhuis tensor is equivalent to the vanishing of the tensor $'M(X, Y, Z)$.

2. Almost Hermite Spaces

Theorem 1—The necessary and sufficient condition for almost Hermite space to be Hermite space is

$$g((D_{\bar{X}} F)(Y), Z)={}'F((D_X F)(Y), Z). \quad \ldots \quad (2.1)$$

PROOF: Let us put

$$'M(X, Y, Z) \overset{\text{def}}{=} g(D_X\bar{Y}-D_X Y, Z)+g(D_X\bar{Y}+D_{\bar{X}}Y, \bar{Z}). \quad .. \quad (2.2)$$

Then we have (Mishra 1967)

$$'M(X, Y, Z) = g((D_{\bar{X}}F)(Y), Z)-'F((D_X F)(Y), Z). \quad .. \quad (2.3)$$

The statement follows from (1.19), (1.20) and (2.3).

Theorem 2—Let us put

$$G(X, Y, Z) \overset{\text{def}}{=} (D_X{'F})(Y, Z)+(D_Y{'F})(X, Z) \quad .. \quad .. \quad .. \quad (2.4)$$

$$J(X, Y, Z) \overset{\text{def}}{=} (D_X{'F})(Y, Z)+(D_Y{'F})(Z, X)+(D_Z{'F})(X, Y). \quad .. \quad (2.5)$$

Then

$$G(X, Y, Z)+G(Y, Z, X)+G(Z, X, Y) = 0 \quad .. \quad .. \quad (2.6)$$

$$G(X, \bar{Y}. \bar{Z})-G(X, \bar{Z}, \bar{Y}) = (D_{\bar{Y}}{'F})(\bar{Z}, X)+(D_{\bar{Z}}{'F})(\bar{X}, Y)-2(D_X{'F})(Y, Z). \quad (2.7)$$

PROOF: Writing two other equations obtained by cyclic permutation of X, Y, Z in (2.4), adding these two equations to (2.4) and using the fact that $'F$ is anti-symmetric we get (2.6). From (2.4) and (2.5) we have

$$G(X, Y, Z)-G(X, Z, Y) = 3(D_X{'F})(Y, Z)-J(X, Y, Z). \quad .. \quad (2.8)$$

Barring Y and Z in (2.8) and using (1.13*b*) and (2.5), we get (2.7).

Corollary 1—We have

$$G(\bar{X}, \bar{Z}, \bar{Y})-G(\bar{X}, \bar{Y}, \bar{Z}) = (D_{\bar{Y}}{'F})(Z, X)+(D_{\bar{Z}}{'F})(X, Y)+2(D_{\bar{X}}{'F})(Y, Z) \quad (2.9a)$$

$$G(X, Y, \bar{Z})-G(X, \bar{Z}, Y) = (D_Y{'F})(\bar{Z}, X)+(D_{\bar{Z}}{'F})(X, Y)+2(D_X{'F})(\bar{Y}, Z) \quad (2.9b)$$

$$G(X, \bar{Y}, Z)-G(X, Z, \bar{Y}) = (D_{\bar{Y}}{'F})(Z, X)+(D_Z{'F})(\bar{X}, Y)+2(D_X{'F})(Y, \bar{Z}) \quad (2.9c)$$

$$G(\bar{X}, \bar{Z}, Y)-G(\bar{X}, Y, \bar{Z}) = (D_Y{'F})(Z, X)-(D_{\bar{Z}}{'F})(X, \bar{Y})-2(D_{\bar{X}}{'F})(\bar{Y}, Z), \text{ etc.} \quad .. \quad (2.9d)$$

PROOF: Barring different vectors in (2.7) and using (1.1) and (1.13), we obtain (2.9 *a*, *b*, *c*, *d*).

Theorem 3—For an almost Kaehler space, we have

$$G(X, \bar{Y}, \bar{Z})+G(Y, \bar{Z}, \bar{X})+G(Z, \bar{X}, \bar{Y}) = 0. \quad .. \quad (2.10)$$

PROOF: Barring Y and Z in (2.4) and using (1.13*b*), we get

$$G(X, \bar{Y}, \bar{Z}) = -(D_X{'F})(Y, Z)+(D_{\bar{Y}}{'F})(X, \bar{Z}). \quad .. \quad .. \quad (2.11)$$

Similarly

$$G(Y, \bar{Z}, \bar{X}) = -(D_Y{'F})(Z, X)+(D_Z{'F})(Y, \bar{X}) \quad .. \quad .. \quad (2.12)$$

$$G(Z, \bar{X}, \bar{Y}) = -(D_Z{'F})(X, Y)+(D_{\bar{X}}{'F})(Z, \bar{Y}). \quad .. \quad .. \quad (2.13)$$

Adding these three equations and using (1.11) and (1.14), we get (2.10).

Corollary 2—We have

$$G(\bar{X}, \bar{Y}, \bar{Z})-G(Y, \bar{Z}, X)-G(Z, X, \bar{Y}) = 0 \quad .. \quad .. \quad (2.14a)$$

$$G(\bar{Y}, \bar{Z}, \bar{X})-G(X, Y, \bar{Z})-G(Z, \bar{X}, Y) = 0 \quad .. \quad .. \quad (2.14b)$$

$$G(\bar{Z}, \bar{X}, \bar{Y})-G(X, \bar{Y}, Z)-G(Y, Z, \bar{X}) = 0 \quad .. \quad .. \quad (2.14c)$$

$$G(Z, X, Y)-G(\bar{X}, Y, \bar{Z})-G(Y, \bar{Z}, X) = 0, \text{ etc.} \quad .. \quad (2.14d)$$

Theorem 4—For an almost Kaehler space, we have

$$G(X, \bar{Y}, \bar{Z})+G(\bar{X}, Y, \bar{Z})+G(X, Y, Z) = 2(D_{\bar{Y}}'F)(X, \bar{Z})+(D_{\bar{Z}}'F)(X, \bar{Y}) \quad .. \quad (2.15)$$

$$G(X, \bar{Z}, \bar{Y})-G(X, \bar{Y}, \bar{Z}) = 2(D_X'F)(Y, Z)+(D_{\bar{X}}'F)(\bar{Y}, Z). \quad .. \quad .. \quad (2.16)$$

Proof: Barring Y and Z in (2.4) and using (1.13*b*), we get

$$G(X, \bar{Y}, \bar{Z}) = -(D_X'F)(Y, Z)+(D_{\bar{Y}}'F)(X, \bar{Z}). \quad .. \quad .. \quad (2.17)$$

Similarly

$$G(\bar{X}, Y, \bar{Z}) = -(D_Y'F)(X, Z)+(D_{\bar{X}}'F)(Y, \bar{Z}). \quad .. \quad .. \quad (2.18)$$

Adding these two equations and using (1.14) and (2.4), we get (2.15). Equation (2.16) is obtained from (2.7) by using (1.14).

Theorem 5—Let us put

$$G(X, Y, Z) \overset{\text{def}}{=} (D_{\bar{X}}'F)(Y, Z)+(D_Y'F)(X, Z) \quad .. \quad .. \quad (2.19)$$

$$'M(X, Y, Z) \overset{\text{def}}{=} (D_{\bar{X}}'F)(Y, Z)+(D_X'F)(Y, \bar{Z}). \quad .. \quad .. \quad (2.20)$$

Then

$$'M(X, Y, Z)+'M(Y, X, Z) = G(X, Y, \bar{Z})-G(\bar{X}, \bar{Y}, \bar{Z}). \quad .. \quad (2.21)$$

Consequently, for almost Kaehler space, we have

$$G(X, Y, \bar{Z})+G(\bar{X}, \bar{Y}, \bar{Z}) - 0. \quad .. \quad .. \quad .. \quad (2.22)$$

Proof: From (2.20), we have

$$'M(X, Y, Z)+'M(Y, X, Z) = (D_{\bar{X}}'F)(Y, Z)+(D_X'F)(Y, \bar{Z})$$
$$+(D_{\bar{Y}}'F)(X, Z)+(D_Y'F)(X, \bar{Z}). \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.23)$$

Barring X, Y and Z in (2.19) and using (1.13*b*), we get

$$G(\bar{X}, \bar{Y}, \bar{Z}) = -\{(D_{\bar{X}}'F)(Y, Z)+(D_{\bar{Y}}'F)(X, Z)\}. \quad .. \quad (2.24)$$

Substituting from (2.24), (2.19) in (2.23), we get (2.21).

Using (1.13*a*) and (1.14) in (2.23), we get

$$'M(X, Y, Z)+'M(Y, X, Z) = -(D_Y'F)(Z, \bar{X})-(D_Z'F)(\bar{X}, Y)+(D_X'F)(Y, \bar{Z})$$
$$-(D_X'F)(Z, \bar{Y})-(D_Z'F)(\bar{Y}, X)+(D_Y'F)(X, \bar{Z})$$
$$= 2(D_X'F)(Y, \bar{Z})+2(D_Y'F)(X, \bar{Z}). \quad .. \quad .. \quad .. \quad .. \quad (2.25)$$

Using (2.19) and (2.21) in (2.25), we get (2.22).

Theorem 6—A necessary and sufficient condition for almost Hermite space to be almost Tachibana space is

$$(D_{\bar{X}}F)(Y)+(D_{\bar{Y}}F)(X) = \overline{(D_XF)(Y)}+\overline{(D_YF)(X)}.$$

Proof: The statement follows from the fact that Nijenhuis tensor is skew-symmetric in all the vectors in almost Tachibana space.

Theorem 7—In almost Tachibana space, we have

$$[\bar{X}, \bar{Y}]+[X, Y] = \overline{D_{\bar{X}}Y}+\overline{D_Y\bar{X}}-\overline{D_{\bar{Y}}X}-\overline{D_X\bar{Y}} \quad .. \quad (2.26a)$$

$$[\bar{X}, Y]-[X, \bar{Y}] = -\overline{D_XY}-\overline{D_YX}-\overline{D_{\bar{Y}}\bar{X}}-\overline{D_{\bar{X}}\bar{Y}}. \quad .. \quad (2.26b)$$

PROOF: We have

$$D_X\bar{Y}+D_Y\bar{X} = (D_XF)(Y)+(D_YF)(X)+\overline{D_XY}+\overline{D_YX}.$$

Substituting from (1.10) in this equation and then barring X in the resulting equation and using (1.1) and (1.16*b*), we obtain (2.26*a*). (2.26*b*) follows from (2.26*a*) by barring X or Y and using (1.1).

Theorem 8—For almost Tachibana space, we have

$$J(X, \bar{Y}, \bar{Z})+J(Y, \bar{X}, \bar{Z}) = 0. \quad \text{(2.27)}$$

PROOF: Barring Y and Z in (2.5) and using (1.13*b*), we get

$$J(X, \bar{Y}, \bar{Z}) = -(D_X'F)(Y, Z)+(D_{\bar{Y}}'F)(\bar{Z}, X)+(D_{\bar{Z}}'F)(X, \bar{Y}). \quad \text{(2.28)}$$

Similarly

$$J(Y, \bar{X}, \bar{Z}) = -(D_Y'F)(X, Z)+(D_{\bar{X}}'F)(\bar{Z}, Y)+(D_{\bar{Z}}'F)(Y, \bar{X}). \quad \text{(2.29)}$$

Adding these equations and using (1.15), we get (2.27).

Corollary 3—We also have

$$J(\bar{X}, \bar{Y}, Z)-J(Y, X, Z) = 0 \quad \text{(2.30a)}$$

$$J(X, \bar{Y}, Z)+J(Y, \bar{X}, Z) = 0 \quad \text{(2.30b)}$$

$$J(\bar{X}, \bar{Y}, \bar{Z})-J(Y, X, \bar{Z}) = 0. \quad \text{(2.30c)}$$

PROOF: Barring X in (2.27) and using (1.1), we get (2.30*a*). (2.30*b*, *c*) follow from (2.30*a*) and (1.1).

Theorem 9—If an almost Hermite space has the following two properties:

(*a*) it is an almost Kaehler space

(*b*) it is an almost Tachibana space

then

$$(D_{\bar{Z}}'F)(X, \bar{Y}) = 2(D_X'F)(Y, Z). \quad \text{(2.31)}$$

PROOF: Let us put

$$J(X, Y, Z) \overset{\text{def}}{=} (D_X'F)(Y, Z)+(D_Y'F)(Z, X)+(D_Z'F)(X, Y) \quad \text{(2.32)}$$

$$G(X, Y, Z) \overset{\text{def}}{=} (D_X'F)(Y, Z)+(D_Y'F)(X, Z). \quad \text{(2.33)}$$

Then

$$2(D_X'F)(Y, Z)+(D_Z'F)(X, Y) = 0 \quad \text{(2.34)}$$

under the conditions (*a*) and (*b*).

Barring Y and Z in (2.34) and using (1.13*b*), we get

$$(D_{\bar{Z}}'F)(X, \bar{Y}) = 2(D_X'F)(Y, Z). \quad \text{(2.35)}$$

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